International Journal of Pure and Applied Mathematics

Volume 41 No. 7 2007, 1019-1032

TAUBERIAN CORE THEOREMS FOR THE POWER SERIES METHOD

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Abstract: The author proves Tauberian Core Theorems for the power series summability method P_p. The core versions of some well known Tauberian Theorems for Abel method and Borel method are also established.

AMS Subject Classification: 40E05, 40C15, 40G10
Key Words: Bonsall core, Knopp core, martix method, power series method, Abel method, Borel method, Tauberian Theorem

1. Introduction

The concept of the core of a sequence $x = (\xi_k)$ of complex numbers has been defined by Knopp in 1930 (see [5], Chapter VI). His definition is equivalent to the following: the core of x is the set of all complex numbers t for which

$$\mathrm{Re}\,(\alpha t) \leq \limsup_{k\to\infty} \mathrm{Re}\,(\alpha \xi_k) \quad \forall \alpha \in \mathbb{C}$$

(Ret is the real part of t).

Let X be a vector space over the field $\mathbb{R} = \mathbb{R}$ or $\mathbb{R} = \mathbb{C}$ and let π be an arbitrarily fixed functional on X with range $[-\infty, \infty]$ such that:

1) $\pi\,(0) = 0$,

2) $\pi\,(\alpha x) = \alpha\pi\,(x) \quad \forall \alpha > 0$,

3) $\pi\,(x + y) \leq \pi\,(x) + \pi\,(y) \quad \forall x : |\pi\,(x)| < \infty$.

Received: November 28, 2007

With the usual conventions for the manipulation of ∞ and $-\infty$, these conditions are always meaningful. This functional π is called *Bonsall functional* and the set

$$K(x) := \{t \in \mathbb{R} \mid \operatorname{Re}(\alpha t) \le \pi(\alpha x) \quad \forall \alpha \in \mathbb{R}\} \tag{1}$$

is called *Bonsall core* of the $x \in X$ (defined by π). If for a certain α holds $\pi(\alpha x) = -\infty$, then $K(x)$ is empty. It is easy to check that

$$K(x) = \{t \in \mathbb{R} \mid -\pi(-\alpha x) \le \operatorname{Re}(\alpha t) \le \pi(\alpha x) \quad \forall \alpha \in \mathbb{R}\}. \tag{2}$$

Let

$$c_\pi := \{x \in X \mid K(x) \text{ is a singleton, } |\pi(\alpha x)| < \infty \; \forall \alpha \in \mathbb{R}\}$$

(the set of all π-*convergent elements*) and let

$$c_{\pi 0} := \{x \in c_\pi \mid \pi(x) = 0\}$$

(the set of all π-*null elements*). The next proposition summarises those results of the Bonsall paper [2] that will be needed for the following inquiry.

Proposition 1. *The sets c_π and $c_{\pi 0}$ are linear subspaces of X. If $x \in X$ and $y \in c_\pi$, then*

$$K(x+y) = K(x) + K(y).$$

If $|\pi(\alpha x_0)| < \infty$ $\forall \alpha \in \mathbb{R}$, then $K(x_0)$ is nonempty closed and convex set in $\mathbb{R}$ and

$$K(\alpha x_0) = \alpha K(x_0) \quad \forall \alpha \in \mathbb{R}.$$

Inclusion relations between different cores are an essential focus of core theory. Let $K_1(x)$ and $K_2(y)$ be two cores defined by Bonsall functionals π_1 and π_2 respectively. An immediate consequence of the definition of core is that if

$$\pi_1(\alpha x) \le \pi_2(\alpha y) \quad \forall \alpha \in \mathbb{R}, \tag{3}$$

then

$$K_1(x) \subset K_2(y).$$

Due to the possibility of empty cores the converse implication is not always true.

Proposition 2. (see [8]) *For every $x \in c_{\pi 0}$ and $y \in X$,*

$$\pi(x+y) = \pi(y) \tag{4}$$

and

$$K(x+y) = K(y). \tag{5}$$

Proof. Let $x \in c_{\pi 0}$ and $\alpha \in \mathbb{R}$. As $c_{\pi 0}$ is a vector space, $\pi(\alpha x) = 0$. For any $y \in X$ we get that

$$\pi(\alpha y) \leq \pi(\alpha x + \alpha y) + \pi(-\alpha x) = \pi(\alpha x + \alpha y) \leq \pi(\alpha x) + \pi(\alpha y) = \pi(\alpha y),$$

i.e.,

$$\pi(\alpha(x+y)) = \pi(\alpha y) \quad \forall \alpha \in \mathbb{R},$$

and (4) is true. The equality of cores (5) follows now from (1). □

Corollary 3. *If $x - y \in c_{\pi 0}$, then $\pi(x) = \pi(y)$ and $K(x) = K(y)$.*

Let ω be the set of all sequences $x = (\xi_k)$, where $\xi_k \in \mathbb{C}$, $k \in \mathbb{N}^0$ and $\mathbb{N}^0 := \{0, 1, 2, ...\}$. Each linear subspace of ω is called a sequence space. In the sequel the cores in sequence spaces are investigated. The following subsets of ω are well known:

$$l_\infty := \left\{ x \in \omega \mid \sup_k |\xi_k| < \infty \right\}$$

(the set of bounded sequences),

$$c := \left\{ x \in \omega \mid \exists \lim_{k\to\infty} \xi_k \right\}$$

(the set of convergent sequences),

$$c_0 := \left\{ x \in c \mid \lim_{k\to\infty} \xi_k = 0 \right\}.$$

Let A be the matrix method that is determined by infinite matrix $A = (a_{nk})$. Let c_A denote the summability domain of the matrix method A, i.e.,

$$c_A := \left\{ x \in \omega \mid \exists \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk}\xi_k \right\}.$$

The set c_A is called also the set of A-convergent sequences. If $x \in c_A$, then the notation

$$A\text{-}\lim x := \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk}\xi_k$$

is used. The set

$$c_{A0} := \{x \in c_A \mid A\text{-}\lim x = 0\}$$

is called the set of A-null sequences.

The functional

$$\pi^\circ(x) := \limsup_{k\to\infty} \mathrm{Re}\xi_k$$

is the Bonsall functional that defines Knopp core $K^\circ(x)$ in ω (cf. [2], [5]). It is obvious that the set $c_{\pi^\circ 0}$ of π°-null elements is c_0 and the set c_{π° of π°-

convergent elements is c. The following lemma describes an important property of Knopp core (see [5], Chapter VI).

Lemma 4. *Let D be the set of all subsequences of $x = (\xi_k)$. If $x \in l_\infty$ then $K^\circ(x)$ is the closure in $\mathbb{C}$ of convex hull of the set*

$$\left\{ \xi \in \mathbb{C} \,|\, \exists\, (\xi_{k_n}) \in D \, \textit{such that} \, \lim_{n\to\infty} \xi_{k_n} = \xi \right\}.$$

Let A and B be two different matrix methods with $c_B \subset c_A$. The problem to determine the subset L of ω, such that $x \in L \cap c_A$ implies $x \in c_B$ have been studied extensively. In summability theory the theorem which gives the description of certain L is called Tauberian Theorem. The condition which determines L is called a Tauberian condition.

Let π_1 and π_2 be two different Bonsall functionals on a sequence space $X \subset \omega$ with $c_{\pi_1} \subset c_{\pi_2}$. Naturally, there arises a question of how to give the description of certain subsets L of X having one of the the following properties:

(a) $x \in L \Longrightarrow K_{\pi_1}(x) = K_{\pi_2}(x)$,

(b) $x \in L \Longrightarrow K_{\pi_1}(x) = K^\circ(x)$,

(c) $x \in L \Longrightarrow K_{\pi_2}(x) \subset K_{\pi_1}(x)$.

We call the theorem which states (a), (b), or (c) *Tauberian (Core) Theorem*. The condition which determines L is called a *Tauberian condition* (cf. [8]).

To formulate some Tauberian conditions we make use of the following Landau order symbols.

Notation 5. For any $\alpha_k > 0$ $(k \geq K)$ and $x = (\xi_k) \in \omega$ we put

$$\xi_k = O(\alpha_k) :\Longleftrightarrow \exists M > 0 : |\xi_k| \leq M\alpha_k \text{ for each } k \geq K,$$

$$\xi_k = O_L(\alpha_k) :\Longleftrightarrow \exists M > 0 : \xi_k \geq -M\alpha_k \text{ for each } k \geq K,$$

and

$$\xi_k = o(\alpha_k) :\Longleftrightarrow \lim_{k\to\infty} \frac{\xi_k}{\alpha_k} = 0.$$

In addition for any $x = (\xi_k)$ and $y = (\zeta_k)$ we put

$$\xi_k \sim \zeta_k :\Longleftrightarrow \lim_{k\to\infty} \frac{\xi_k}{\zeta_k} = 1.$$

2. Definition of Core for the Power Series Method

Throughout this paper let (p_k) be a real sequence with $p_0 > 0$ and $p_k \geq 0$ $(k \in \mathbb{N})$ and such that the corresponding power series

$$p(t) := \sum_{k=0}^{\infty} p_k t^k$$

have radius of convergence R with $0 < R \leq \infty$. Let for $x = (\xi_k) \in \omega$

$$p_x(t) := \sum_{k=0}^{\infty} p_k \xi_k t^k$$

and let

$$\omega_p := \{x \in \omega \,|\, \text{radius of convergence of } p_x(t) \text{ is equal or greater than } R\}.$$

It is obvious that ω_p is a sequence space.

Definition 6. The sequence $x \in \omega_p$ is said to be summable by power series method P_p to a number a (or P_p-summable to a) if

$$\lim_{t \to R-} \frac{p_x(t)}{p(t)} = a =: P_p\text{-}\lim x.$$

The set of all sequences x, that are summable by a power series method P_p is denoted by c_{P_p}.

It is said that power series method P_p is defined by $p = (p_k)$.

Example 7. The well known Abel summability method J_1 is the power series method defined by $p = (p_k)$, where $p_k = 1$ $\left(k \in \mathbb{N}^0\right)$. Then $R = 1$ and $p(t) = \frac{1}{1-t}$, for $t \in (-1, 1)$.

If we have $p_k = \frac{1}{k!}$, then $p = (p_k)$ defines the Borel method B_1. For Borel method $p(t) = e^t$ and $R = \infty$.

Let

$$W := \{w = (t_k) \,|\, 0 < t_k \to R-\}. \tag{6}$$

The matrix method corresponding to the infinite matrix $A_w = (a_{nk})$, where

$$a_{nk} = \frac{p_k t_n^k}{p(t_n)},$$

is called a *discrete power series method* (with respect to $p = (p_k)$ and $w = (t_n) \in W$). An immediate consequence of the sequential criterion for the existence of a limit is

$$c_{P_p} = \bigcap_{w \in W} c_{A_w},$$

where c_{A_w} is the set of A_w-convergent sequences.

Proposition 8. *The following statements are equivalent:*

(a) P_p is regular (that is $c \subset c_{P_p}$ and P_p-$\lim x = \lim x$ for all $x \in c$),

(b) for each $w \in W$ the discrete power series method A_w is regular,

Proof. For the proof see [3] p. 160. □

Corollary 9. *If the power series method P_p is regular, then for every discrete power series method A_w*

$$\pi^{\circ}(A_w x) \leq \pi^{\circ}(x) \text{ for each } x \in \omega_p$$

and

$$K^{\circ}(A_w x) \subset K^{\circ}(x) \text{ for each } x \in \omega_p,$$

i.e., A_w is core-shrinking for Knopp core.

Proof. The proof of this corollary follows directly from the Knopp core theorem, by which every positive regular method is core-shrinking (see [6], Chapter III). □

We now introduce the notion of core for a power series method (cf. [7]).

Definition 10. The core $K_p(x)$ of $x \in \omega_p$ defined by Bonsall functional

$$\pi_p(x) = \limsup_{t \to R-} \frac{\mathrm{Re}\, p_x(t)}{p(t)}$$

on ω_p is called power series Knopp core induced by $p = (p_k)$.

It is easy to see that π_p is a Bonsall functional and

$$\pi_p(x) = \sup_{w \in W} \pi^{\circ}(A_w x).$$

Thus, by Lemma 4, we get that if $K_p(x)$ is a bounded set, then it is the closure of the convex hull of E in $\mathbb{C}$, where

$$E = \{\xi \mid \exists w \in W, \quad K^{\circ}(A_w x) = \{\xi\}\}.$$

In case of the real sequence x,

$$K_p(x) = \left\{ \tau \in \mathbb{R} \mid \liminf_{t \to R-} \frac{p_x(t)}{p(t)} \leq \tau \leq \limsup_{t \to R-} \frac{p_x(t)}{p(t)} \right\}. \tag{7}$$

Proposition 11. *If P_p is a regular power series method, then:*

(a) $K^{\circ}(A_w x) \subset K_p(x)$ *for every* $x \in \omega_p$ *and* $w \in W$,

(b) $K_p(x) \subset K^{\circ}(x)$ *for every* $x \in \omega_p$,

(c) $c_{\pi_p} = \bigcap_{w \in W} c_{A_w} = c_{P_p}$.

Proof. These properties are immediate consequences of the definition of power series Knopp core $K_p(x)$ and of Corollary 9. □

Let P_p be a regular power series method and let w_0 be an arbitrarily fixed element of W. It is well known, that

$$c_{P_p} \subset c_{A_{w_0}} \neq c_{P_p} \text{ and } c \neq c_{P_p}.$$

This implies that there exist x_1 and x_2 such that

$$K^{\circ}(A_{w_0} x_1) \neq K_p(x_1) \text{ and } K_p(x_2) \neq K^{\circ}(x_2).$$

That is why we are interested finding the Tauberian conditions for the implications:

1. $x \in L \Longrightarrow K_p(x) = K^{\circ}(A_{w_0} x)$;
2. $x \in L \Longrightarrow K_p(x) = K^{\circ}(x)$.

3. Tauberian Core Theorems for P_p

In this section we establish two Tauberian Core Theorems for general power series method P_p and apply them to both Abel method and Borel method.

Let P_p be a power series method and let $\Delta_k := \inf_{0<t<R} p(t)\, t^{-k}$. For every method P_p there exists a sequence $w = (t_k^*) \in W$ with the property

$$p(t_k)\,(t_k^*)^{-k} = \Delta_k \text{ for all } k = 0, 1, 2, ... \tag{8}$$

(see [3], p. 187). In case of ordinary Tauberian Theorems, Tauberian conditions are frequently connected with the sequence (Δ_k). It turns out that this sequence is useful for Tauberian Core Theorems also.

What follows is some information about the sequence (Δ_k) that corresponds to the given power series method P_p.

Example 12. 1. If $p = \left(\frac{1}{k+1}\right)$, then $\Delta_k \sim \log k$.

2. If $\alpha > 0$ and $p = \left(\binom{k+\alpha-1}{k}\right)$, then $\Delta_k \sim k^{\alpha} \frac{e^{\alpha}}{\alpha^{\alpha}}$.

3. If $\beta \in (0,1)$ and $p = \left(e^{k^{\beta}}\right)$, then $\Delta_k \sim p_k \sqrt{\frac{2\pi}{\beta(1-\beta)}} k^{1-\beta}$,

See [3], p. 191.

Lemma 13. *Let P be a regular power series method and let $w^* = (t_k^*) \in W$ have the property (8). Each $x = (\xi_k) \in \omega_p$ which satisfies the Tauberian condition*

$$\xi_k - \xi_{k-1} = o\left(\frac{p_k}{\Delta_k}\right), \tag{9}$$

has the following property

$$\lim_k \left(\frac{p_x(t_k^*)}{p(t_k^*)} - \xi_k\right) = 0.$$

Proof. The proof of this lemma is a part of the proof of the o-Tauberian Theorem for P_p (see [3], Theorem 4.3.5 p. 188-190). □

Theorem 14. *Let P_p be a regular power series method. Then*

$$K_p(x) = K^\circ(x) \tag{10}$$

for each $x = (\xi_k)$ which satisfies the Tauberian condition (9).

Proof. Let $x = (\xi_k)$ satisfy the Tauberian condition (9) and let $w^* = (t_k^*) \in W$ have the property (8). By Corollary 3 and Lemma 13 we get that

$$\pi^\circ(A_{w^*}(x)) = \pi^\circ(x) \text{ and } K^\circ(A_{w^*}x) = K^\circ(x).$$

Due to Proposition 11 we get

$$K^\circ(A_{w^*}x) \subset K_p(x) \subset K^\circ(x),$$

i.e., (10) holds. □

We are now in a position to show that for specific cases of the power series method the well known Tauberian conditions for ordinary Tauberian Theorems are also the conditions for the Tauberian Core Theorems.

Corollary 15. *Abel method J_1 is core-preserving, i.e.,*

$$K_{J_1}(x) = K^\circ(x),$$

for every $x = (\xi_k)$, which satisfies the condition

$$\xi_k - \xi_{k-1} = o\left(\frac{1}{k}\right). \tag{11}$$

Proof. For the Abel method we have $p_k = 1$ $(k \in \mathbb{N}^0)$ and $\Delta_k \sim ek$ (see [3] p. 186). Therefore, due to Theorem 14, we get the condition (11). □

The condition (11) is the original Tauber's condition of 1897.

Corollary 16. *Borel method B_1 is core-preserving, i.e.,*

$$K_{B_1}(x) = K^\circ(x),$$

for every $x = (\xi_k)$, which satisfies the condition

$$\xi_k - \xi_{k-1} = o\left(\frac{1}{\sqrt{k}}\right). \tag{12}$$

Proof. In the case of Borel method $p_k = \frac{1}{k!}$ $\left(k \in \mathbb{N}^0\right)$ and $\Delta_k \sim c\frac{\sqrt{k}}{k!}$ (see [3] p. 186). Therefore, due to Theorem 14, we get the condition (12). □

The well known result due to Hardy and Littlewood from 1916 tells that if $x = (\xi_k) \in c_{B_1}$ and

$$\xi_k - \xi_{k-1} = O\left(\frac{1}{\sqrt{k}}\right), \tag{13}$$

then $x \in c$ (cf. [4]). The condition (12) is a stonger version of Tauberian condition (13).

We now apply Theorem 14 to the particular cases of power series methods P_p for which the corresponding sequences (Δ_k) are described in Example 12.

Corollary 17. *Let $p = (p_k)$. Then the Tauberian condition* (9), *for P_p to satisfy* (10), *is as follows:*

1. $\xi_k - \xi_{k-1} = o\left(\frac{1}{(k+1)\log n}\right)$, *if* $p = \left(\frac{1}{k+1}\right)$.
2. $\xi_k - \xi_{k-1} = o\left(\binom{k+\alpha-1}{k}k^{-\alpha}\right)$, *if* $\alpha > 0$ *and* $p = \left(\binom{k+\alpha-1}{k}\right)$.
3. $\xi_k - \xi_{k-1} = o\left(k^{-1+\beta/2}\right)$, *if* $\beta \in (0,1)$ *and* $p = \left(e^{k^\beta}\right)$.

Theorem 18. *Let P_p and P_q be two regular power series methods defined by (p_k) and (q_k) respectively. Let $\varepsilon_k = 1 - \frac{q_k}{p_k}$ $(k = 0, 1, ...)$ and let $\varepsilon_k = o(1)$. Each sequence $x = (\xi_k) \in \omega_p$ which satisfies the condition $(\varepsilon_k \xi_k) \in c_{0p}$ has the property*

$$K_p(x) = K_q(x). \tag{14}$$

Proof. Let $q(t)=\sum\limits_{k=0}^{\infty}q_kt^k$. Our proof starts with the observation that

$$\lim_{t\to R-}\frac{q(t)}{p(t)}=\lim_{t\to R-}\frac{1}{p(t)}\sum_{k=0}^{\infty}q_kt^k=\lim_{t\to R-}\frac{1}{p(t)}\sum_{k=0}^{\infty}p_k\frac{q_k}{p_k}t^k=1, \qquad (15)$$

as $\varepsilon_k=o(1)$ and P_p is a regular power series method. Let $x=(\xi_k)\in\omega_p$ satisfy the condition $(\varepsilon_k\xi_k)\in c_{0p}$. Then

$$\lim_{t\to R-}\frac{1}{p(t)}\left(\sum_{k=0}^{\infty}p_k\xi_kt^k-\sum_{k=0}^{\infty}q_k\xi_kt^k\right)=\lim_{t\to R-}\frac{1}{p(t)}\sum_{k=0}^{\infty}p_k(\varepsilon_k\xi_k)t^k=0.$$

For this reason, due to (15),

$$\begin{aligned}\pi_p(\alpha x) &= \limsup_{t\to R-}\frac{1}{p(t)}\mathrm{Re}\alpha\sum_{k=0}^{\infty}q_k\xi_kt^k\\ &= \limsup_{t\to R-}\frac{q(t)}{p(t)}\left(\frac{1}{q(t)}\mathrm{Re}\alpha\sum_{k=0}^{\infty}q_k\xi_kt^k\right)=\pi_q(\alpha x)\end{aligned}$$

for every $\alpha\in\mathbb{C}$. An immediate consequence of the definition of Bonsall core gives (14). □

Let $p_k>0$ $(k\in\mathbb{N}^0)$ and $\zeta_k\geq 0$ $\left(k\in\mathbb{N}^0\right)$ and let (Δ_k) be the sequence that correspond to the power series method P_p. It is well known that in such case $y=(\zeta_k)\in c_{0p}$ implies $\left(\frac{\zeta_kp_k}{\Delta_k}\right)\in c_0$ (see [3] p. 191). Therefore, if $(\varepsilon_k\xi_k)\in c_{0p}$ and if $\varepsilon_k\xi_k\geq 0$ $(k\in N)$, then $\left(\frac{p_k-q_k}{\Delta_k}\xi_k\right)\in c_0$.

What follows are the direct consequences of the Theorem 18.

Corollary 19. *Let P_p and P_q be two regular power series methods defined by (p_k) and (q_k) respectively. If*

$$p_k\sim q_k,$$

then

$$K_p(x)=K_q(x)\quad \forall x\in l_\infty.$$

Corollary 20. *Let P_p be a regular power series methods defined by (p_k). If $\lim\limits_k p_k=1$, then*

$$K_p(x)=K_{J_1}(x)$$

for every $x=(\xi_k)$ which satisfies the condition

$$\lim_k(p_k-1)\xi_k=0.$$

4. A Tauberian Core Theorem for Abel Means

Let W be the set that is defined for Abel method J_1 by (6). Let $\lambda = (\lambda_k)$ be a given sequence such that $1 \leq \lambda_0 < \lambda_1 < ... < \lambda_n \to \infty$ and define a sequence $w_\lambda = (t_n)$ by $t_n = 1 - \lambda_n^{-1}$ $\left(n \in \mathbb{N}^0\right)$. Clearly $0 \leq t_0 < t_1 < ... < t_n \to 1$. As the radius of convergence of Abel method J_1 is 1, the sequence $w_\lambda = (t_n)$ is element of W. The method $A_{w_\lambda} = (a_{nk})$, where $a_{nk} = \lambda_n \left(1 - \lambda_n^{-1}\right)^k$, is the discrete Abel method which is defined by w_λ. The main result of this section is Tauberian conditions for A_{w_λ} to held the equality

$$K^\circ \left(A_{w_\lambda} x\right) = K_{J_1}(x)$$

provided that $\lambda_n \sim \lambda_{n+1}$.

Lemma 21. *Let $P_p = J_1$. If $x = (\xi_k) \in \omega_{J_1}$, then*

$$\frac{p_x(t)}{p(t)} = (1-t) \sum_{k=1}^{\infty} \xi_k t^k = \sum_{k=1}^{\infty} h_k \left(\frac{t^k}{k} - \frac{t^{k+1}}{k+1} \right), \tag{16}$$

where

$$h_k = \sum_{j=0}^{k} (\xi_k - \xi_j).$$

Proof. For the proof of this lemma see [1]. □

Applying (16), an easy computation shows that if $x = (\xi_k) \in \omega_{J_1}$, then

$$\frac{p_x(t_2)}{p(t_2)} - \frac{p_x(t_1)}{p(t_1)} = \sum_{k=0}^{\infty} h_k \int_{t_1}^{t_2} t^{k-1} (1-t)\, dt \tag{17}$$

for every $0 < t_1 < t_2 < 1$.

Theorem 22. *Let $\lambda_n \sim \lambda_{n+1}$. If for a real sequence $x = (\xi_k)$ the sequence $\left(h_k k^{-1}\right)$ is bounded below, i.e., $h_k = O_L(k)$, then*

$$K^\circ \left(A_{w_\lambda} x\right) = K_{J_1}(x). \tag{18}$$

Proof. Let $\lambda = (\lambda_k)$ and let $x = (\xi_k)$ satisfy the above assumptions. As $h_k = O_L(k)$, there exists $H > 0$ such that $h_k > -Hk$ for all k. Now, if we put

$$f(t) = \frac{p_x(t)}{p(t)} \quad \forall t \in (0,1),$$

we get by (17), that for every $0 < t_1 < t_2 < 1$

$$f(t_2) - f(t_1) \geq -H \sum_{k=0}^{\infty} k \int_{t_1}^{t_2} t^{k-1} (1-t)\, dt$$

$$= -H \int_{t_1}^{t_2} (1-t) \sum_{k=0}^{\infty} k t^{k-1} dt = -H \int_{t_1}^{t_2} (1-t)^{-1} dt = -H \ln \frac{1-t_1}{1-t_2}. \quad (19)$$

Let $w = (t_k)$ be an arbitrarily fixed element of W and let k_0 be such that $t_k > 1 - \lambda_0^{-1}$ for every $k \geq k_0$. Then for every $k \geq k_0$ there exists $j_k \in \mathbb{N}^0$ such that

$$1 - \lambda_{j_k}^{-1} \leq t_k \leq 1 - \lambda_{j_k+1}^{-1}.$$

Clearly $w_{\lambda_1} = \left(1 - \lambda_{j_k}^{-1}\right) \in W$ and

$$K^{\circ}\left(A_{w_{\lambda_1}} x\right) \subset K^{\circ}\left(A_{w_\lambda} x\right).$$

From (19) it follows that

$$f(t_k) - f\left(1 - \lambda_{j_k}^{-1}\right) \geq H \ln \lambda_{j_k} (1 - t_k) \geq -H \ln \frac{\lambda_{j_k+1}}{\lambda_{j_k}}$$

and

$$f\left(1 - \lambda_{j_k+1}^{-1}\right) - f(t_k) \geq -H \ln \lambda_{j_k+1} (1 - t_k) \geq -H \ln \frac{\lambda_{j_k+1}}{\lambda_{j_k}}.$$

Hence

$$f\left(1 - \lambda_{j_k}^{-1}\right) - H \ln \frac{\lambda_{j_k+1}}{\lambda_{j_k}} \leq f(t_k) \leq f\left(1 - \lambda_{j_k+1}^{-1}\right) + H \ln \frac{\lambda_{j_k+1}}{\lambda_{j_k}}.$$

This gives

$$\liminf_{k\to\infty} f\left(1 - \lambda_{j_k}^{-1}\right) \leq \liminf_{k\to\infty} f(t_k)$$

$$\leq \limsup_{k\to\infty} f(t_k) \leq \limsup_{k\to\infty} f\left(1 - \lambda_{j_k+1}^{-1}\right).$$

In consequence,

$$\begin{aligned} \liminf_{k\to\infty} f\left(1 - \lambda_k^{-1}\right) &\leq \inf_{(t_k)\in W} \liminf_{k\to\infty} f(t_k) \\ \leq \sup_{(t_k)\in W} \limsup_{k\to\infty} f(t_k) &\leq \limsup_{k\to\infty} f\left(1 - \lambda_k^{-1}\right) \end{aligned}$$

and by (2), (7) and (3) we have that

$$K_{J_1}(x) \subset K^{\circ}\left(A_{w_\lambda}(x)\right).$$

Abel method J_1 is regular, therefore in account of Proposition 11

$$K_{J_1}(x) \supset K^{\circ}\left(A_{w_\lambda}(x)\right).$$

This gives (18). □

This theorem states that the boundedness below of the sequence $\left(h_k k^{-1}\right)$ is a Tauberian condition for A_{w_λ}. Let L be now the set of all real sequences $x = (\xi_k)$ for which the sequence $\left(h_k k^{-1}\right)$ is bounded below, i.e.,

$$L := \{x \in \omega(\mathbb{R}) \mid h_k = O_L(k)\}.$$

We shall now determine some subsets of L. It means that we find some stronger Tauberian conditions for A_{w_λ} to have (18).

Corollary 23. *If $\lambda_n \sim \lambda_{n+1}$, then $l_\infty(\mathbb{R}) \subset L$ and the equality* (18) *holds for every bounded real sequence x.*

Proof. Let $x = (\xi_k)$ and $\sup_k |\xi_k| = H < \infty$. Then

$$|h_k| = \left|\sum_{j=0}^{k} (\xi_k - \xi_j)\right| \leq \left|(k+1)\xi_k - \sum_{j=0}^{k} \xi_j\right| \leq 2(k+1)H,$$

consequently $h_k = O_L(k)$, and the statement of this corollary follows. □

Definition 24. (see [3], p. 169) A real sequence $x = (\xi_k)$ is said to be slowly decreasing if

$$\liminf_{1 \leq \frac{r}{n} \to 1, n \to \infty} (\xi_r - \xi_n) \geq 0$$

Let S be the set of all slowly decreasing sequences. Every increasing sequence is obviously slowly decreasing. A bounded sequence need not to be slowly decreasing. For instance the sequence $x = (\xi_k) = \left((-1)^k\right)$ has

$$\liminf_{r \to \infty} (\xi_r - \xi_{r-1}) = -2.$$

Corollary 25. *If $\lambda_n \sim \lambda_{n+1}$, then $S \subset L$ and the equality* (18) *holds for every real sequence $x = (\xi_k) \in S$.*

Proof. The proof of this corollary follows from the fact, that if $x = (\xi_k)$ is slowly decreasing, then $h_k = O_L(k)$ (see [1]). □

Let T be the set of all real $x = (\xi_k)$ sequences that satisfy the one-side local Tauberian condition, i.e.,

$$T = \left\{x = (\xi_k) : \xi_k - \xi_{k-1} = O_L\left(k^{-1}\right)\right\}.$$

The set T is the subset of S (see [3] p. 171). For this reason the following corollary holds.

Corollary 26. *If* $\lambda_n \sim \lambda_{n+1}$, *then* $T \subset S$ *and therefore* (18) *holds for every real sequence* $x = (\xi_k) \in T$.

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