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THE DOUBLE INEQUALITY RELATED TO THE VOLUME OF THE UNIT BALL IN R^n

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Abstract: Let $\Omega_n = \pi^{n/2}/\Gamma(1+\frac{n}{2})$ be the volume of the unit ball in R^n. In this paper, we prove that for any positive integer k, there exists positive integer $N(k) \geq k$, such that $\left(1+\frac{k+3}{n}\right)^{\frac{k^2}{2(k+3)}} < \frac{\Omega_n^2}{\Omega_{n-k}\Omega_{n+k}} < (1+\frac{1}{n})^{\frac{k^2}{2}}$ for all $n \geq N(k)$.

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1. Introduction

For real and positive values of x, the Euler gamma function Γ and its logarithmic derivative ψ, the so-called digamma function, are defined as

$$\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}dt, \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

For extensions of these functions to complex variables and for basic properties see [6].

In the recent past, several authors presented interesting monotonicity properties of the volume of the nuit ball in R^n,

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$$\Omega_n = \pi^{n/2}/\Gamma(1+\frac{n}{2}), \quad n = 1, 2, 3 \cdots .$$

The sequence itself is not monotonic, it was shown in [4] by J. Böhm and E. Hertel that Ω_n attains its maimunm at $n = 5$. But $\Omega_n^{1/n}$ $(n = 1, 2, \cdots)$ is strictly decreasing with $\lim_{n\to\infty} \Omega_n^{1/n} = 0$, as was proved by G.D. Anderson, M.K. Vamanamurthy and M. Vuorinen [3]. In 1997, G.D. Anderson and S.L. Qiu [2] showed that even $\Omega_n^{1/(n\log(n))}$ $(n = 2, 3, \cdots)$ is strictly decreasing with $\lim_{n\to\infty} \Omega_n^{1/(n\log(n))} = e^{-\frac{1}{2}}$, and in the same year D.A. Klain and G.C. Rota [5] established that $(n + 1)\Omega_{n+1}/\Omega_n$ $(n = 1, 2, \cdots)$ is strictly increasing. The monotonicity theorems given in [3] and [5] lead to inequality

$$\Omega_n^2/(\Omega_{n-1}\Omega_{n+1}) < 1 + \frac{1}{n}, \quad n = 1, 2, \cdots . \tag{1}$$

Since the gamma function is logarithmically convex we conclude that the sequence $\Omega_n (n = 0, 1, \cdots)$ is logarithmically concave. Hence, the following converse of (1) is true:

$$1 < \Omega_n^2/(\Omega_{n-1}\Omega_{n+1}), \quad n = 1, 2, \cdots . \tag{2}$$

In 2000, H. Alzer [1] obtain the refinements of inequalities (1) and (2) as follows.

Theorem A. *For all integers $n \geq 1$ we have*

$$(1+\frac{1}{n})^{\alpha} \leq \Omega_n^2/\Omega_{n-1}\Omega_{n+1} < (1+\frac{1}{n})^{\beta}$$

with the best possible constants $\alpha = 2 - (\log\pi)/\log 2 = 0.34850\cdots$ and $\beta = \frac{1}{2}$.

The main purpose of this paper is to prove the following theorem.

Theorem 1. *For any integer $k \geq 1$, there exists integer $N(k) \geq k$, such that*

$$\left(1+\frac{k+3}{n}\right)^{\alpha(k)} < \frac{\Omega_n^2}{\Omega_{n-k}\Omega_{n+k}} < (1+\frac{1}{n})^{\beta(k)}$$

with $\alpha(k) = \frac{k^2}{2(k+3)}$ and $\beta(k) = \frac{k^2}{2}$ for all $n \geq N(k)$.

2. Proof of Theorem 1

We shall first introduce the following three lemmas, they will be used in the proof of our main result.

Lemma 1. (see [1]) *For all $x > 0$, we have*

$$\log\Gamma(x+1) - \log\Gamma(x+\frac{1}{2}) < \frac{1}{2}\log\left(x + \frac{8x+3}{8(4x+1)}\right). \tag{3}$$

Moreover, we have the asymptotic expansions

$$\log\Gamma(x) = (x-\frac{1}{2})\log x - x + \log\sqrt{2\pi} + \frac{1}{12x} + O(\frac{1}{x^3}) \quad (x\to\infty). \tag{4}$$

Lemma 2. (see [1]) *For all $x > 0$, we have*

$$\psi(x+\frac{1}{2}) - \psi(x) > \frac{2x+1}{x(4x+1)}. \tag{5}$$

The following lemma is well-known.

Lemma 3. *If $x \to \infty$, then*

$$\log(1+\frac{1}{x}) = \frac{1}{x} - \frac{1}{2x^2} + o(\frac{1}{x^2}). \tag{6}$$

Proof of Theorem 1. First we shall prove the right hand side inequality in Theorem 1. Set

$$x_n = [2\log\Omega_n - \log\Omega_{n-k} - \log\Omega_{n+k}]/\log(1+\frac{1}{n}).$$

Then we need only to prove that there exists $N(k)$ such that $x_n < \beta(k)$ for $n \geq N(k)$. If we define function u as following:

$$u(x) = [\log\Gamma(x+1-\frac{k}{2}) + \log\Gamma(x+1+\frac{k}{2}) - 2\log\Gamma(x+1)]/\log(1+\frac{1}{2x}), \tag{7}$$

then

$$x_n = u(n/2) \quad n = 1, 2, \cdots.$$

Next we shall show that for any integer $k \geq 1$, there exists $M(k)$ such that u is strictly increasing on $[M(k), \infty)$, with $\lim\limits_{x\to\infty} u(x) = \frac{k^2}{2}$. This implies that for any integer $k \geq 1$, there exists $N(k)$ such that $x_n < \frac{k^2}{2}$ for $n \geq N(k)$. The following two cases will complete the proof.

Case 1. If $k = 2m \quad (m = 1, 2, 3, \cdots)$. Then (7) and $\Gamma(x+1) = x\Gamma(x)$ lead to

$$u(x) = \left[\sum_{i=1}^{m}\log(x+i) - \sum_{i=1}^{m}\log(x+1-i)\right]/\log(1+\frac{1}{2x}).$$

Differentiation yields

$$\log^2(1+\frac{1}{2x})u'(x)=\sum_{i=1}^{m}g_i(x),$$

where

$$g_i(x)=\frac{1}{x(2x+1)}\log(1+\frac{2i-1}{x+1-i})-\frac{2i-1}{(x+i)(x+1-i)}\log(1+\frac{1}{2x}),$$

then (6) implies

$$\begin{aligned}g_i(x)&=\frac{1}{x(2x+1)}[\frac{2i-1}{x+1-i}-\frac{(2i-1)^2}{2(x+1-i)^2}+o(\frac{1}{x^2})]\\&\qquad-\frac{2i-1}{(x+i)(x+1-i)}[\frac{1}{2x}-\frac{1}{8x^2}+o(\frac{1}{x^2})]\\&=\frac{1}{x^4}\left[\frac{(4i-2)-(32i^3-44i^2+16i-1)\frac{1}{x}-(2i^2-3i+1)\frac{1}{x^2}}{8(2+\frac{1}{x})(1+\frac{i}{x})(1+\frac{1-i}{x})^2}+o(1)\right]\\&\qquad>0\quad(x\to\infty).\end{aligned}$$

Case 2. If $k=2m-1\quad(m=1,2,3,\cdots)$. Then (7) and $\Gamma(x+1)=x\Gamma(x)$ leads to

$$\begin{aligned}u(x)&=[\log\Gamma(x+\frac{3}{2}-m)+\log\Gamma(x+\frac{1}{2}+m)-2\log\Gamma(x+1)]/\log(1+\frac{1}{2x})\\&=\frac{\sum_{i=1}^{m-1}\log(1+\frac{2i}{x+\frac{1}{2}-i})+\log(x+\frac{1}{2})+2[\log\Gamma(x+\frac{1}{2})-\log\Gamma(x+1)]}{\log(1+\frac{1}{2x})}.\end{aligned}$$

Making use of $\psi(x+1)=\frac{1}{x}+\psi(x)$ and differentiation yield

$$x\log^2(1+\frac{1}{2x})u'(x)=f(x)+\sum_{i=1}^{m-1}h_i(x),$$

where

$$\begin{aligned}f(x)&=2x\log(1+\frac{1}{2x})[\psi(x+\frac{1}{2})-\psi(x)]-\log(1+\frac{1}{2x})\\&\qquad+\frac{1}{2x+1}\log x+\frac{1}{x+\frac{1}{2}}[\log\Gamma(x+\frac{1}{2})-\log\Gamma(x+1)],\end{aligned}$$

and

$$h_i(x)=\frac{1}{2x+1}\log(1+\frac{2i}{x+\frac{1}{2}-i})-\frac{2ix}{(x+\frac{1}{2}+i)(x+\frac{1}{2}-i)}\log(1+\frac{1}{2x}).$$

Then for $x > 0$, (3) and (5) yield

$$f(x) > \frac{1}{4x+1}\log(1+\frac{1}{2x}) - \frac{1}{2x+1}\log\left(1+\frac{8x+3}{8x(4x+1)}\right) = v(x).$$

If for $y > 1$, let $x = \frac{1}{2(y-1)}$, then we have

$$\begin{aligned} v(x) = v\left(\frac{1}{2(y-1)}\right) &= \frac{y-1}{y}\left[\frac{y}{y+1}\log y - \log\frac{3y^2+2y+3}{4(y+1)}\right] \\ &= \frac{y-1}{y}w(y). \end{aligned} \tag{8}$$

Differentiation leads to

$$(y+1)^2 w'(y) = \log y - \frac{4(y^2-1)}{3y^2+2y+3}.$$

Since

$$((y+1)^2 w'(y))' = \frac{(y-1)^2(9y^2+22y+9)}{y(3y^2+2y+3)^2} > 0,$$

we conclude that

$$(y+1)^2 w'(y) > 4w'(1) = 0 \quad (y > 1).$$

Hence, we have

$$w(y) > w(1) = 0 \quad (y > 1),$$

so that (8) implies $f(x) > v(x) > 0$.

For $h_i(x)$, making use of (6) we obtain

$$\begin{aligned} h_i(x) = &\frac{1}{2x+1}\left[\frac{2i}{x+\frac{1}{2}-i} - \frac{2i^2}{(x+\frac{1}{2}-i)^2} + o(\frac{1}{x^2})\right] \\ &- \frac{2ix}{(x+\frac{1}{2}+i)(x+\frac{1}{2}-i)}\left[\frac{1}{2x} - \frac{1}{8x^2} + o(\frac{1}{x^2})\right] \\ = &\frac{1}{x^3}\left[\frac{i + (i - i^2 - 8i^3)\frac{1}{x} + (\frac{i}{4} - \frac{i^2}{2})\frac{1}{x^2}}{4(1+\frac{1}{2x})(1+\frac{1}{2x}+\frac{i}{x})(1+\frac{1}{2x}-\frac{i}{x})^2} + o(1)\right] > 0 \quad (x \to \infty). \end{aligned}$$

At last $\lim_{x\to\infty} u(x) = \frac{k^2}{2}$ follows from (4), (7) and the fact that

$$\lim_{x\to\infty} \frac{(x+\frac{1}{2}+\frac{k}{2})\log(1+\frac{k}{2(x+1)}) - (x+\frac{1}{2}-\frac{k}{2})\log(1+\frac{k}{2(x+1-\frac{k}{2})})}{\log(1+\frac{1}{2x})} = \frac{k^2}{2},$$

and

$$\lim_{x\to\infty} \frac{k^2}{24(1+x)(x+1+\frac{k}{2})(x+1-\frac{k}{2})\log(1+\frac{1}{2x})} = 0.$$

Next we shall prove the left hand side inequality in Theorem 1.

Set

$$y_n = [2\log\Omega_n - \log\Omega_{n-k} - \log\Omega_{n+k}]/\log(1+\frac{k+3}{n})$$

and

$$u(x) = [\log\Gamma(x+1-\frac{k}{2}) + \log\Gamma(x+1+\frac{k}{2}) - 2\log\Gamma(x+1)]/\log(1+\frac{k+3}{2x}). \quad (9)$$

Then

$$y_n = u(n/2), \quad n = 1, 2, \cdots.$$

Case 1. If $k = 2m \quad (m = 1, 2, 3, \cdots)$. Then (9) and $\Gamma(x+1) = x\Gamma(x)$ lead to

$$u(x) = \left[\sum_{i=1}^{m}\log(x+i) - \sum_{i=1}^{m}\log(x+1-i)\right]/\log(1+\frac{2m+3}{2x}).$$

Differentiation yields $\log^2(1+\frac{1}{2x})u'(x) = \sum\limits_{i=1}^{m} f_i(x)$, where

$$\begin{aligned} &f_i(x) \\ &= \frac{2m+3}{x(2x+2m+3)}\log(1+\frac{2i-1}{x+1-i}) - \frac{2i-1}{(x+i)(x+1-i)}\log(1+\frac{2m+3}{2x}). \end{aligned}$$

(6) implies

$$\begin{aligned} f_i(x) = &\frac{2m+3}{x(2x+2m+3)}\left[\frac{2i-1}{x+1-i} - \frac{(2i-1)^2}{2(x+1-i)^2} + o(\frac{1}{x^2})\right] \\ &- \frac{2i-1}{(x+i)(x+1-i)}\left[\frac{2m+3}{2x} - \frac{(2m+3)^2}{8x^2} + o(\frac{1}{x^2})\right] \\ &= -\frac{1}{x^4}\left[\frac{F_i(x)}{G_i(x)} + o(1)\right] < 0 \quad (x\to\infty), \end{aligned}$$

where

$$\begin{aligned} F_i(x) = &(2i-1)(2m+3)[(4m+2) \\ &- (4m^2+8m+4mi-16i^2+18i+3)\frac{1}{x} + (i-1)(2m+3)^2\frac{1}{x^2}] \end{aligned}$$

and $G_i(x) = 8(2+\frac{2m+3}{x})(1+\frac{i}{x})(1+\frac{1-i}{x})^2$.

Case 2. If $k = 2m-1 \quad (m = 1, 2, 3, \cdots)$. Then (9) and $\Gamma(x+1) = x\Gamma(x)$ lead to

$$u(x) = \frac{\sum\limits_{i=1}^{m-1}\log(1+\frac{2i}{x+\frac{1}{2}-i}) - \log(x+\frac{1}{2}) + 2[\log\Gamma(x+\frac{3}{2}) - \log\Gamma(x+1)]}{\log(1+\frac{m+1}{x})}.$$

Making use of $\psi(x+1) = \frac{1}{x} + \psi(x)$ and differentiation yield

$$\log^2(1+\frac{m+1}{x})u'(x) = p(x) + \sum_{i=1}^{m-1} q_i(x),$$

where

$$p(x) = \left[\frac{1}{x+\frac{1}{2}} + 2(\psi(x+\frac{1}{2}) - \psi(x+1))\right]\log(1+\frac{m+1}{x})$$
$$+ \frac{m+1}{x(x+m+1)}\left[2(\log\Gamma(x+\frac{3}{2}) - \log\Gamma(x+1)) - \log(x+\frac{1}{2})\right]$$

and

$$q_i(x) = \frac{m+1}{x(x+m+1)}\log(1+\frac{2i}{x+\frac{1}{2}-i})$$
$$- \frac{2i}{(x+\frac{1}{2}+i)(x+\frac{1}{2}-i)}\log(1+\frac{m+1}{x}).$$

(3), (5) and (6) lead to

$$p(x) < \left[\frac{1}{x+\frac{1}{2}} - \frac{4x+4}{(x+\frac{1}{2})(4x+3)}\right]\log(1+\frac{m+1}{x}) + \frac{m+1}{x(x+m+1)}$$
$$\times\left[\log(x+\frac{1}{2}+\frac{8x+7}{8(4x+3)}) - \log(x+\frac{1}{2})\right] < -\frac{1}{(x+\frac{1}{2})(4x+3)}$$
$$\times\log(1+\frac{m+1}{x}) + \frac{m+1}{x(x+m+1)}\log(1+\frac{x+1}{(x+\frac{1}{2})(4x+3)})$$
$$= -\frac{1}{x^4}\left[\frac{D_i(x)}{E_i(x)} + o(1)\right] < 0 \qquad (x\to\infty),$$

where

$$D_i(x) = (m+1)[(4m-3) - (4m^2+3m+7)\frac{1}{x}$$
$$- (5m^2 + \frac{17}{2}m + \frac{11}{2})\frac{1}{x^2} - \frac{3}{2}(m+1)^2\frac{1}{x^3}],$$

and

$$E_i(x) = 2(1+\frac{m+1}{x})(1+\frac{1}{2x})^2(4+\frac{3}{x})^2.$$

For $1 \le i \le m-1$, (6) implies

$$q_i(x) = \frac{m+1}{x(x+m+1)}\left[\frac{2i}{x+\frac{1}{2}-i} - \frac{2i^2}{(x+\frac{1}{2}-i)^2} + o(\frac{1}{x^2})\right]$$

$$-\frac{2i}{(x+\frac{1}{2}+i)(x+\frac{1}{2}-i)}\left[\frac{m+1}{x}-\frac{(m+1)^2}{2x^2}+o(\frac{1}{x^2})\right]$$
$$=-\frac{1}{x^4}\left[\frac{H_i(x)}{J_i(x)}+o(1)\right]<0 \quad (x\to\infty),$$

where

$$H_i(x)=(m+1)i[m-(m^2+mi+\frac{3}{2}m-4i^2+1)\frac{1}{x}+(i-\frac{1}{2})(m+1)^2\frac{1}{x^2}]$$

and

$$J_i(x)=(1+\frac{m+1}{x})(1+\frac{1}{2x}+\frac{i}{x})(1+\frac{1}{2x}-\frac{i}{x})^2.$$

At last $\lim_{x\to\infty}u(x)=\frac{k^2}{2(k+3)}$ follows from (4), (9) and the fact

$$\lim_{x\to\infty}\frac{(x+\frac{1}{2}+\frac{k}{2})\log(1+\frac{k}{2(x+1)})-(x+\frac{1}{2}-\frac{k}{2})\log(1+\frac{k}{2(x+1-\frac{k}{2})})}{\log(1+\frac{k+3}{2x})}=\frac{k^2}{2(k+3)}$$

and

$$\lim_{x\to\infty}\frac{k^2}{24(1+x)(x+1+\frac{k}{2})(x+1-\frac{k}{2})\log(1+\frac{k+3}{2x})}=0.$$

References

[1] H. Alzer, Inequalities for the volume of the unit ball in R^n, *J. Math. Anal. Appl.*, **252** (2000), 353-363.

[2] G.D. Anderson, S.L. Qiu, A monotonicity property of the gamma function, *Proc. Amer. Math. Soc.*, **125**, No. 11 (1997), 3355-3362.

[3] G.D. Anderson, M.K. Vamanamurthy, M. Vuorinen, Special functions of quasiconformal theory, *Exposition. Math.*, **7**, No. 2 (1989), 97-136.

[4] J. Böhm, E. Hertel, *Polyedergeometrie in n−Dimensionalen Räumen Konstanter Krümmung*, Birkhäuser, Basel (1981).

[5] D.A. Klain, G.C. Rota, A continuous analogue of Sperner's Theorem, *Comm. Pure Appl. Math.*, **50**, No. 3 (1997), 205-223.

[6] E.T. Whittaker, G.N. Watson, *A Course of Morden Analysis*, Cambridge Univ. Press, Cambridge (1958).