International Journal of Pure and Applied Mathematics
Volume 41 No. 5 2007, 735-746

NONCLASSICAL SYMMETRIES AND DIRECT SIMILARITY ANALYSIS OF A NONLOCAL GASEOUS IGNITION MODEL

R. Asokan[1 §], M. Pitchaimani[2]
[1,2]School of Mathematics
Madurai Kamaraj University
Tamil Nadu, Madurai, 625 021, INDIA
e-mail: rasoka_mku@yahoo.co.in

Abstract: The determining equations for the nonclassical reductions of a nonlocal gaseous ignition problem are obtained. It is shown that requiring compatibility with a first order quasilinear partial differential equation, the determining equations are also obtainable. Finally we discuss the similarity analysis of a nonlocal gaseous ignition problem by the direct method of Clarkson and Kruskal to obtain its similarity solution.

AMS Subject Classification: 35K55, 34C14, 37L20, 70G65, 35F25, 53C70
Key Words: nonlinear parabolic equation, determining equations, nonclassical symmetries, invariant surface condition, compatibility condition, quasilinear evolution equations, direct method and first integral

1. Introduction

Exact solutions of nonlinear partial differential equations are often obtained by means of Lie‘s [10] symmetry approach. Lie’s method and its generalizations are presented in Bluman and Anco [4], Olver [12], Rogers and Ames [13], Clarkson and Kruskal [7], Simon Hood [8], and Sachdev and Mayil Vaganan [14].

Received: August 17, 2007
§Correspondence author

If

$$\Delta(t, y, u, u_t, u_y, u_{tt}, u_{ty}, ...) = 0 \tag{1}$$

is invariant under the one parameter a group of infinitesimal transformations

$$\begin{aligned} t^* &= t + \epsilon T(y, t, u) + O(\epsilon^2), \\ y^* &= y + \epsilon Y(y, t, u) + O(\epsilon^2), \\ u^* &= u + \epsilon U(y, t, u) + O(\epsilon^2), \end{aligned} \tag{2}$$

then the infinitesimals T, Y, U are found to satisfy a set of so-called determining equations, which, when solved, gives rise to the symmetries of (1). The invariant surface condition

$$Tu_t + Yu_y = U, \tag{3}$$

then yields the similarity form of solutions which, when substituted into equation (1) gives an ordinary differential equation. Another generalization of the "classical method" of Lie was proposed by Bluman and Cole [5] and is known as the "nonclassical method". The idea of the nonclassical method is to seek the invariance of (1) together with the invariant surface condition (3).

It is a proven fact that all exact solutions obtained by their classical method could also be obtained by the nonclassical method. Unlike the determining equations for the classical method which are linear, the determining equations for the nonclassical method are nonlinear.

Mansfield [11] gave the general solution of the determining equations for the heat equation; then Arrigo and Hickling [2] were able to show that these determining equations belong to a class of matrix Burger's equation and was solved using a matrix Hope-Cole transformation.

In this paper, we derive the determining equations of the nonlinear parabolic equation Stephen Bricher [6]

$$uu_{yy} - u_y^2 - \frac{1}{2}yuu_y - (T_* - t)uu_t + u^3 - u^2 + (T_* - t)u^2 g(t) = 0, \tag{4}$$

using both the nonclassical method and the method of Arrigo and Beckham [1].

The rest of the paper is organised as follows: In Section 2 we derive the determining equations of (4) using the nonclassical method. The recent approach of Arrigo and Beckham is employed to (4) to rederive the determining equations in Section 3. In Section 4 the direct method as presented in an easily applicable form by Sachdev and Mayil Vaganan [14] is used to reduce the nonlinear parabolic equation to an ordinary differential equation. The result of the present study is set forth in Section 5.

2. The Nonclassical Method

If we seek equation (4) to be invariant under the Lie's group of infinitesimal transformations (2), then

$$\left(\frac{U}{u} + U_u - 2Y_y - 3Y_u u_y - T_u u_t\right)\left[u_y^2 + \frac{1}{2}yuu_y + (T_* - t)uu_t - u^3 + u^2\right.$$
$$\left.-(T_* - t)u^2 g(t)\right] + U\left[-\frac{1}{2}yu_y - (T_* - t)u_t + 3u^2 - 2u + 2(T_* - t)ug(t)\right]$$
$$+ T\left[uu_t - u^2 g(t) + (T_* - t)u^2 g'(t)\right] - Y\frac{1}{2}uu_y - (T_* - t)u\left[U_t + (U_u - T_t)u_t\right.$$
$$\left.-Y_t u_y - T_u u_t^2 - Y_u u_y u_t\right] - \left(2u_y + \frac{1}{2}uy\right)\left[U_y + (U_u - Y_y)u_y\right.$$
$$\left.-T_y u_t - Y_u u_y^2 - T_u u_y u_t\right] + \left[U_{yy} + (2U_{yu} - Y_{yy})u_y - T_{yy}u_t + (U_{uu} - 2Y_{yu})u_y^2\right.$$
$$\left.- 2T_{yu}u_y u_t - Y_{uu}u_y^3 - T_{uu}u_y^2 u_t - 2T_y u_{yt} - 2T_u u_{yt}u_y\right] u = 0. \quad (5)$$

We denote the nonlinear parabolic equation (4) by Δ_1 and the invariant surface condition (3) with $T = 1$ by Δ_2:

$$\Delta_1 = uu_{yy} - u_y^2 - \frac{1}{2}yuu_y - (T_* - t)uu_t + u^3 - u^2 + (T_* - t)u^2 g(t), \quad (6)$$
$$\Delta_2 = Tu_t + Yu_y - U. \quad (7)$$

Now we put $T = 1$ and use $\Delta_1 = 0$ and $\Delta_2 = 0$ into equation (5) to obtain

$$\left(\frac{U}{u} + U_u - 2Y_y - 3Y_u u_y\right)\left[u_y^2 + \frac{1}{2}yuu_y + (T_* - t)u(U - Yu_y)\right.$$
$$\left.-u^3 + u^2 - (T_* - t)u^2 g\right] + U\left[-\frac{1}{2}yu_y - (T_* - t)(U - Yu_y)\right.$$
$$\left.+3u^2 - 2u + 2(T_* - t)ug\right] + \left[u(U - Yu_y) - u^2 g + (T_* - t)\right.$$
$$\left.u^2 g'(t)\right] - Y\frac{1}{2}uu_y - (T_* - t)u\left[U_t + U_u(U - Yu_y) - Y_t u_y\right.$$
$$\left.-Y_u u_y(U - Yu_y)\right] - \left(2u_y + \frac{1}{2}uy\right)\left[U_y + (U_u - Y_y)u_y\right.$$
$$\left.-Y_u u_y^2\right] + \left[U_{yy} + (2U_{yu} - Y_{yy})u_y + (U_{uu} - 2Y_{yu})u_y^2 - Y_{uu}u_y^3\right] u = 0. \quad (8)$$

Equating the coefficients of u_y, u_y^2, u_y^3 and rest to zero gives rise to the determining equations

$$\left[2(T_* - t)Y - \frac{1}{2}y\right] Y_y - 2(T_* - t)UY_u + 3uY_u[u - 1$$

$$+ (T_* - t)g] + (T_* - t)Y_t - \frac{2}{u}U_y + 2U_{yu} - Y_{yy} = 0, \quad (9)$$

$$u^2 U_{uu} - uU_u + U - u^2 \left[y - 2(T_* - t)Y\right] Y_u - 2u^2 Y_{uy} = 0, \quad (10)$$

$$Y_u + uY_{uu} = 0, \quad (11)$$

$$u(2Y_y - U_u)[u - 1 + (T_* - t)g] + [2u + (T_* - t)g$$
$$- 2(T_* - t)Y_y]U - ug + (T_* - t)ug' - (T_* - t)U_t - \frac{1}{2}yU_y + U_{yy} = 0. \quad (12)$$

3. Method of Arrigo and Beckham

In this section we derive the determining equations (9)-(12) for the nonclassical symmetries of the nonlinear parabolic equation (4) via the compatibility with the invariant surface condition (3), with $T = 1$.

The determining equations for the nonclassical symmetries of the nonlinear parabolic equation (4) are obtained by requiring that

$$\Gamma^{(2)}\Delta_1\big|_{\Delta_1=0,\Delta_2=0} = 0. \quad (13)$$

The infinitesimal generator Γ is given by

$$\Gamma = T\frac{\partial}{\partial t} + Y\frac{\partial}{\partial y} + U\frac{\partial}{\partial u}. \quad (14)$$

Its first and second extensions are

$$\Gamma^{(1)} = \Gamma + U_{[t]}\frac{\partial}{\partial u_t} + U_{[y]}\frac{\partial}{\partial u_y}, \quad (15)$$

$$\Gamma^{(2)} = \Gamma^{(1)} + U_{[tt]}\frac{\partial}{\partial u_{tt}} + U_{[ty]}\frac{\partial}{\partial u_{ty}} + U_{[yy]}\frac{\partial}{\partial u_{yy}}, \quad (16)$$

where

$$U_{[t]} = D_t U - u_t D_t T - u_y D_t Y, \quad (17)$$

$$U_{[y]} = D_y U - u_t D_y T - u_y D_y Y, \quad (18)$$

$$U_{[tt]} = D_t U_{[t]} - u_{tt} D_t T - u_{ty} D_t Y, \quad (19)$$

$$U_{[ty]} = D_y U_{[t]} - u_{tt} D_y T - u_{ty} D_y Y, \quad (20)$$

$$U_{[yy]} = D_y U_{[y]} - u_{ty} D_y T - u_{yy} D_y Y. \quad (21)$$

The total differential operators D_t and D_y are given by

$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tt}\frac{\partial}{\partial u_t} + u_{ty}\frac{\partial}{\partial u_y} + \cdots, \quad (22)$$

$$D_y = \frac{\partial}{\partial y} + u_y\frac{\partial}{\partial u} + u_{ty}\frac{\partial}{\partial u_t} + u_{yy}\frac{\partial}{\partial u_y} + \cdots. \quad (23)$$

The parabolic equation and the invariant surface condition are rewritten as

$$
\begin{aligned}
u_{yy} &= \frac{u_y^2}{u} + \frac{1}{2} y u_y + (T_* - t) u_t - u^2 + u - (T_* - t) u g(t), && (24)\\
u_t &= U - Y u_y. && (25)
\end{aligned}
$$

The compatibility condition, then takes the form

$$
D_t(u_{yy}) - D_y^2(u_t) = 0. \qquad (26)
$$

We insert equations (24)-(25) into equation (26) to obtain

$$
D_t\left[\frac{u_y^2}{u} + \frac{1}{2} y u_y + (T_* - t) u_t - u^2 + u - (T_* - t) u g(t)\right] - D_y^2\left[U - Y u_y\right] = 0, \qquad (27)
$$

which implies

$$
\begin{aligned}
&\left(2\frac{u_y}{u} + \frac{1}{2} y\right)\left[U_y + (U_u - Y_y) u_y - Y_u u_y^2 - Y\left(\frac{u_y^2}{u} + \frac{1}{2} y u_y + (T_* - t) u_t\right.\right.\\
&\left.\left. - u^2 + u - (T_* - t) u g(t)\right)\right] - \left[\frac{u_y^2}{u^2} + 2u + (T_* - t) g\right](U - Y u_y) + u g(t)\\
&- (T_* - t) u g'(t) + (T_* - t)\left[U_t + U_u(U - Y u_y) - Y_t u_y - Y_u (U - Y u_y) u_y\right.\\
&- Y\left(U_y + (U_u - Y_y) u_y - Y_u u_y^2 - Y\left(\frac{u_y^2}{u} + \frac{1}{2} y u_y + (T_* - t) u_t - u^2 + u\right.\right.\\
&\left.\left.\left. - (T_* - t) u g(t)\right)\right)\right] - U_{yy} - 2U_{uy} u_y - (U_{uu} - 2Y_{uy}) u_y^2 + Y_{uu} u_y^3 + Y_{yy} u_y\\
&+ (2Y_y + 3Y_u u_y - U_u)\left[\frac{u_y^2}{u} + \frac{1}{2} y u_y + (T_* - t) u_t - u^2 + u\right.\\
&\left. - (T_* - t) u g(t)\right] + Y\left[\left(2\frac{u_y}{u} + \frac{1}{2} y\right)\left(\frac{u_y^2}{u}\right.\right.\\
&\left. + \frac{1}{2} y u_y + (T_* - t) u_t - u^2 + u - (T_* - t) u g\right)\\
&- \frac{u_y^3}{u^2} + u_y\left(\frac{3}{2} - 2u - (T_* - t) g\right) + (T_* - t)\\
&\left(U_y + (U_u - Y_y) u_y - Y_u u_y^2 - Y\left(\frac{u_y^2}{u} + \frac{1}{2} y u_y\right.\right.\\
&\left.\left.\left. + (T_* - t) u_t - u^2 + u - (T_* - t) u g(t)\right)\right)\right] = 0. \qquad (28)
\end{aligned}
$$

Setting the coefficients of u_y, u_y^2, u_y^3 and rest to zero gives rise to exactly the same determining equations (9)-(12).

4. Direct Similarity Analysis of (4)

The semi-linear parabolic equation

$$w_t = \Delta w + f(w), \tag{29}$$

where $f(w)$ is either e^w or w^p $(p > 1)$, [3] models the thermal combustion process in a solid fuel, where heat transfer by conduction is constant and the reaction rate depends on temperature.

For an ideal gaseous fuel in a bounded container, the motion caused by the compressibility of the gas leads to the addition of a nonlocal integral term that complicates the model. Indeed the ignition period of a thermal event is modeled by the integro-parabolic equation [3]

$$w_t = \Delta w + e^w + \frac{\gamma - 1}{\mid \Omega \mid} \int_{\Omega} e^w \, dy, \quad (x,t) \in \Omega \times (0, \infty), \tag{30}$$

where $\Omega \subset \mathbf{R^n}$ is a bounded open container, w is the temperature perturbation of the gas, $\gamma > 1$ is the gas parameter and $\mid \Omega \mid \equiv \mathrm{vol}(\Omega)$. Another model equation is

$$w_t = \Delta w + e^w + \frac{\gamma - 1}{\gamma \mid \Omega \mid} \int_{\Omega} w_t \, dy. \tag{31}$$

However, (30) may be obtained from equation (31) by integrating the later over Ω, applying Green's first identity to $\int_{\Omega} \Delta w \; dy$ and omitting the nonlocal gradient term. Thus the gaseous ignition model (30) depicites the thermal behaviour for a gaseous fuel whose flux of the temperature's spatial rate of change across the container's boundary is negligible.

Souplet [15] considered the nonlocal parabolic equation

$$w_t - \Delta w = \int_{\Omega} e^{w(y,t)} \, dy \tag{32}$$

and proved that

$$\lim_{t \to T_*^-} \frac{\parallel w(t) \parallel_\infty}{\mid \log(\mathrm{T}_* - \mathrm{t}) \mid} = 1, \tag{33}$$

where T_* is the final time.

Similar behaviour for equation (30), which includes the same nonlocal term as well as the ignition term e^w is observed also Stephen Bricher [6]. Further, Bricher briefly reviwed comparison methods for nonlocal problems as discussed [6] and used them to obtain monotonicity results for a class of integro-parabolic equations that includes (30) as well as a comparison result for solution to equation (30) and equation (31). Stephen Bricher made the hot-spot change of variables

$$\tau = -\log(T_* - t), \quad y = x(T_* - t)^{\frac{-1}{2}}, \qquad \omega(y,\tau) = w(x,t) + \log(T_* - t), \tag{34}$$

to (30) in order to obtain the parabolic equation for $\omega(y,\tau)$:

$$\omega_\tau = \Delta\omega - \frac{1}{2}y.\nabla\omega + e^{\omega} - 1 + e^{-\tau}g(T_* - e^{-\tau}) \tag{35}$$

on the set $(y,\tau) \in \mathbf{B}(\tau) \times (\tau_0, \infty)$ with $\mathbf{B}(\tau) \equiv \{y \in \mathbf{R^n} :\mid y \mid < R\ e^{\tau/2}\}$.

Langlias and Philips [**?**] have shown that $\omega \to 0$ as $\tau \to \infty$, using a stabilization technique.

In the present section, we give a detailed account of the similarity reductions of the parabolic equation (Stephen Bricher [6])

$$\omega_\tau = \omega_{yy} - \frac{1}{2}y\omega_y + e^{\omega} - 1 + e^{-\tau}g(T_* - e^{-\tau}), \tag{36}$$

by the direct method of Clarkson and Kruskal [7].

We transform equation (36) to

$$v_{yy} - \frac{1}{2}yv_y - (T_* - t)v_t + e^v - 1 + (T_* - t)g(t) = 0, \tag{37}$$

through

$$t = T_* - e^{-\tau}, \quad \omega(\tau, y) = v(t, y). \tag{38}$$

Further equation (37) can be changed via $u(y,t) = e^{v(y,t)}$ to obtain equation (4).

Now we make the ansatz

$$u(x,t) = A(x,t) + B(x,t)f(z), \quad B(x,t) \neq 0, \tag{39}$$

where $z = z(x,t)$ is the similarity variable.

Putting equation (39) into equation (4), we have

$$AA_{yy} - {A_y}^2 - \frac{1}{2}yAA_y - (T_* - t)AA_t + A^3 - A^2 + (T_* - t)A^2g(t)$$
$$+ \Big[AB_{yy} + BA_{yy} - 2A_yB_y - \frac{1}{2}y(AB_y + BA_y) - (T_* - t)(AB_t$$
$$+BA_t) + 3A^2B - 2AB + 2(T_* - t)ABg(t)\Big] f + [2AB_yz_y$$

$$+ABz_{yy} - 2A_yB - \frac{1}{2}yABz_y - (T_* - t)ABz_t\Big] f' + [2BB_yz_y$$
$$+B^2z_{yy} - 2BB_yz_y - \frac{1}{2}yB^2z_y - (T_* - t)B^2z_t\Big] ff' + [BB_{yy}$$
$$-{B_y}^2 - \frac{1}{2}yBB_y - (T_* - t)BB_t + 3AB^2 - B^2$$
$$+(T_* - t)B^2g(t)\big] f^2 - B^2z_y^2f'^2 + B^3f^3 + ABz_y^2f'' + B^2z_y^2ff'' = 0. \quad (40)$$

For equation (40) to represent an ordinary differential equation governing $f(z)$, we introduce functions $\Gamma_n(z),\ n = 1, 2, \cdots, 8$ in the following manner:

$$AA_{yy} - {A_y}^2 - \frac{1}{2}yAA_y - (T_* - t)AA_t + A^3 - A^2 + (T_* - t)A^2g(t) = B^2z_y^2\Gamma_1(z), \quad (41)$$

$$AB_{yy} + BA_{yy} - 2A_yB_y - \frac{1}{2}y(AB_y + BA_y) - (T_* - t)(AB_t + BA_t) + 3A^2B - 2AB + 2(T - t)ABg(t) = B^2z_y^2\Gamma_2(z), \quad (42)$$

$$2AB_yz_y + ABz_{yy} - 2A_yB - \frac{1}{2}yABz_y - (T_* - t)ABz_t = B^2z_y^2\Gamma_3(z), \quad (43)$$

$$2BB_yz_y + B^2z_{yy} - 2BB_yz_y - \frac{1}{2}yB^2z_y - (T_* - t)B^2z_t = B^2z_y^2\Gamma_4(z), \quad (44)$$

$$BB_{yy} - {B_y}^2 - \frac{1}{2}yBB_y - (T_* - t)BB_t + 3AB^2 - B^2 + (T_* - t)B^2g(t) = B^2z_y^2\Gamma_5(z), \quad (45)$$

$$-1 = \Gamma_6(z), \quad (46)$$

$$B = z_y^2\Gamma_7(z), \quad (47)$$

$$A = B\Gamma_8(z). \quad (48)$$

In view of equations (41)-(48), equation (40) takes the form

$$\Gamma_1(z) + \Gamma_2(z)f + \Gamma_3(z)f' + \Gamma_4(z)ff' + \Gamma_5(z)f^2 + \Gamma_6(z)f'^2 + \Gamma_7(z)f^3 + \Gamma_8(z)f'' + ff'' = 0. \quad (49)$$

The following remarks are used in determination of A, B, z and $\Gamma_n(z),\ n = 1, 2, \cdots, 8$ from the system (41)-(48).

Remark 1. If $A(x,t)$ has the form $A(x,t) = \hat{A}(x,t) + B(x,t)\Gamma(z)$, then we put $\Gamma(z) \equiv 0$.

Remark 2. If $B(x,t)$ is found to have the form $B(x,t) = \hat{B}(x,t)\Gamma(z)$, then we may choose $\Gamma(z) \equiv 1$.

Remark 3. If $z(x,t)$ is to be determined from the relation $F(z) = \hat{z}(x,t)$, where $F(z)$ is an invertible function, then without loss of generality, we may take $F(z) \equiv z$.

Using Remark 1 in equation (48), we obtain $\Gamma_8(z) \equiv 0$ and therefore

$$A = 0. \tag{50}$$

By substituting equation (50) into equations (41)-(43) and equation (45) to give

$$\Gamma_1(z) = \Gamma_2(z) = \Gamma_3(z) = 0$$

and we have

$$BB_{yy} - {B_y}^2 - \frac{1}{2}yBB_y - (T-t)BB_t \\ - B^2 + (T_* - t)B^2 g(t) = B^2 z_y^2 \Gamma_5(z)\,. \tag{51}$$

From Remark 2, equation (47) gives $\Gamma_7(z) \equiv 1$ and

$$B = {z_y}^2. \tag{52}$$

Writing equation (44) as

$$z_{yy} = \frac{1}{2}yz_y + (T_* - t)z_t + {z_y}^2\Gamma_4(z). \tag{53}$$

Now inserting equation (53) into equation (45), we get

$$4\frac{z_{yy}}{{z_y}^2}\Gamma_4(z) + 2\Gamma_4(z)' - 2\frac{z_{yy}}{{z_y}^4} + \frac{(T_* - t)g(t)}{{z_y}^2} \quad = \quad \Gamma_5(z). \tag{54}$$

From equation (53) can be written as

$$\frac{1}{2}z_y + (T_* - t)z_t = 0. \tag{55}$$

The above equation (55) can be written as

$$\frac{2dy}{y} = \frac{dt}{T_* - t} = \frac{dz}{0}. \tag{56}$$

Solving equation (56), we obtain

$$z = y(T_* - t)^{1/2} \tag{57}$$

is the similarity variable.

Assuming $\Gamma_4 = 0, \Gamma_5 = l_5$ and using equation (57) into equation (54), we get

$$g(t) = l_5. \tag{58}$$

Applying equation (57) into equation (52), we have

$$B(y,t) = T_* - t. \tag{59}$$

On using equation (50) and equation (59) into equation (39), we get the similarity transformation of (4) as

$$u(y,t) \quad = \quad (T_* - t)f(z)\,. \tag{60}$$

Applying $\Gamma_1 = \Gamma_2 = \Gamma_3 = \Gamma_4 = \Gamma_8 = 0$, $\Gamma_5 = l_5, \Gamma_6 = -1$ and $\Gamma_7 = 1$ in equation (5), we get the ordinary differential equation for $f(z)$ appearing in the similarity transformation (60) which is given by

$$ff'' - f'^2 + l_5 f^2 + f^3 \quad = \quad 0\,. \tag{61}$$

Equation (61) gives

$$fpp'(f) \quad = \quad p^2 - l_5 f^2 - f^3. \tag{62}$$

The above equation (62) can be written as

$$p'(f) = \frac{1}{f}p + (-l_5 f - f^2)p^{-1}, \tag{63}$$

which is the form of Bernoulli equation. Its solution is

$$p = \sqrt{w}\,. \tag{64}$$

Now inserting equation (64) into equation (63), we obtain

$$w'(f) - \frac{-2}{f}wf = -2(l_5 f + f^2) \tag{65}$$

which is linear in w. Its integrating factor is

$$w(f) = \frac{1}{y^2}.$$

The general solution of equation (65) is

$$f'^2 = p^2 = w = Mf^2 - 2f^2\left(\log f^{l_5} + f\right), \tag{66}$$

where M is a constant of integration.

The variable coefficient nonlinear equation (4) is reduced to the ordinary differential equation (61) and a first integral (66) of it is obtained.

5. Results

In the present work we extent the applicability of the recent method of Arrigo and Beckham to the nonlinear parabolic equation (4) which is of the form

$$u_t - \frac{1}{T_* - t}u_{yy} \quad = \quad R(t, y, u, u_y). \tag{67}$$

The form (67) is much more general than the form

$$u_t - u_{yy} = R(u, u_y) \tag{68}$$

considered by Arrigo and Beckham.

Indeed the determining equations (9)-(12) of (4) obtained by the application of the nonclassical method to (4) are quickly and easily recovered by the use of the invariant surface condition (3) and the compatibility condition (26).

References

[1] D.J. Arrigo, J.R. Beckham, Nonclassical symmetries of evolutionary partial differential equations and compatibility, *J. Math. Anal. Appl.*, **289** (2004), 55-65.

[2] D.J. Arrigo, F. Hickling, On the determining equations for the nonclassical reductions of the heat and Burgers‘ equation, *J. Math. Anal. Appl.*, **270** (2002), 582-589.

[3] Bebernes Eberly, *Mathematical Problems from Combustion Theory*, Appl. Math. Sci., **83**, Springer-Verlag, New York (1989).

[4] G.W. Bluman, S.C. Anco, *Symmetries and Differential Equations*, Springer-Verlag, New York (1989).

[5] G.W. Bluman, J.D. Cole, The general similarity solution of the heat equation, *J. Math. Phys.*, **18** (1969), 1025-1042.

[6] Stephen Bricher, Total versus single point blow-up for a nonlocal gaseous ignition model, *Rocky Mountain J. Math.*, **32** (2002), 25-43.

[7] P.A. Clarkson, M.D. Kruskal, New similarity reductions of the Boussinesq equation, *J. Math. Phys.*, **30** (1989), 2201-2213.

[8] Simon Hood, New exact solutions of Burger's equation-an extension to the direct method of Clarkson and Kruskal, *J. Math. Phys.*, **36**, No. 4 (1995), 1971-1990.

[9] Langlias Phillips, Stabilization of solutions of nonlinear and degenerate evolution equations, *Nonlinear Anal. Methods Appl.*, **9** (1985), 321-333.

[10] S. Lie, Klassification and Integration von gewohnlichen Differentialgleichen zwischen x, y die eine Gruppe von Transformation gestatten, *Math. Ann.*, **32** (1888), 213-281.

[11] E.L. Mansfield, Nonclassical group analysis of the heat equation, *J. Math. Anal. Appl.*, **231** (1999), 526-542.

[12] P.J. Olver, *Applications of Lie Groups to Differential Equations*, Second Edition, Springer-Verlag, New York (1993).

[13] C. Rogers, W.F. Ames, *Nonlinear Boundary Value Problems in Science and Engineering*, Academic Press (1989).

[14] P.L. Sachdev, B. Mayil Vaganan, Exact free surface flows for shallow water equations: Part I: The incompressible case, *Stud. Appl. Math.*, **93** (1994), 251-274.

[15] Souplet, Uniform blow-up profiles and boundary behavior for diffusion equations with nonlocal source, *J. Diff. Eqs.*, **153** (1999), 374-406.