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THE DOMINATION NUMBER OF CONNECTED GRAPHS

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Abstract: Let $\gamma(G)$ be the domination number of a graph G and $\mathcal{CG}(m,n)$ be the class of all non-isomorphic connected graphs of order n and size m. Punnim [3] proved that for any positive integers n and m with $n-1 \leq m \leq \binom{n}{2}$, there exist positive integers a and b in which there exists a graph $G \in \mathcal{CG}(m,n)$ such that $\gamma(G) = c$ if and only if c is an integer and $a \leq c \leq b$. Thus for any positive integers m and n, $\{\gamma(G) : G \in \mathcal{CG}(m,n)\}$ is uniquely determined by

$$a := \min(\gamma; m,n) = \min\{\gamma(G) : G \in \mathcal{CG}(m,n)\}, \quad \text{and}$$
$$b := \max(\gamma; m,n) = \max\{\gamma(G) : G \in \mathcal{CG}(m,n)\}.$$

We are able to find the values of $\min(\gamma; m,n)$ and $\max(\gamma; m,n)$ in all situations.

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1. Introduction

We limit our discussion to graphs that are simple and finite. For the most part,

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our notation and terminology follows that of Chartrand and Zhang [1]. Let $G = (V, E)$ denote a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. We use $|S|$ to denote the cardinality of a set S. We define $n = |V|$ to be the *order* of G and $m = |E|$ to be the *size* of G. We simply write $e = uv$ for the *edge* e that joins the vertices u and v. Let $e = uv$ be an edge of G. We say that u and v are *joined* by the edge e. The vertices u and v are referred to as *neighbors* of each other. The *degree* of a vertex v in a graph G is denoted by $\deg_G v$, or simply written $\deg v$ if the graph G is clear from the context. A vertex of degree 0 is referred as an *isolated vertex*. The maximum degree and the minimum degree of a graph G are usually denoted by $\Delta(G)$ and $\delta(G)$, respectively. If $\delta(G) = \Delta(G)$, then G is called *regular*. If $\deg v = r$ for every vertex v of G, where $0 \leq r \leq n-1$, then G is r*-regular*. A graph G is *complete* if every two distinct vertices of G are adjacent. A complete graph of order n is denoted by K_n. The graph $\overline{K}_n$ has n vertices and no edges; it is called the *empty graph* of order n. A spanning subgraph H of a connected graph G such that H is a tree is called a *spanning tree* of G. For a graph G and $X \subseteq E(G)$, we denote by $G - X$ the graph obtained from G by removing all edges in X. If $X = \{e\}$, we write $G-e$ for $G-\{e\}$. For a graph G and $X \subseteq V(G)$, $G-X$ is the graph obtained from G by removing all vertices in X and all edges incident with vertices in X. For a graph G and $X \subseteq E(\overline{G})$, $G+X$ denotes the graph obtained from G by adding all edges in X. If $X = \{e\}$, we simply write $G+e$ for $G+\{e\}$. Let G be a graph of order n and let $V(G) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of G. The sequence $(\deg v_1, \deg v_2, \ldots, \deg v_n)$ is called a *degree sequence* of G. An r-regular graph G has degree sequence $d = r^n := (r, r, \ldots, r)$. A sequence $d = (d_1, d_2, \ldots, d_n)$ of non-negative integers is a *graphic degree sequence* if it is a degree sequence of some graph G. In this case, G is called a *realization* of d. For a vertex v of a graph G, recall that a *neighbor* of v is a vertex adjacent to v in G. Also, the *neighborhood* (or open neighborhood) $N(v)$ of v is the set of neighbors of v. The *closed neighborhood* $N[v]$ is defined as $N[v] = N(v) \cup \{v\}$. A vertex v in a graph G is said to *dominate* itself and each of its neighbors, that is, v dominates the vertices in its closed neighborhood $N[v]$. Therefore v dominates $1+\deg v$ vertices of G. A set S of vertices of G is a *dominating set* of G if every vertex of G is dominated by some vertex in S. Equivalently, a set S of vertices of G is a dominating set of G if every vertex in $V(G)-S$ is adjacent to some vertex in S. A *minimum dominating set* in a graph G is a dominating set of minimum cardinality. The cardinality of a minimum dominating set is called the *domination number* of G and is denoted by $\gamma(G)$. We denote $\mathcal{CG}(m, n)$ the class of all non-isomorphic connected graphs of order n and size m.

2. Extremal Problem for γ in $\mathcal{CG}(m, n)$

Let G be the class of all graphs. A function $\pi : G \to Z$ is called a *graph parameter* if $\pi(G) = \pi(H)$ for all isomorphic graphs G and H. A study of extremal problem for graph parameters was motivated by Dirac's Conjecture as state in the following.

In the graph-theoretic colloquium at Smolenice in 1963, Dirac Conjectured that the chromatic number of a proper regular subgraph of a complete n-gon is at most $\frac{3n}{5}$. Erdős and Gallai answered this conjecture immediately and presented their result during the conference. Their article was entitled "Solution to a problem of Dirac" [2], and it was appeared in the proceedings of the symposium, Smolenice, in 1964. For the graph parameter γ, Punnim [3] proved that for any positive integers n and m with $n-1 \leq m \leq \binom{n}{2}$, there exist positive integers a and b in which there exists a graph $G \in \mathcal{CG}(m, n)$ such that $\gamma(G) = c$ if and only if c is an integer and $a \leq c \leq b$. Thus for any positive integers m and n, $\{\gamma(G) : G \in \mathcal{CG}(m, n)\}$ is unique determined by

$$a := \min(\gamma; m, n) = \min\{\gamma(G) : G \in \mathcal{CG}(m, n)\}, \quad \text{and}$$
$$b := \max(\gamma; m, n) = \max\{\gamma(G) : G \in \mathcal{CG}(m, n)\}.$$

Let π be a graph parameter and $\mathcal{J}$ be a class of graphs. The problem of finding $\min(\pi, \mathcal{J}) := \min\{\pi(G) : G \in \mathcal{J}\}$ and $\max(\pi, \mathcal{J}) := \max\{\pi(G) : G \in \mathcal{J}\}$ is so called an *extremal problem* in graph theory.

Remark 1. Proving that $A = \min(\pi, \mathcal{J})$ requires showing two things:

1. $\pi(G) \geq A$ for all $G \in \mathcal{J}$, and
2. $\pi(G) = A$ for some $G \in \mathcal{J}$.

The proof of the bound must apply to every $G \in \mathcal{J}$. For equality it suffices to obtain an example in $\mathcal{J}$ with the desired value of π. Changing " $\geq$ " to " $\leq$ " yields the criteria for a maximum.

Since the vertex set of a graph is always a dominating set, the domination number is well defined for every graph. If G is a graph of order n, then $1 \leq \gamma(G) \leq n$. A graph G of order n with $\gamma(G) = 1$ if and only if G contains a vertex v of degree $n-1$, in which case $\{v\}$ is a minimum dominating set; while $\gamma(G) = n$ if and only if $G \cong \overline{K}_n$, in which case $V(G)$ is the unique minimum dominating set. If G is a cycle of order $n \geq 3$, then $\gamma(G) = \lceil \frac{n}{3} \rceil$. It is easy to see that any connected graph G of order $n = 2, 3$, $\gamma(G) = 1$.

In order to obtain the values of $\min(\gamma; m, n)$ and $\max(\gamma; m, n)$, we first consider for small n with the observation of the following facts.

For $4 \le n \le 7$, $\min(\gamma; m, n) = 1$ for all m.

For $n = 4$, $\max(\gamma; m, n) = \begin{cases} 2 & \text{if } m = 3, 4, \\ 1 & \text{if } m = 5, 6. \end{cases}$

For $n = 5$, $\max(\gamma; m, n) = \begin{cases} 2 & \text{if } 4 \le m \le 7, \\ 1 & \text{if } 8 \le m \le 10. \end{cases}$

For $n = 6$, $\max(\gamma; m, n) = \begin{cases} 3 & \text{if } m = 5, 6, \\ 2 & \text{if } 7 \le m \le 12, \\ 1 & \text{if } 13 \le m \le 15. \end{cases}$

For $n = 7$, $\max(\gamma; m, n) = \begin{cases} 3 & \text{if } 7 \le m \le 10, \\ 2 & \text{if } 11 \le m \le 17, \\ 1 & \text{if } 18 \le m \le 21. \end{cases}$

The following theorems are useful for our proof.

Theorem 2. (see [1], p. 364) *If G is a graph of order n, then*

$$\frac{n}{1 + \Delta(G)} \le \gamma(G) \le n - \Delta(G).$$

Theorem 3. (see [4]) *Let G be a graph with n vertices, domination number d, where $3 \le d \le \frac{n}{2}$, and no isolated vertices. Then the number of edges of G is at most $\binom{n-d+1}{2}$. If G has exactly this number of edges then it must be of the following form:*

1. An $(n-d)$-clique, together with an independent set of size d, such that each of the vertices in the $(n-d)$-clique is adjacent to exactly one of the vertices in the independent set, and such that each of these d vertices has at least one vertex adjacent to it.

2. For $d = 3$, G may consist of a clique of $n - 5$ vertices, together with 5 vertices x_1, x_2, x_3, x_4, x_5, with edges x_1x_3, x_2x_4, x_2x_5, such that every vertex in the $(n-5)$-clique is adjacent to x_4 and x_5, and in addition adjacent to either x_1 or x_3. Moreover, at least one of these vertices is adjacent to x_1 and at least one to x_3.

Suppose that $n \ge 8$ and $n - 1 \le m \le \binom{n}{2}$. The following theorem gives the value of $\min(\gamma; m, n)$.

Theorem 4. *For an integer $n \ge 8$ and $n - 1 \le m \le \binom{n}{2}$*, $\min(\gamma; m, n) = 1$.

Proof. The proof follows easily by the fact that there exists a graph $G \in \mathcal{CG}(m, n)$ containing $K_{1,n-1}$ as its spanning tree. □

To determine $\max(\gamma; m, n)$, we need the following theorems.

Theorem 5. (see [1], p. 367) *Let G be a graph without isolated vertices. If S is a minimal dominating set of G, then $V(G) - S$ is a dominating set of G.*

Theorem 6. (see [1], p. 367) *If G is a graph of order n without isolated vertices, then $\gamma(G) \le \frac{n}{2}$.*

A graph G of order n and size $m \ge \frac{n(n-2)+1}{2}$ must contain a vertex of degree $n-1$. Thus we have the following propositions.

Proposition 7. $\max(\gamma; m, n) = 1$ *if and only if* $\frac{n(n-2)+1}{2} \le m \le \binom{n}{2}$.

Proof. If $\frac{n(n-2)+1}{2} \le m \le \binom{n}{2}$, then for each $G \in \mathcal{CG}(m,n)$, G contains a vertex of degree $n-1$. Thus $\max(\gamma; m, n) = 1$. Conversely, suppose that $m < \frac{n(n-2)+1}{2}$. If n is even, then there exists an $(n-2)$-regular graph G of order n and size $\frac{n(n-2)}{2}$. Thus $\gamma(G) = 2$. Let $H \in \mathcal{CG}(m,n)$ be obtained from G by removing some edges of G. Thus $\gamma(H) \ge 2$. If n is odd, then there exists a graph G of order n having $n-1$ vertices of degree $n-2$ and one vertex of degree $n-3$. Thus $\gamma(G) = 2$. Let $H \in \mathcal{CG}(m,n)$ be obtained from G by removing some edges of G. Thus $\gamma(H) \ge 2$. □

Proposition 8. $\max(\gamma; m, n) = 2$ *if and only if* $\binom{n-2}{2} + 1 \le m < \frac{n(n-2)+1}{2}$.

Proof. Suppose that $\binom{n-2}{2} + 1 \le m < \frac{n(n-2)+1}{2}$. Let $G \in \mathcal{CG}(m,n)$. Then, by Proposition 7, $\gamma(G) \ge 2$. If $\gamma(G) = d \ge 3$, then, by Theorem 3, $m \le \binom{n-d+1}{2} < \binom{n-2}{2} + 1$. Thus $\gamma(G) = 2$ and $\max(\gamma; m, n) = 2$. Conversely, suppose that $m \le \binom{n-2}{2}$. Let G be a graph of order n obtained from an $(n-3)$-clique and three independent vertices such that each of the vertices in the $(n-3)$-clique is adjacent to exactly one of the vertices in the independent set, and such that each of these d vertices has at least one vertex adjacent to it. Thus G has $\binom{n-2}{2}$ edges and $\gamma(G) = 3$. By appropriate removing some edges from G yields a graph H of order n, size m and $\gamma(H) \ge 3$. The proof is complete. □

Proposition 9. *If* $\binom{\lceil n/2 \rceil + 1}{2} < m \le \binom{n-2}{2}$, *then*

$$\max(\gamma; m, n) = \left\lfloor \frac{(2n+1) - \sqrt{8m+1}}{2} \right\rfloor .$$

Proof. Let S be a minimum dominating set of $G \in \mathcal{CG}(m,n)$ with cardinality $d \ge 3$. By Theorem 3, we have $m \le \binom{n-d+1}{2}$. That is $d^2 - (2n+1)d + (n^2 + n - 2m) \ge 0$. By using the quadratic formula, we obtain $d \le \frac{(2n+1)-\sqrt{8m+1}}{2}$.

Thus $\gamma(G) = |S| = d \leq \left\lfloor \frac{(2n+1)-\sqrt{8m+1}}{2} \right\rfloor$. We claim that this bound is sharp. Consider $d = \left\lfloor \frac{(2n+1)-\sqrt{8m+1}}{2} \right\rfloor$. We now construct a graph $G \in \mathcal{CG}(m,n)$ with $\gamma(G) = d$ as follows. Let $\{v_1, v_2, \ldots, v_{n-d}\}$ be the vertex set of K_{n-d}. Adding d new vertices $u_1, u_2, \ldots, u_d$ to K_{n-d} and joining each pair of vertices u_i, v_i for $1 \leq i \leq d$ yields a connected graph H of order n and size $\binom{n-d}{2} + d$. Thus $\{v_1, v_2, \ldots, v_d\}$ is a dominating set of H, by the fact that $d \leq n-d$, that is $\gamma(H) \leq d$. Since each vertex u_i is adjacent to exactly one of vertices v_i's for $1 \leq i \leq d$ which are distinct, so $\gamma(H) \geq d$. That is $\gamma(H) = d$. To construct a connected graph $G \in \mathcal{CG}(m,n)$, where $\binom{\lceil n/2 \rceil+1}{2} < m \leq \binom{n-2}{2}$ with $\gamma(G) = d$, we consider two cases according to m.

Case 1. Assume that $\binom{\lceil n/2 \rceil+1}{2} < m < \binom{n-d}{2} + d$. Let T be a spanning tree of K_{n-d}. Since $\binom{n-d}{2} - (n-d-1) > \binom{n-d}{2} + d - m$, so the connected graph G with size m can be obtained from H by deleting $\binom{n-d}{2} + d - m$ edges from $E(K_{n-d}) - E(T)$. We have that $\{v_1, v_2, \ldots, v_d\}$ is also a dominating set of G, that is $\gamma(G) \leq d$. Since every dominating set of G is also a dominating set of H, so $\gamma(G) \geq \gamma(H) = d$. That is $\gamma(G) = d$.

Case 2. Assume that $\binom{n-d}{2} + d \leq m \leq \binom{n-2}{2}$. Now we have the graph H with size $\binom{n-d}{2} + d$ and $\gamma(H) = d$. Consider $\binom{n-d}{2} + d < m \leq \binom{n-2}{2}$. Note that for a graph with size m and the domination number d, $m \leq \binom{n-d+1}{2}$. We have $m - \binom{n-d}{2} \leq n-d$. Then the connected graph G with size m can be obtained from the graph H by joining u_d to v_j for $d+1 \leq j \leq m - \binom{n-d}{2}$. Hence $G \in \mathcal{CG}(m,n)$. We have that $\{v_1, v_2, \ldots, v_d\}$ is also a dominating set of G, that is $\gamma(G) \leq d$. Since each vertex u_i is adjacent to exactly one of vertex v_i's for $1 \leq i \leq d-1$ and u_d adjacent to v_j for $d+1 \leq j \leq m - \binom{n-d}{2}$ which are distinct, $\gamma(G) \geq d$. That is $\gamma(G) = d$. □

Proposition 10. *If* $n-1 \leq m \leq \binom{\lceil n/2 \rceil+1}{2}$, *then* $\max(\gamma; m, n) = \left\lfloor \frac{n}{2} \right\rfloor$.

Proof. Let $G \in \mathcal{CG}(m,n)$. By Theorem 6, $\gamma(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$. To construct a connected graph of order n and size m, where $n-1 \leq m \leq \binom{\lceil n/2 \rceil+1}{2}$ with the domination number $\left\lfloor \frac{n}{2} \right\rfloor$, we divide into two cases.

Case 1. Suppose that n is even, $n = 2k$. Let $V(K_k) = \{v_1, v_2, \ldots, v_k\}$ and let T be a spanning tree of K_k. Let G be a graph obtained from K_k by adding k new vertices u_i; $1 \leq i \leq k$ and k new edges u_iv_i for $1 \leq i \leq k$. Then G is a connected graph of order n and size $\binom{k}{2} + k$. Thus $\{v_1, v_2, \ldots, v_k\}$ is a dominating set of G, that is $\gamma(G) \leq k$. Since each vertex u_i is adjacent to exactly one of vertices v_i's for $1 \leq i \leq k$ which are distinct, so $\gamma(G) \geq k$. That is $\gamma(G) = k$. Let $n-1 \leq m \leq \binom{\lceil n/2 \rceil+1}{2}$. Since $\binom{k}{2} - (k-1) \geq \binom{k}{2} + k - m$, so a graph $H \in \mathcal{CG}(m,n)$ with $\gamma(H) = k$ can be obtained from G by deleting $\binom{k}{2} + k - m$ edges from $K_k - E(T)$. It is clear that $H \in \mathcal{CG}(m,n)$. We have that $\{v_1, v_2, \ldots, v_k\}$ is a dominating set of H, that is $\gamma(H) \le k$. Since every dominating set of H is also a dominating set of G, so $\gamma(H) \ge \gamma(G) = k$. That is $\gamma(H) = k$.

Case 2. Suppose that n is odd, $n = 2k + 1$. Let

$$V(K_{k+1}) = \{v_1, v_2, \ldots, v_{k+1}\}$$

and let T' be a spanning tree of K_{k+1}. Let G' be a graph obtained from K_{k+1} by adding k new vertices u_i; $1 \le i \le k$ and k new edges $u_i v_i$ for $1 \le i \le k$. Then G' is a connected graph of order n and size $\binom{k+1}{2} + k$. Thus $\{v_1, v_2, \ldots, v_k\}$ is a dominating set of G', that is $\gamma(G') \le k$. Since each vertex u_i is adjacent to exactly one of vertices v_i's for $1 \le i \le k$ which are distinct, so $\gamma(G') \ge k$. That is $\gamma(G') = k$. Let $n - 1 \le m \le \binom{\lceil n/2 \rceil + 1}{2}$. Since $\binom{k+1}{2} - k \ge \binom{k+1}{2} + k - m$, so a graph $H' \in \mathcal{CG}(m,n)$ with $\gamma(H') = k$ can be obtained from G' by deleting $\binom{k+1}{2} + k - m$ edges from $K_{k+1} - E(T')$. It is clear that $H' \in \mathcal{CG}(m,n)$. We have that $\{v_1, v_2, \ldots, v_k\}$ is also a dominating set of H', that is $\gamma(H') \le k$. Since every dominating set of H' is also a dominating set of G', $\gamma(H') \ge \gamma(G') = k$. That is $\gamma(H') = k$. □

Theorem 11. *For an integer $n \ge 8$ and $n - 1 \le m \le \binom{n}{2}$,*

1. $\max(\gamma; m, n) = \lfloor \frac{n}{2} \rfloor$ *if* $n - 1 \le m \le \binom{\lceil n/2 \rceil + 1}{2}$,

2. $\max(\gamma; m, n) = \left\lfloor \frac{(2n+1) - \sqrt{8m+1}}{2} \right\rfloor$ *if* $\binom{\lceil n/2 \rceil + 1}{2} < m \le \binom{n-2}{2}$,

3. $\max(\gamma; m, n) = 2$ *if* $\binom{n-2}{2} < m < \binom{n}{2} - \lfloor \frac{n-1}{2} \rfloor$,

4. $\max(\gamma; m, n) = 1$ *if* $\binom{n}{2} - \lfloor \frac{n-1}{2} \rfloor \le m \le \binom{n}{2}$.

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