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ZERO-DIMENSIONAL SCHEMES, BLOWING-UP AND ASYMPTOTIC POSTULATION

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Abstract: Here we look at the postulation of general unions of zero-dimensional schemes more general than fat points with respect to a very positive linear system.

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Fix an integral n-dimensional projective variety X and integers $h > 0$ and $s_1 \geq \cdots \geq s_h \geq 0$. Set $X_0 := X$. Fix s_1 distinct points $P_i, \dots, P_{s_1} \in X_{reg}$ and let $f_1 : X_1 \to X$ be the blowing-up of the points $P_1, \dots, P_{s_1}$. Set $P_{i,0} := P_i$. Let $D_{i,1} := f_1^{-1}(P_{i,0})$, $1 \leq i \leq s_1$, denote the exceptional divisors of f_1. If $h \geq 2$, then fix $P_{i,1} \in D_{i,1}$, $1 \leq i \leq s_2$, and let $f_2 : X_2 \to X_1$ the blowing-up of the points $P_{1,1}, \dots, P_{1,s_2} \in X_1$. Let $D_{i,2} := f_2^{-1}(P_{i,1})$, $1 \leq i \leq s_2$, denote the exceptional divisors of f_1. For $s_2 < i \leq s_1$ set $D_{i,2} := f_2^{-1}(D_{i,1})$. Hence, if $h \geq 3$, then fix $P_{i,2} \in D_{i,2}$, $1 \leq i \leq s_3$, and let $f_3 : X_3 \to X_2$ the blowing-up of the points $P_{1,2}, \dots, P_{s_2,2} \in X_1$. Let $D_{i,3} := f_2^{-1}(P_i)$, $1 \leq i \leq s_2$, denote the exceptional divisors of f_2. As in the first blowing-up use a similar convection for $s_2 < i \leq s_1$, and so on, defining for every integer i such that $1 \leq i \leq h$ blowing-ups $f_i : X_i \to X_{i-1}$, disjoint integral divisors $D_{j,i} \subset X_i$, $1 \leq j \leq s_h$, with $D_{j,i} = f_i^{-1}(P_{j,i-1})$ for some $P_{j,i-1} \in X_{i-1}$; if $i \geq 2$, then we also have $P_{j,i-1} \in D_{j,i-2}$. For $0 \leq j < i \leq h$ set $g_{i,j} := f_1 \circ \cdots \circ f_i : X_i \to X_j$. Set $g_i := g_{i,0}$. For all integers $1 \leq j < i \leq h$ and $1 \leq x \leq s_j$ let $D_{x,j,i} \subset X_i$ denote the strict

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transform by $g_{i,j}$ of the divisor $D_{x,j} \subset X_j$. Hence each $D_{x,j,i}$ is irreducible. For every integer x such that $1 \le x \le s_1$ let c_x be the maximal integer y such that $1 \le y \le h$ and $1 \le x \le s_y$. Hence $g_h^{-1}(P_{x,1}) = \sum_{j=1} c_x D_{x,j,h}$ (both scheme-theoretically and as effective Cartier divisors) for all $x \in \{1, \dots, s_1\}$. Set $\Phi := \{(x,y) \in \mathbb{Z}^{\oplus 2} : 1 \le x \le s_1 \text{ and } 1 \le y \le c_x\}$. For all $(x,y) \in \Phi$ fix a non-negative integer $m_{x,y}$ and call $\widetilde{m}$ the set of all $(x,y,m_{x,y})$. Call Ψ the set of all data $\{(x,y,m_{x,y} : (x,y) \in \Phi$ and $m_{x,y} \le m_{x,y+1}$ if $y < c_x\}$. For any $\psi \in \Psi$, say $\{(x,y,m_{y,x})\}$ set $I_\psi := \mathcal{O}_{X_h}(-\sum_{x=1}^{s_1}\sum_{y=1}^{c_x} m_{x,y} D_{x,y,h})$. Hence $I_\psi \in \mathrm{Pic}(X_h)$ and its dual has a non-zero section whose zero-locus is a linear combination of the exceptional components $D_{x,y,h}$, $1 \le x \le s_1$, $1 \le y \le c_x$, of g_h. Since g_{h*} is a left exact functor, $\mathcal{I}_\psi$ is an ideal sheaf and we will write $Z(\psi)$ for the associated zero-dimensional subscheme of X. Write $\mathrm{length}(\psi) = \sum_{x=1}^{s_1}\sum_{y=1}^{c_x}\binom{n+m_{x,y}-m_{x,y-1}-1}{n}$ with the convention $m_{x,0} = 0$ for all x and $\binom{n-1}{n} = 0$. Set $\mathcal{I}_\psi := g_{h*}(I_\psi)$. Under this assumption we have $\mathrm{length}(Z(\psi) = \mathrm{length}(\psi)$ and I_ψ is the blowing-up of the ideal sheaf $\mathcal{I}_\psi$. Fix any $i \in \{1, \dots, s_1\}$ and assume $i_c = 2$. Set $m_1 := m_{1,i}$ and $m_2 := m_{2,i}$. Let $Z(P_i, m_1, m_2)$ denote the connected component of $Z(\psi)$ supported by P_i. We will say that $Z(P_i, m_1, m_2)$ is a subscheme of type $[m_1, m_2]$ of X. Notice that if $m_1 = m_2$, then $Z(P_i, m_1, m_1)$ is the fat point $\{m_1 P_i, X\}$, i.e. the closed subscheme of X with $(\mathcal{I}_{P_i})^{m_1}$ as its ideal sheaf. We will say that an $[0,x]$-scheme, $x > 0$, is an $[x,x]$-scheme, while the empty set is the only $[0,0]$-scheme. Here we will generalize [1], Theorem 1.1, and prove the following two statements which are easily seen equivalent.

Theorem 1. *Fix an integer $m > 0$, an integral projective variety, a zero-dimensional scheme $A \subset X$ and $M, R \in \mathrm{Pic}(X)$ with R ample. There is an integer $\tau > 0$ with the following property. Fix any integer $t \ge \tau$. Let $\Theta(m)$ denote the set of all pairs of integers $m_2 \ge m_1 > 0$ such that $m_2 < 2m_1$ and $m_2 \le m$. For all (m_1, m_2) fix an integer $x_{m_1,m_2} \ge 0$. Let $Z \subset X$ be a general union for all $(m_1, m_2) \in \Theta(m)$ of x_{m_1,m_2} subschemes of type $[m_1, m_2]$. Then either $h^0(X, \mathcal{I}_{A\cup Z} \otimes M \otimes R^{\otimes t}) = 0$ or $h^1(X, \mathcal{I}_{A\cup Z} \otimes M \otimes R^{\otimes t}) = 0$.*

Theorem 2. *Fix an integer $m > 0$, an integral projective variety, a zero-dimensional scheme $A \subset X$ and $M, R \in \mathrm{Pic}(X)$ with R ample. There is an integer $\delta > 0$ with the following property. Let $\Theta(m)$ denote the set of all pairs of integers $m_2 \ge m_1 > 0$ such that $m_2 < 2m_1$ and $m_2 \le m$. For all (m_1, m_2) fix an integer $x_{m_1,m_2} \ge 0$. Let $Z \subset X$ be a general union for all $(m_1, m_2) \in \Theta(m)$ of x_{m_1,m_2} subschemes of type $[m_1, m_2]$. Asume* $\mathrm{length}(Z) \ge \delta$. *Fix an integer $t \ge 0$. Then either $h^0(X, \mathcal{I}_{A\cup Z} \otimes M \otimes R^{\otimes t}) = 0$ or $h^1(X, \mathcal{I}_{A\cup Z} \otimes M \otimes R^{\otimes t}) = 0$*

Different connected components of Z may be associated to different integers

m_1, m_2, with the only restriction $m \ge m_2 \ge m_1 > 0$. The case $m_1 = m_2$ for all connected components of Z is just [1], Theorem 1.1 and Remark 7.2. The restriction "$m_2 < 2m_1$" in the statements of Theorems 1 and 2 is very unpleasant, but we do not know how to avoid it (at least for many of the points of Z_{red}). It is used to get $e_{m_1} = 1$ in the set-up of Remark 2.

Notation 1. Fix a an integral projective scheme X, an effective Cartier divisor $D \subset X$, $P \in D_{reg}$ and a zero-dimensional scheme $A \subset X$ such that $A_{red} = \{P\}$. Set $A_0 := A$. Define inductively the schemes A_i and B_i, $i \ge 1$, by the formulas $A_i := \mathrm{Res}_D(A_{i-1})$ and $B_i := A_i \cap D$. For evey integer $i \ge 1$, set $e_i := \mathrm{length}(B_{i-1})$. We will say that A has type $(e_1, e_2, \dots)$ with respect to D. Since A is zero-dimensional and $A_{red} \subseteq D$, we have $e_i = 0$ for $i \gg 0$, $e_{i+1} \le e_i$ for all i, and $\mathrm{length}(A) = \sum_{i \ge 1} c_i$. Instead of a sequence $(e_1, e_2, \dots)$ we will usually write a finite string $(e_1, \dots, e_s)$ if $e_{s+1} = 0$, i.e. if $c_i = 0$ for all $i > s$.

Remark 1. Fix $i \in \{1, \dots, h\}$ and set $P := P_{i,0}$, $c := c_i$ and $m_j := m_{i,j}$. Let $D \subset X$ such that $P \in D_{reg}$ and that $P_{i,1}$ is not contained in the strict transform of D in X_1. Let $Z(\psi, P)$ (or $Z_X(\psi, P)$ denote the connected component of $Z(\psi)$ supported by P and $(e_1, \dots)$ its associated sequence with respect to D. The integers e_i, $i \ge 1$, are determined in the following way. Obviously, $e_i \ge e_j$ if $i \le j$, and $e_i \ne 0$ if and only if $i \le m_c$. Set $A_0 := Z(\psi, P)$ and $B_0 := A_0 \cap D$. For all integers $i \ge 1$ set $A_i := \mathrm{Res}_H(A_{i-1})$ and $B_i := A_i \cap D$. Since B_0 is an For all $1 \le i \le m_c$ there is a non-negative integer d_i such that $e_i = \binom{n+d_j-1}{n-1}$ and the values of the integers $d_1, \dots, d_{m_c}$ are determined in the following way. Set $m_0 := 0$. Each integer $m_i - m_{i-1}$, $1 \le i \le c$, gives $m_i - m_{i-1}$ integers d_j, which are the integers t such that $1 \le t \le m_i - m_{i-1}$. For instance in the case $c = 2$ we get $c_i = \binom{n+m_2-i}{n-1}$ for $1 \le i \le m_2 - m_1$ and $c_{m_2-m_1+2j-1} = c_{m_2-m_1+2j} = \binom{n+m_1-j}{n-1}$ for $1 \le j \le m_1$ and $c_i = 0$ for all $i > m_2$. Now assume $c \le 2$. If $m_1 = m_2$, i.e. if $c = 1$, then the situation is as in the case $D = H$, because in this case $Z(\psi, P) = m_1 P$. Now assume $c = 2$, i.e. assume $m_2 > m_1$. Let H be an effective Cartier divisor of X such that $P \in H_{reg}$ and the tangent space $T_P H$ of H at P contains the tangent vector of X at P corresponding to the blowing-up $X_2 \to X_1$. We have $Z(\psi, P) \cap H = Z(\psi, P)_H$ and $\mathrm{Res}_H(Z(\psi, P)) = Z(\psi', P)$, where ψ' is associated to the sequence $(m_1 - 1, m_2 - 2)$, with the convention that if $m_2 = m_1 + 1$, then $Z(\psi', P)$ is the fat point xP. If $m_2 \ge m_1 + 2$, then we continue. If $m_2 \le 2m_1$, we are sure that after $m_2 - m_1$ steps we arrive at the fat point $(2m_1 - m_2)P$. Then we continue $2m_1 - m_2$ times as in the case $c = 1$.

Remark 2. Fix integers $2m_1 > m_2 \ge m_1 > 0$ and X, P, H as in the last part of Remark 1. Let $(e_1, \dots,)$ be sequences associated to $Z(\psi, P)$ with respect to H. We saw that $e_i = 0$ for all $i > m_1$ and that $e_{m_1} = 1$. Notice that $\binom{n+i-1}{n-1}/\binom{n+i-2}{n-1} = (n+i-1)/(i-1)$ for all integers $i \geq 2$. Hence $(n+m_2-2)/2(m_2-1) \leq e_{i+1}/e_i < 1$ for all integers $1 \leq i \leq m_2 - 1$.

Remark 3. Fix integers $a \geq 0$, $m_2 \geq m_1 > 0$ and X, P, H as in the last part of Remark 1. Fix $L \in \mathrm{Pic}(X)$ and a zero-dimensional scheme $W \subset X$ such that $P \notin W_{red}$, and $h^1(X, L) = h^1(X, L(-H)) = 0$. Fix a general $Q \in X$ and let $Z(m_1, m_2, Q)$ denote the zero-dimensional scheme obtained taking Q instead of P in Remark 1. To check $h^1(X, \mathcal{I}_{W \cup Z(m_1,m_2,Q)} \otimes L) \leq a$ (resp. $h^0(X, \mathcal{I}_{Z \cup Z(m_1,m_2,Q)} \otimes L) \leq a$) it is sufficient to prove $h^1(X, \mathcal{I}_{\mathrm{Res}_H(W) \cup B} \otimes L(-D)) + h^1(H, \mathcal{I}_{W \cap H \cup \{P\}} \otimes (L|H)) \leq a$ (resp. $h^0(X, \mathcal{I}_{\mathrm{Res}_H(W) \cup B} \otimes L(-D)) + h^0(H, \mathcal{I}_{Z \cap H \cup \{P\}} \otimes (L|H)) \leq a$), where B is a virtual scheme supported by P and with type $(e_1, \dots)$ with respect to D, where e_i is the same as the integer e_i associated to $Z(\psi, P)$ for $i \neq m_1$, while $e_{m_1} = 0$. Hence $\mathrm{length}(B) = \mathrm{length}(Z(m_1, m_2, Q)) - 1$. We will say that B is a virtual scheme of type $[m_2, m_1, H]$ and that we applied Remark 2 with respect to the integers m_2, m_1.

Outline of Proofs of Theorems 1 and 2. It is easy to check that the two statements are equivalent. Set $n := \dim(X)$. As in [1] we will only write down the case $A = \emptyset$. As in [1] we easily reduce to the case $M = \mathcal{O}_X$, R very ample, and $h^i(X, R^{\otimes t}) = 0$ for all $i > 0$ and all $t > 0$. We use induction on n. We follow step by step the proof of [1], Theorem 1.1, with the same numerical aymptotic extimates using the lower bounds for the integers e_{i+1}/e_i given in Remark 2 (see [1], Lemmas 4.2, 5.2 and 6.2). At each step we insert inside a divisor H_a of X only $[m_1, m_2]$-schemes such that, if $m_1 \neq m_2$, then H_a contains the tagent direction corresponding to this scheme. The condition $e_{m_1} = 1$ allows us to use at each step the differential Horace Lemma with perfect length supported by the given divisor. The termination is as in [1], Proposition 7.1. □

We work over an algebraically closed field $\mathbb{K}$ with either $\mathrm{char}(\mathbb{K}) = 0$ or $\mathrm{char}(\mathbb{K}) > m$.

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