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REDUCTION (2+1)-DIMENSIONAL KP EQUATION TO P-II TYPE EQUATION

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Abstract: In this paper, we will propose a connection between (2+1) dimensional Kadomtsev-Petviashvili (KP) equation and the ordinary differential equation Painlevé II type equation constructively. Painlevé II equation had been well researched. Through our connection, many properties of KP equation could be transformed by the well known properties of Painlevé II equation, we cite some as examples, such as infinite rational solutions, nonlinear superposition formula in part, and transcendent solutions expressed by Airy functions and Bessel functions.

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1. Introduction

Many phenomena in engineering and science could be described by nonlinear evolution equations, and nonlinear wave phenomena appears in many fields, such as fluid mechanics, plasma physics, biology, hydrodynamics, solid state

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physics and optical fibres, etc. These nonlinear phenomena are often related to nonlinear wave equations [2]. In order to better understand these nonlinear phenomena as well as further application of them in the practical life, it is important to seek their more exact solutions, if available. Exact solutions not only certify whether or not the obtained numerical solutions is better, but also are used to watch the sport rule of the wave by making the graphs of the exact solutions. Because of the complexity of nonlinear wave equations, there does not exist an uniform method to find all solutions of all nonlinear differential equations.

Painlevé equations were introduced in the research of pure mathematics, but now many properties of Painlevé equations have been discovered by many scientists in the past years, such as A. Hinkkanen et al [15] proves the solutions of the first and second Painlevé equations are meromorphic, in some special conditions, the solution of Painlevé equation could be expressed by classical special functions, algebraic functions and rational functions. V.I. Gromok do much work on the Bäcklund transformation of the solution hierarchies of the Painlevé equation which has special parameters [13]. We have knew that there is a consanguineou connection between nonlinear evolution equations and Painlevé equations. M.J. Ablowitz et al [4, 5] proved that no solution of an ODE, obtained by solving a linear integral equation of a certain kind, can have any movable critical points. and given an algorithm to test whether a given ODE satisfies necessary conditions to be of Painlevé equations. Many results in the Painlevé theory of partial differential equations and to test whether or not a given nonlinear evolution equation may be completely integrable are obtained by M.J. Ablowitz, A. Ramani, H. Segur [2, 4, 5, 6, 3] and P.A. Clarkson [9, 10]. Fokas [11, 12] gives a unify method to determine transcendent and elementary solutions of Painlevé equations. S. Kawomoto [17], Tajiri [20] and Boiti [8] presented that nonlinear Schrödinger equation, the KdV equation and cylindrical KdV equation are all reducible to Painlevé I equation.

In this paper, we will reduce (2+1) dimensional KP equation and (2+1) dimensional Boussinesq equation into the second Painlevé equation. The rest of this paper was organized as what follows: In Section 2, we will reduce (2+1) dimensional KP equation to the second Painlevé equation and two connections between them are given. In Section 3, we will reduce (2+1) dimensional KP equation to the second Painlevé equation and some connections between them are given. In Section 4, we will conclude this paper.

2. Connection Between (2+1) KP Equation and P-II Equation

(2+1) dimensional KP equation

$$(u_t + uu_x + u_{xxx})_x + \alpha^2 u_{yy} = 0 \tag{1}$$

is called KP-I when $\alpha^2 = 1$ and KP-II when $\alpha^2 = -1$ (subscripts x, y and t denote partial differentiations). It is well established now that the Kadomtsev-Petviashvili (KP) equation is the key ingredient in a number of remarkable nonlinear problems, both in physics and mathematics [2, 18]. Finding special solutions is an important aspect for understanding these problems. The solutions of KP equation have been extensively studied since they were first found. Various methods have been tried and many special solutions were given in [14]. Recently, [7, 1], the authors found a novel class of solutions of the KPI equation and discussed the properties of these solutions. It suggested that the class of reflectionless potentials be far richer than what was previous known.

The second Painlevé equation is in the form of

$$\frac{d^2\omega}{dz^2} = 2\omega^3 + z\omega + \alpha\,, \tag{2}$$

where z is the independent variable.

In the following we will give a constructive method to obtain the connection between (2+1) dimensional KP equation and the second Painlevé equation.

We assume the solutions to KP equation in the form of

$$u(x,y,t) = a(x,y,t)W(\xi(x,y,t)) + b(x,y,t) + c(x,y,t)W^2(\xi(x,y,t)) + d(x,y,t)W'(\xi(x,y,t))\,, \tag{3}$$

where $W(\xi)$ satisfies the second Painlevé equation,

$$W''(\xi(x,y,t)) = 2W^3(\xi(x,y,t)) + \xi(x,y,t)W(\xi(x,y,t)) + \alpha\,. \tag{4}$$

$a(x,y,t), b(x,y,t), c(x,y,t), d(x,y,t)$ and $\xi(x,y,t)$ are all arbitrary functions which will be determined later, α is an arbitrary constant.

Substituting equation (3) in to (2+1) dimensional KP equation and eliminating all the terms that the order of $W(\xi)$ over one by equation (4), we obtain a equation that can be regarded as a polynomial of $W(\xi(x,y,t))$ and $W'(\xi(x,y,t))$. Setting all coefficients of $W(\xi(x,y,t))$ and $W'(\xi(x,y,t))$ yields a system of differential equation (for simple we call it connection system) of $a(x,y,t)$, $b(x,y,t)c(x,y,t)$, $d(x,y,t)$ and $\xi(x,y,t)$ as follows:

$$18d\xi_x^2(6\xi_x^2 + c) = 0, \tag{5.1}$$

$$6d^2\xi_x^2 + 72c\xi_x^4 + 6c^2\xi_x^2 = 0, \tag{5.2}$$

$$48d_x\xi_x^3 + 4cd_x\xi_x + 6ac\xi_x^2 + 12a\xi_x^4 + 72d\xi_x^2\xi_{xx} + 2cd\xi_{xx} + 4dc_x\xi_x = 0, \tag{5.3}$$

$$12ad\xi_x^2 + 2c^2\xi_{xx} + 2d^2\xi_{xx} + 96c_x\xi_x^3 + 8dd_x\xi_x + 8cc_x\xi_x + 144c\xi_x^2\xi_{xx} = 0, \tag{5.4}$$

$$c_x^2 + 4\beta^2\xi_y^2 + 4c\xi_x\xi_t + 4ad_x\xi_x + cc_{xx} + 4bc\xi_x^2 + 2a^2\xi_x^2 + 2c^2\xi\xi_x^2 + 40c\xi\xi_x^4 + 2ad\xi_{xx}4d^2\xi\xi_x^2 + 16c\xi_x\xi_{xxx} + 4a_xd\xi_x + 12c\xi_{xx}^2 + 24c_{xx}\xi_x^2 + 48c_x\xi_x\xi_{xx} = 0, \tag{5.5}$$

$$\vdots$$

From the equation (5. 1), we will easy to get

$$c(x,y,t) = -6\,\xi(x,y,t)_x^2, \tag{6}$$

then equation (5. 2) is eliminated into

$$d(x,y,t) = \pm 6\,\xi_x^2. \tag{7}$$

Firstly, we consider $d(x,y,t) = 6\,\xi_x^2$, then equation (5. 3) and equation (5. 4) are reduced in to

$$-720\xi_x^4\xi_{xx} + 72a\xi_x^4 = 0, \tag{8}$$

$$360\xi_x^4\xi_{xx} - 24a\xi_x^4 = 0. \tag{9}$$

Obviously,

$$a(x,y,t) = 0, \quad \xi_{xx} = 0. \tag{10}$$

We could replace $\xi(x,y,t)$ in equations (5) in the form of

$$\xi(x,y,t) = F_1(y,t)\,x + F_2(y,t). \tag{11}$$

Substituting equations (6), (7), (10) and equation (11) in to connection system equations (5), we get the following differential equation system in terms of $F_1(y,t), F_2(y,t)$ and $b(x,y,t)$

$$F_1^5x + F_2F_1^4 + bF_1^2 + F_1(F_{2t} + F_{1t}x) + x^2F_{1yy}\beta^2 + (2F_{1y}+1)F_{2y}\beta^2 = 0, \tag{12}$$

$$\vdots$$

For simple, we choose $F_{1yy} = 0$, then $b(x,y,t)$ should be in the form of

$$b(x,y,t) = F_4(y,t)\,x + F_5(y,t). \tag{13}$$

Substituting equation (13) into equations (12), we get some differential equations in terms of $F_1(y,t)$, $F_2(y,t)$, $F_3(y,t)$ and $F_4(y,t)$, which would also be regarded as some polynomials in terms of x with coefficients of $F_1(y,t), F_2(y,t), F_3(y,t)$

and $F_4(y,t)$.

Collecting all the coefficients of x and setting to zero yields a partial differential equation system in terms of $F_1(y,t), F_2(y,t), F_3(y,t)$ and $F_4(y,t)$. And this equation system admits the following solution

$$F_1(y,t) = F_6(t), \tag{14}$$

$$F_2(y,t) = \left(-\frac{F_6'(t)}{2\beta^2} - \frac{{F_6}^4(t)}{2\beta^2}\right) y^2 + F_7(t)\, y + F_8(t), \tag{15}$$

$$F_4(y,t) = \frac{-F_6'(t) - {F_6}^4(t)}{F_6(t)}, \tag{16}$$

$$\begin{aligned} F_5(y,t) = \frac{1}{2\beta^2 F_6^2(t)} \Big[& F_6(t)\, y^2 F_6''(t) + \left(y^2 {F_6}^4(t) + 4\, y F_7(t)\beta^2\right) F_6'(t) \\ & - 2\, y^2 {F_6'}^2(t) - 2\, F_6(t)\, F_7'(t) y\beta^2 - 2\, F_6(t)\, F_8'(t)\, \beta^2 \\ & - y^2 {F_6}^8(t) + 2\,\beta^2 \left(F_7(t)\, y - F_8(t)\right) {F_6}^4(t) - 2\, {F_7}^2(t)\, \beta^4 \Big], \end{aligned} \tag{17}$$

where $F_6(t), F_7(t)$ and $F_8(t)$ are all arbitrary functions of t. So, $c(x,y,t)$, $d(x,y,t)$ and $\xi(x,y,t)$ could in the form of

$$c(x,y,t) = 6\, F_6^2(t), \tag{18}$$

$$d(x,y,t) = 6\, F_6^2(t), \tag{19}$$

$$\xi(x,y,t) = F_6(t)x + \left(-\frac{F_6'(t)}{2\beta^2} - \frac{{F_6}^4(t)}{2\beta^2}\right) y^2 + F_7(t)\, y + F_8(t). \tag{20}$$

and $b(x,y,t)$ is in the form of

$$b(x,y,t) = \frac{-F_6'(t) - {F_6}^4(t)}{F_6(t)} x + F_5(y,t), \tag{21}$$

where $F_5(y,t)$ is given in equation (17), and $F_6(t), F_7(t)$ and $F_8(t)$ are all arbitrary functions of t.

For the second cases of $d(x,y,t) = -6\, F_6^2(t)$, after a similar procedure, we also get equations (14)-(17).

From the above discussion and computation, we conclude that:

Proposition 1. *If $W(\xi)$ satisfies Painleé II equation, then*

$$u(x,y,t) = 6\, F_6^2(t)[W'(\xi) \pm W(\xi)^2] + \frac{-F_6'(t) - {F_6}^4(t)}{F_6(t)} x + F_5(y,t) \tag{22}$$

satisfies (2+1) dimensional KP equation, where W' denotes the derivation of the second Painlevé equation, $\xi(x,y,t)$ and $F_5(x,y,t)$ are given in equation (20) and equation (17) respectively, and $F_6(t), F_7(t)$ and $F_8(t)$ are all arbitrary *functions in terms of* t.

3. Some Results from the Connection

In this section, we will reduce some interesting properties of (2+1) dimensional KP equation along with the connection presented in Section 2.

3.1. Rational Solutions

The nonzero rational solutions to the second Painlevé equation are written as

$$W(\xi) = \frac{P(\xi)}{Q(\xi)}\,, \tag{23}$$

where $P(\xi)$ and $Q(\xi)$ are polynomials in terms of ξ and the degree of $P(\xi)$ and $Q(\xi)$ are $n-1$ and n respectively.

We might as well suppose

$$P_l(\xi) = \prod_{j=1}^{l}(\xi - \xi_j), Q_{n-l}(\xi) = \prod_{j=l+1}^{n}(\xi - \xi_j). \tag{24}$$

We assume that if $l = 0$, then $p_0 = 1$; if $n - l = 0$, then $Q_0 = 0$ and

$$W(\xi) = \sum_{j=1}^{l}(\xi - \xi_j)^{-1} - \sum_{j=l+1}^{n}(\xi - \xi_j)^{-1} = \frac{P_l'(\xi)}{P_l(\xi)} - \frac{Q_{n-l}'(\xi)}{Q_{n-l}(\xi)} \tag{25}$$

A.P. Vorob'ev [21] has researched the equivalence equation system

$$\begin{cases} P_l''Q_{n-l} - 2P_l'Q_{n-l}' + P_lQ_{n-l}'' = 0, \\ P_l'''Q_{n-l} - 3P_l''Q_{n-l}' + 3P_l'Q_{n-l}'' - P_lQ_{n-l}''' = \xi P_l'Q_{n-l}\,, \\ -\xi P_lQ_{n-l}' + \alpha P_lQ_{n-l}. \end{cases} \tag{26}$$

If P_α and Q_α are different nonzero solutions to equation (26) in the form of polynomial for the parameter α, then

$$\begin{cases} P_{\alpha+1} = Q_\alpha, \\ Q_{\alpha+1} = P_\alpha^{-1}(\xi Q_\alpha^2 + 4Q_\alpha'^2 - 4Q_\alpha Q_\alpha''). \end{cases} \tag{27}$$

gives a polynomial solution to equation (26) for the parameter $\alpha + 1$

In fact, when $\alpha = 0$, then $W(\xi) \equiv 0$; when $\alpha = \pm 1$, then $W(\xi) = \pm\frac{1}{\xi}$; when $\alpha = \pm 2$, then $W(\xi) = \mp\frac{2\xi^3-4}{\xi^4+4\xi}$. For more information see [21].

According to the rational solution to Painlevé II equation, we could get many rational solutions to (2+1) dimensional KP equation. Here we only illustrate two rational solutions to KP equation for example

When $\alpha = \pm 1$, we have a solution to (2+1) dimensional KP equation

$$u(x,y,t) = \frac{-F_6'(t) - {F_6}^4(t)}{F_6(t)} x + F_5(y,t)\,, \tag{28}$$

and

$$u(x,y,t) = \pm 6\, F_6^2(t) \frac{2}{\xi^2} + \frac{-F_6'(t) - {F_6}^4(t)}{F_6(t)} x + F_5(y,t)\,, \tag{29}$$

where $\xi(x,y,t)$ and $F_5(x,y,t)$ are given in equation (20) and equation (17) respectively, and $F_6(t), F_7(t)$ and $F_8(t)$ are all arbitrary functions.

When $\alpha = \pm 2$, we get another solution to (2+1) dimensional KP equation

$$u(x,y,t) = 6\, F_6^2(t) \frac{6\xi(\xi^3 - 8)}{(\xi^3 + 4)^2} + \frac{-F_6'(t) - {F_6}^4(t)}{F_6(t)} x + F_5(y,t) \tag{30}$$

and

$$u(x,y,t) = \pm 6\, F_6^2(t) \frac{2}{\xi^2} + \frac{-F_6'(t) - {F_6}^4(t)}{F_6(t)} x + F_5(y,t)\,, \tag{31}$$

where $\xi(x,y,t)$ and $F_5(x,y,t)$ are given in equation (20) and equation (17) respectively, and $F_6(t), F_7(t)$ and $F_8(t)$ are all arbitrary functions.

3.2. Nonlinear Superposition Formula

Yablonskii [22] introduced an equivalence equation system and found a Bäcklund transformation of Painlevé II equation

$$\begin{aligned} W(\xi, \alpha\epsilon - 1) &= -\epsilon W(\xi, \alpha) \\ &+ \frac{(2\alpha\epsilon - 1) exp\left[2\epsilon \int_{\xi_0}^{\xi} W(\xi,\alpha) d\xi\right]}{2W_0'\epsilon - 2W_0^2 - \xi_0 + (2\alpha\epsilon - 1)\int_{\xi_0}^{\xi} exp\left[2\epsilon \int_{\xi_0}^{\xi} W(\xi,\alpha) d\xi\right] d\xi}\,. \end{aligned} \tag{32}$$

There also exists a nonlinear superposition formula of Painlevé II equation

$$W_{\alpha+1} = -\frac{(W_{\alpha-1} + W_\alpha)(4W_\alpha^3 + 2\xi W_\alpha + 2\alpha + 1) + (2\alpha - 1)W_\alpha}{2(W_{\alpha-1} + W_\alpha)(2W_\alpha^2 + \xi) + 2\alpha - 1}\,. \tag{33}$$

Combined with the nonlinear superposition formula and the connection between KP equation and the second Painlevé equation, we could give a nonlinear superposition formula to (2+1) dimensional KP equation, as a consequently result, infinite solutions to (2+1) dimensional KP equation expressed by the second Painlevé equation could be obtained.

We shall note that this nonlinear superposition formula is not fixed to any solutions of KP equation, since it is transformed from the Painlevé II equation. It is only applicable to the solutions that expressed by Painlevé II equation. In other words, this superposition formula gives a string of special solutions to (2+1) dimensional KP equation.

3.3. Transcendent Solutions

When $\alpha = \frac{\epsilon}{2}$, the solution of variant Riccati equation

$$W' = \epsilon W^2 + \frac{\epsilon\xi}{2} \tag{34}$$

gives a one-parameter solution hierarchy to Painlevé II equation equation (2), where $\epsilon^2 = 1$. Taking a transformation

$$W = -\epsilon u' u^{-1} \tag{35}$$

for equation (34), we get a linear equation, i.e. Airy equation

$$-u(2u'' + \xi u) = 0\,. \tag{36}$$

We can write the solution of Airy function as

$$u = MAi(-\frac{\xi}{\sqrt[3]{2}}) + NBi(-\frac{\xi}{\sqrt[3]{2}})\,, \tag{37}$$

where Ai and Bi are a basis solution to Airy equation, M and N are arbitrary constants. So,

$$W = -\frac{\epsilon}{\sqrt[3]{2}} \frac{MAi'(-\frac{\xi}{\sqrt[3]{2}}) + NBi'(-\frac{\xi}{\sqrt[3]{2}})}{MAi(-\frac{\xi}{\sqrt[3]{2}}) + NBi(-\frac{\xi}{\sqrt[3]{2}})} \tag{38}$$

is a solution to Painlevé II equation.

In other words, we get a solution to KP equation expressed by Airy functions $Ai(\xi)$ and $Bi(\xi)$ along with equation (22) and equation (38).

There is a connection between Ariy function and Bessel functions

$$Ai(z) = \begin{cases} \frac{1}{3}\sqrt{|z|}\,[J_{1/3}(\xi) + J_{-1/3}(\xi)], & z \le 0\,, \\ \frac{1}{3}\sqrt{z}\,[I_{-1/3}(\xi) - I_{1/3}(\xi)], & z \ge 0\,, \end{cases}$$

and

$$Bi(z) = \begin{cases} \sqrt{\frac{|z|}{3}}\,[-J_{1/3}(\xi) + J_{-1/3}(\xi)], & z \ge 0\,, \\ \sqrt{\frac{z}{3}}\,[I_{-1/3}(\xi) + I_{1/3}(\xi)], & z \le 0\,, \end{cases}$$

where ξ equals $\frac{2}{3}|z|^{3/2}$.

Note that the transcendent solutions to Painlevé II equation could also be obtained an infinite string by the nonlinear superposition formula.

4. Conclusion

In this paper, we successfully reduce (2+1) dimensional KP equation into Painlevé II equation. The obtained connections between (2+1) dimensional KP equation and Painlevé II equation consist three arbitrary functions. We can select convenience values of arbitrary functions according to conditions. In our knowledge, this connection is first presented in this paper. From the connection we could get many properties of the solutions to the (2+1) dimensional KP equation, such as rational solutions to KP equation, nonlinear superposition formula, transcendent solutions including Airy function and Bessel function. There are still many other properties to the second Painlevé equation.

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