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ANALYSIS OF CHAOTIC MOTION OF NONLINEAR OSCILLATOR UNDER BOUNDED NOISE EXCITATION

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Abstract: A harmonic function with constant amplitude and random frequency and phase is called bounded noise. In this paper, the effect of hounded noise on the chaotic behavior of the nonlinear oscillator under parametric dynamics in the system. It is found that the threshold of hounded noise amplitude for the on set of chaos in the system increases as the in density of the noise in frequency increase. The threshold of bounded noise amplitude for the onset of chaos is also determined by the numerical calculation of the largest Lyapunov exponent. The numerical results qualitatively confirm the conclusion drawn by using the random Melnikov process with mean-square criterion for larger noise intensity.

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1. Introduction

The deterministic and stochastic nonlinear dynamical systems has attracted more and more attention for many years. The chaotic phenomena appear due to sensitive dependence upon initial conditions. The system becomes unpredictable. But the Melnikov method is an effective approach to detect chaotic dynamics and to analyze near homoclinic motion with deterministic or random perturbation. The method was first applied by Holmes [2] to study a

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periodically forced Duffing oscillator with negative linear stiffness. And by Ariaratnam et al [1] to investigate the chaotic behavior of a parametrically excited system such as the transverse vibration of a bucked column under axial periodic excitation. It was found that the noise increases the homoclinic threshold Lyapunov exponents were studied and is was found that the critical value of periodic forcing amplitude for been drawn. Melnikov method was extend by Frey and Simiu [10]. To study the effect of additive noise on near-integrable second-order dynamical systems. The mean-square criterion for the Melnikov process was used to study the periodically forced Duffing system with additive random perturbation by Y.K. Lin and Yim [4].

In the present paper, we consider a dynamical system with quadric and cubic term that is its nonlinear tem. Bounded noise is briefly introduced firstly. We study the homoclinic bifurcation and chaos of the nonlinear system that be given above. Under external excitation of bounded noise is investigated. Random Melnikov process with mean-square criterion and vanishing the largest Lyapunov exponent are used to determine the threshold amplitude of bounded noise excitation for the onset of chaos in the system. On the other hand. Another threshold of bounded noise amplitude for the onset of chaos is obtained by calculating the largest Lyapunov exponents numerically. A qualitatively consistent conclusion is drawn from the two thresholds obtained by using the analytical method and digital simulation.

2. Bounded Noise

A bounded noise is a harmonic function with constant amplitude and random frequency and phase. The mathematical expression for the noise is

$$\xi(t) = \mu \cos(\Omega\, t + \Psi)\,, \tag{1}$$

$$\Psi = \sigma B(t) + \Gamma\,, \tag{2}$$

where μ, Ω are positive constants. representing the amplitude and averaged frequency of bounded noise, respectively; $B(t)$ is unit Wiener process; σ is a constant, representing intensity of random frequency; Γ is a random variable uniformly distributed in $[0, 2\pi]$ representing random phase angle. $\xi(t)$ is a stationary random process in wide sense with zero mean. Its covariance function is

$$C_\xi(\tau) = \mu^2 \exp(-\sigma^2 \tau) \cos(\Omega \tau)\,, \tag{3}$$

and its spectral density is

$$S_\xi(\omega) = \frac{(\mu\sigma)^2}{2\pi}\left(\frac{1}{4(\omega-\Omega)^2+\sigma^4} + \frac{1}{4(\omega+\Omega)^2+\sigma^4}\right). \tag{4}$$

The variance of the bounded noise is

$$C(0) = \frac{mu^2}{2}\,, \tag{5}$$

which implies that the noise has finite power. The shape of spectral density depends on μ, Ω, σ while the bandwidth of the noise depends mainly on σ. It is a narrow-band process when $\sigma$is small and it approaches to white noise. When $\sigma \longrightarrow \infty$. It is easy to show that the sample functions of the noise are continuous and bounded which are required in the derivation of Melnikov function [4].

3. Random Melnikov Process

Consider a single degree-of-freedom Hamilton system subject to light damping and external or parametric perturbation of bounded noise. The equation of motion of the system is of the form

$$\begin{aligned} \dot{Q} &= \frac{\partial H}{\partial P}\,, \\ \dot{P} &= -\frac{\partial H}{\partial Q} - \epsilon c(Q,P)\frac{\partial H}{\partial P} + \epsilon f(Q,P)\xi(t)\,, \end{aligned} \tag{6}$$

where $Q$and P are generalized displacement and momentum. respectively; $H = H(Q,P)$ is Hamiltonian with continuous first-order derivatives; ϵ is a small positive parameter; $\xi(t)$ is bounded noise; $c(Q,P)$ represents the coefficient of quasi-linear damping; $f(Q,P)$ represents the amplitude of excitation. $f\xi(t)$ is an external excitation when f is a constant; it is a parametric excitation if $f$is a function of Q and (or) P. It is assumed that the unperturbed Hamiltonian system (6) with $\epsilon = 0$, system (6) with $\epsilon = 0$, possesses a hyperbolic fixed point connected to itself by homoclinic orbits $(q(t_0)p(t_0)$.

A bounded noise with spectral density (4) can be approximated by a sum of many harmonic functions with different frequencies and random phase. As in [8], the random Melnikov process for system (6) is

$$\begin{aligned} M(t_0) &= \int_{-\infty}^{+\infty} \frac{\partial H}{\partial P}\left[-c(Q,P)\frac{\partial H}{\partial P} + f(Q,P)\xi(t+t_0)\right]\mathrm{d}t \\ &= M_d + Z(t_0)\,. \end{aligned} \tag{7}$$

Here M_d is the component of Melnikov processduo to damping and $Z(t_0)$ is that due to bounded noise excitation. A deterministic Melnikov function having simple zeros implies transversal intersection of stable and unstable manifolds, giving rise to Smale horseshoes and hyperbolic in variant set. However, in (7) is a random process rather than a deterministic function and it can be treated only in some statistical sense. Obviously, the mean of random Melnikov process is

$$E[M(t_0)] = -\int_{-\infty}^{+\infty} c(Q,P)\left(\frac{\partial H}{\partial P}\right)^2 \mathrm{d}t\,. \tag{8}$$

Here $E[\cdot]$ denotes the expectation operator. For positive damping, (8) always yield negative value, which implies that system (6) cannot be chaotic in mean sense. So we consider if random process (7) has simple zeros in mean-square sense. The mean-square values of random Melnikov process (7) are

$$M_d^2 = \left(\int_{-\infty}^{+\infty} c(Q,P)(\frac{\partial H}{\partial P})^2 \mathrm{d}t\right)^2 , \tag{9}$$

$$\sigma_d^2 = E[Z^2(t_0)] = E\left(\int_{-\infty}^{+\infty} \frac{\partial H}{\partial P} f(Q,P)(t)\xi(t+t_0)]\mathrm{d}t\right)^2 . \tag{10}$$

The integral in (9) yield a positive constant since $Q = q_0(t), P = p_0(t)$, The integral in (10) is a convolution one and it can be rewritten as

$$Z(t_0) = \int_{\infty}^{+\infty} f(Q,P)\frac{\partial H}{\partial P} f(Q,P)\xi(t+t_0)]\mathrm{d}t = h(t) * \xi(t)\,, \tag{11}$$

where $h(t) = f(Q,P)\frac{\partial H}{\partial P}|_{Q=q_0(t),\ P=p_0(t)}$, $h(t)$ can be regarded as the impulse response function of a time-invariant linear system while $\xi(t)$ is an input of the system. Thus, the variance of system can be obtained in frequency domain as follows:

$$\sigma_Z^2 = \int_{-\infty}^{+\infty} |H(\omega)|^2 S_\xi(\omega)\mathrm{d}\omega\,, \tag{12}$$

where $H(\omega)$ is the frequency response function of the system, which is the Fourier transformation of the impulse response function $h(t)$, and $S_\xi(\omega)$ is the spectral density of $\xi(t)$. Note that $\xi(t)$ has zero meal. Random Melnikov process has simple zeros in mean-square sense when

$$M_d^2 = \sigma_Z^2\,, \tag{13}$$

which yield the threshold amplitude of bounded noise excitation for the onset of chaos in system (6).

Now let us apply the randomMelnikov process with mean-square criterion to a nonlinear dynamic system of three-dimensional sing 4-layer lattice shell. with linear damping and subject to wheal external and(or) parametric excitation of bounded noise. The Hamilton equation of the system is of the form.

$$\ddot{x} + x - \alpha x^2 + x^3 + \epsilon\beta\dot{x} = \mu\cos(\Omega t + \sigma B(t) + \Gamma)\,. \tag{14}$$

Here α, β are constant, Let $x = Q, \dot{x} = P$, (14) can be rewritten in the form of perturbed Hamiltonian system.

$$\dot{Q} = P\,, \tag{15}$$
$$\dot{P} = -Q + \alpha Q^2 - Q^3 - \epsilon\beta P + \mu\cos(\Omega t + \Psi)\,.$$

The Hamiltonian of equation of the unperturbed Hamilton system with $\epsilon, \mu = 0$ is

$$H(Q,P) = \frac{1}{2}Q^2 - \frac{\alpha}{3}Q^3 + \frac{1}{3}Q^4 + \frac{1}{2}P^2\,. \tag{16}$$

It is known that the unperturbed Hamiltonian system possesses saddle point $A(\frac{\alpha}{2} \pm \sqrt{\frac{\alpha^2}{4} - 1}, 0)(\alpha > 2)$. And if let $\alpha = 3$ then a homoclinic orbits is

$$q_0(t) = \frac{2}{2 - \sqrt{2}\cos t}\,, \quad p_0(t) = \frac{-2\sqrt{2}\sin t}{(2 - \sqrt{2}\cos t)^2} \tag{17}$$

The random Melnikov process for system (14) can be obtained by using formula (7) as follows:

$$\begin{aligned} M(t_0) &= \int_{-\infty}^{+\infty} [-\beta p_0^2(t) + \mu p_0(t)\cos(\Omega(t + t_0) + \Psi)]dt \\ &= -2\sqrt{2}\beta\frac{\pi}{8} + \int_{-\infty}^{+\infty} \mu p_0(t)\cos(\Omega(t + t_0) + \Psi)dt\,. \end{aligned} \tag{18}$$

Here

$$M_d = -2\sqrt{2}\beta\frac{\pi}{8}, Z_{t_0} = \int_{-\infty}^{+\infty} \mu p_0(t)\cos(\Omega(t + t_0) + \Psi)dt\,. \tag{19}$$

In this case, the impulse response function is

$$h(t) = q_0(t)p_0(t)\,. \tag{20}$$

The associated frequently response function is

$$H(\omega) = \int_{-\infty}^{+\infty} h(t)e^{-i\omega t d\omega}\,. \tag{21}$$

Variance σ_Z^2 is thus

$$\sigma_Z^2 = \int_{-\infty}^{+\infty} |H(\omega)|^2 \frac{(\mu\sigma)^2}{2\pi}\left(\frac{1}{4(\omega - \Omega)^2 + \sigma^4} + \frac{1}{4(\omega + \Omega)^2 + \sigma^4}\right)\mathrm{d}\omega\,. \tag{22}$$

The integral σ_Z^2 in (22) can be completed only numerically. Numerical computation has been made for the following parameter value. The mean-square criterion in (13) yield the threshold of bounded noise amplitude for the onset

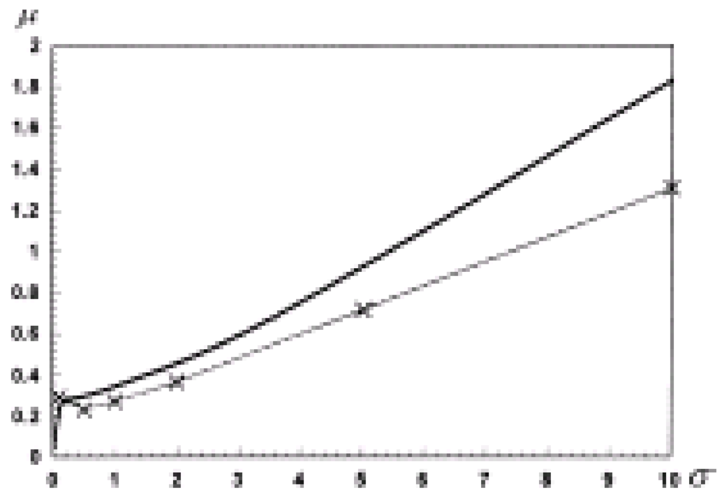

Figure 1: The relation between σ and μ threshold ($-$: analytical results; $\times$: digital simulation)

of chaos in system (14), which is shown in Figure 1.

4. Digital Simulation

4.1. The Wiener process increments

$$\mathrm{d}B_i = B(t_{i+1}) - B(t_i), \qquad i = 1, 2, \cdots,$$

of unit Wiener process $B(t)$ are used to generate the sample functions of the Wiener process in bounded noise. $\mathrm{d}B_i$ are in dependently and identically distributed Gaussian random variable with zero mean and variance dt. For one sample of bounded noise, γ phase is a constant randomly selected from interval $[0, 2\pi]$ with equal probability.

The largest Lyapunov exponent. The Lyapunov exponent represents the asymptotic rate of exponential convergence of divergence of nearby orbits of a dynamical system in phase space and is one of most important characteristics of system behavior. Exponential divergence of nearly orbit implies that the behavior of a dynamical system is sensitive to initial condition. Which is a characteristic of chaotic dynamical system. A dynamical system with a positive largest Lyapunov exponent is usually identified as chaotic. Thus the condition for the onset of chaos in a dynamical system can also be determined by numerically calculating the larges Lyapunov exponent.

4.2. The largest Lyapunov exponent for system (14) is computed to check the threshold of bounded noise amplitude for the onset of chaos obtain by using random Melnikov process with mean-square criterion. For an n-dimensional continuous dynamical system, and initial n-dimensional sphere in phase space will be come an n-dimensional ellipsoid due to the local deformation of the phase flow. The ith Lyapunov exponent is defined in terms of the length of ith ellipsoid principal axis $a_i(t)$ as follows.

$$\lambda_i = \lim_t \to \infty \frac{1}{t} \log \frac{a_i(t)}{a_i(0)}, \quad i = 1, 2, \cdots, n\,,$$

λ_i are usually arranged in a decreasing order and λ_1 is the largest Lyapunov exponent Wolf et al [9]. Developed a powerful algorithm for systematically computing all the Lyapunov exponents of a dynamical system. Here, their algorithm is used to calculate the largest Lyapunov exponent of system (14). The largest Lyapunov exponent as function of bounded noise amplitude for a serious of σ values is shown in Figure 2. The threshold of bounded noise amplitude for the onset of chaos in system (14) given by Figure 2 is

It is seen from the Figures 2. that λ_1 is negative for small value of μ. As μ increase. λ_1 Changes from negative to positive, which signifying the presence of chaotic motion in the system.

In the absence of noise, beyond the threshold μ for onset of chaotic motion. There are many “windows” or intervals, in which λ_1 becomes negative again and the system is then periodic. For larger noise intensity, there are no “periodic windows” presented for the parameter range of μ simulated. The effect of noise is to wash out or diminish these periodic windows or intervals. It is clear that the largest Lyapunov exponents characterize quantitatively the dynamics of the system.

It is also seen from Figure 1. that for smaller noise intensity σ, the threshold for the onset of chaos decreases as the noise intensity σincrease. For larger noise intensity, both the random Melnikov process with mean-square criterion and the numerical calculation of the largest Lyapunov exponent yield the same variation trend for the threshold μ as noise intensity σ increases. The threshold increases as noise intensity increases because the diffusion of frequency reduces the effect of bounded noise on triggering chaos in the system. The threshold of bounded criterion is larger than that given by calculating the largest Lyapunov exponent numerically. It is reasonable since the mean-square criterion underestimates the effect of bounded noise on the onset of chaos. We note easily that the bounded noise even has no effect on the onset of chaos if mean criterion is used.

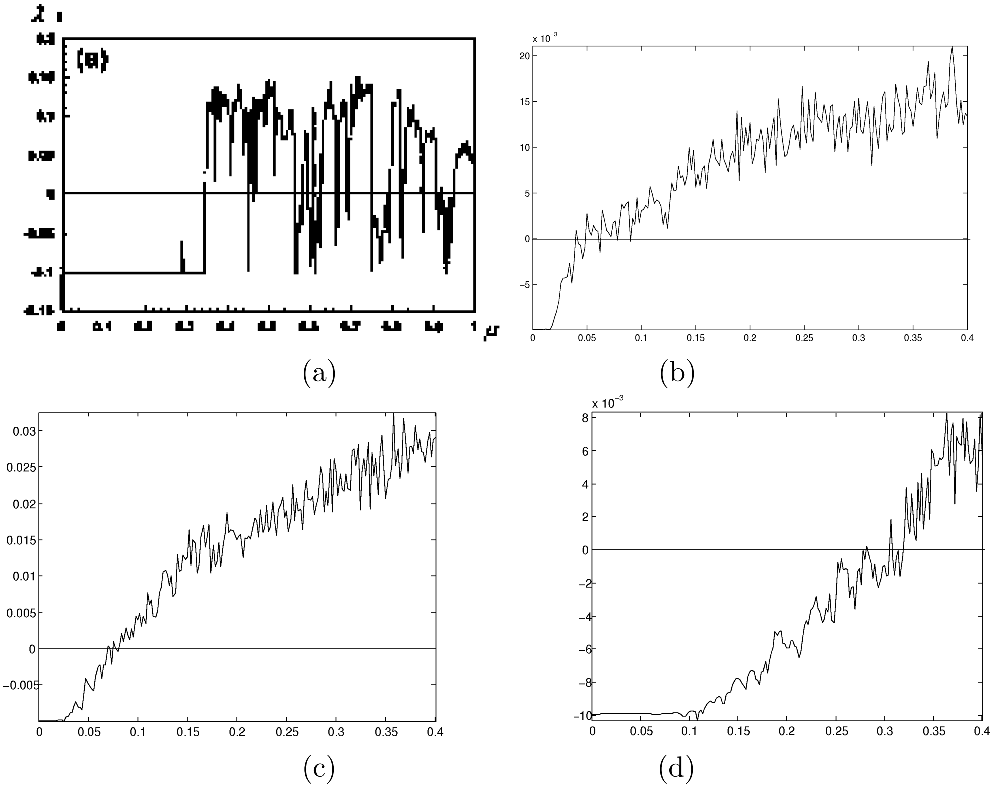

Figure 2: The largest Lyapunov exponent for noise-system with (a) $\psi = 0.0$, and for noisy system with (b) $\sigma = 0.5$, (c) $\sigma = 1.0$, (d) $\sigma = 5.0$

Therefore, it can be said that the random Melnikov process with mean-square criterion noise intensity, however, the two methods give inconsistent results and further investigation is needed to clarify this in consistency

5. Conclusions

In the present paper, the homoclinic bifurcation and chaos behavior of a nonlinear dynamical with quatic and cublic tem subject to external and (or)parametric excitations of bounded noise have been studied by using random Melnikov process, the largest Lyapunov exponent. The random Melnikov process with mean-square criterion has been used to establish the threshold of bounded noise amplitude for the onset of chaos. It is founded that for larger noise intensity, noise perturbation increases the threshold. The result has been verified by the numerical results for the largest Lyapunov exponent. When the amplitude of bounded noise is less than the threshold value, the motion of the system is periodic equilibrium at origin. When the amplitude of bounded noise is larger than the threshold value, the motion of the system is chaotic or random chaotic, when the intensity of noise is increased, the "periodic windows" or in intervals of the largest Lyapunov exponent beyond the onset of chaotic motion are gradually washed out or diminished. The effect of noise is to reduce the harmonics and to increase the background noise of the power spectra.

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