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CONTINUOUS DEPENDENCE OF SOLUTIONS TO NONLINEAR ELLIPTIC-PARABOLIC-HYPERBOLIC PROBLEM IN ONE DIMENSION

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Abstract: We consider a nonlinear elliptic-parabolic-hyperbolic equation of the form: $b(u)_t - a(u, \varphi(u)_x)_x = f$. We show the continuous dependence of the "mild solution" of the associated Cauchy problem with respect to the data a, f and v_0 without Alt and Luckhaus structure condition.

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1. Introduction

We consider the Cauchy problem

$$\begin{cases} b(u)_t - a(u, \varphi(u)_x)_x = f \text{ in } \mathbb{Q} = (0,T) \times \mathbb{R}\,, \\ b(u)(0,.) = v_0 \text{ on } \mathbb{R}\,, \end{cases} \tag{EP}$$

where $f \in L^1(\mathbb{Q}), v_0 \in L^1(\mathbb{R}), a : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and $a(k,.)$ nondecreasing, φ and $b : \mathbb{R} \to \mathbb{R}$ are continuous, nondecreasing and b surjective.

Whenever u takes a value such that $b(u)$ is constant, (EP) degenerates into an elliptic problem of the form:

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$$\begin{cases} -a(u,\varphi(u)_x)_x = f \text{ in } \mathbb{Q} = (0,T)\times\mathbb{R}\,, \\ b(u)(0,.) = v_0 \text{ on } \mathbb{R}\,. \end{cases} \tag{1.1}$$

Take $b = id$, on each part where u takes a value such that $\varphi(u)$ is constant, (EP) degenerates to a scalar conservation law of the form:

$$\begin{cases} u_t - a(u,0)_x = f \text{ in } \mathbb{Q} = (0,T)\times\mathbb{R}\,, \\ u(0,.) = u_0 \text{ on } \mathbb{R}\,. \end{cases} \tag{1.2}$$

Elliptic-parabolic-hyperbolic problems arise as model in many applications, for example as a model of flow through porous media (cf. [4], [10]), and also in electromagnetic field theory (cf. [12]).

According to what precede, it is clear that we include in (EP), some first order hyperbolic problems, for which (even under assumptions of regularity on data) there is no hope of getting classical global solutions. Such models are of interest and are not so easy to treat due to the possible loss of regularity (due to possible degeneracy), combined with the dependence on nonlinear coefficients. To treat this kind of problem, the structure condition is some time needed. Recall that the structure condition is equivalent to the condition

$$a(r,\xi) = \widetilde{a}(b(r),\xi) \text{ for all } r,\xi \in \mathbb{R},$$

for some nondecreasing with respect to the second variable and continuous function $\widetilde{a} : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$. This structure condition seemed to be essential in order to get existence of weak solutions (cf. [1], [7], [8], [14], [16]), and has been assumed to be the best according to our knowledge, in all articles dedicated to general elliptic-parabolic-hyperbolic problems. Roughly speaking, the reason is that, in the standard approximation process (semigroup approach, parabolic regularization, Galerkin method, etc.) used in order to get an existence result for the degenerate elliptic-parabolic-hyperbolic problem, it is possible to prove a strong relative compactness of the approximate solutions $b(u)$ in t and x, but only some weak relative compactness of u. If the structure condition is satisfied, this allows to pass to the limit in the nonlinearity $a(u,\varphi(u)_x) = \widetilde{a}(b(u),\varphi(u)_x)$ by using a pseudomonotonicity argument (cf. [11]).

Note also that the structure condition is, in particular, satisfied if b is strictly increasing or a does not depend on r, i.e. $a(r,\xi) = a(\xi)$.

Using nonlinear semigroup theory in L^1, we deduce the results for the "evolution problem" (EP) of the properties of the " stationary problem":

$$b(u) - a(u,\varphi(u)_x)_x = f \text{ on } \mathbb{R}. \tag{SP}$$

The study of (SP) has been presented in a first paper [15]; we will quickly recall the results in Section 2. In Section 3, we develop the concept of mild solution of (EP). We study in Section 4 the continuous dependence of entropy solution of (SP) with respect to a and f and we deduce in Section 5 of this paper by using the result of Section 4 the continuous dependence of mild solution of (EP) with respect to a, f and v_0.

2. The Stationary Problem

We recall in this section some results obtained in the study of the stationary problem (SP) (cf. [15]). Let us note to start that

$$\operatorname{sign}_0(r) = \begin{cases} 1 & \text{if } r > 0\,, \\ 0 & \text{if } r = 0\,, \\ -1 & \text{if } r < 0\,, \end{cases} \quad \operatorname{sign}_0^+(r) = \begin{cases} 1 & \text{if } r > 0\,, \\ 0 & \text{if } r \leq 0\,, \end{cases}$$

$$\operatorname{sign}^+(r) = \begin{cases} 1 & \text{if } r > 0\,, \\ [0,1] & \text{if } r = 0\,, \\ 0 & \text{if } r < 0\,, \end{cases} \quad \operatorname{sign}(r) = \begin{cases} 1 & \text{if } r > 0\,, \\ [-1,1] & \text{if } r = 0\,, \\ -1 & \text{if } r < 0. \end{cases}$$

We denote by

$$H(k) = a(k,0) \text{ for } k \in \mathbb{R} \text{ and } h = a(u, \varphi(u)_x).$$

Our main assumption is the coerciveness of a with respect to ξ, for k bounded; more precisely:

$$\lim_{|\xi|\to\infty} \inf_{|k|<R} \mid a(k,\xi) \mid = +\infty \text{ for all } \; R > 0. \tag{H_1}$$

Remark 2.1. Assumption (H_1) is essential in order to get *a priori estimates* for existence of solutions result in [15] and for the continuous dependence results that we prove in this work. This essential argument is specific to the one dimension and cannot be used in several dimension since it is not equivalent to the coerciveness condition in several dimension.

Let us now give the definitions of weak and entropy solutions of (SP).

Definition 2.2. Let $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$; a weak solution of (SP) is a measurable function $u \in L^\infty(\mathbb{R})$ such that $\varphi(u) \in W^{1,\infty}(\mathbb{R})$ and

$$b(u) - a(u, \varphi(u)_x)_x = f \text{ in } D'(\mathbb{R}).$$

Definition 2.3. Let $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$; an entropy solution of (SP) is a weak solution u satisfying:

(i) there exists $h \in C(\mathbb{R})$ such that $h = a(u, \varphi(u)_x)$ a.e on $\mathbb{R}$;

(ii) for any $\xi \in D^+(\mathbb{R}), k \in \mathbb{R}$, the following entropy inequalities

$$\int_{\mathbb{R}} \text{sign}_0(b(u) - b(k)) \{\xi_x(H(k) - h) + \xi(f - b(u))\} \, dx \geq 0, \tag{2.1}$$

$$\int_{\mathbb{R}} \text{sign}_0(b(k) - b(u)) \{\xi_x(H(k) - h) + \xi(f - b(u))\} \, dx \leq 0. \tag{2.2}$$

In [15] we have proved under general assumptions on the data that (SP) admits a unique entropy solution which permits us to define the $L^1(\mathbb{R})$ operator A_b associated with the evolution problem (EP) by $A_b b(u) = -a(u, \varphi(u)_x)_x$ satisfying:

$$\begin{cases} v \in A_b \Longleftrightarrow b(u) \in L^1(\mathbb{R}), v \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \\ \text{and u is entropy solution of (SP) with } f = v + b(u). \end{cases}$$

We have shown for this operator, the following result that we present as lemma.

Lemma 2.4. *Under assumptions* (H_1), *the operator* A_b *defined above satisfies the following:*

1. A_b *is T-accretive in* $L^1(\mathbb{R})$, *i.e.*

$$\left\|(x - \widetilde{x})^+\right\|_1 \leq \left\|(x - \widetilde{x} + \lambda(A_b x - A_b \widetilde{x}))^+\right\|_1,$$

$\forall \lambda \geq 0$ *and* $x, \widetilde{x} \in D(A_b)$.

2. For any $\lambda > 0$, *the range* $R(I + \lambda A_b)$ *of* $I + \lambda A_b$ *is dense in* $L^1(\mathbb{R})$.

3. $D(A_b)$ *is dense in* $L^1(\mathbb{R})$.

3. Mild Solution of Evolution Problem

As usual in the theory of evolution equations governed by accretive operators, we consider an approximation of (EP) by an implicit time discretization (also used by Alt and Luckhaus [1])

$$\begin{cases} \dfrac{b(u_i) - b(u_{i-1})}{t_i - t_{i-1}} = a(u_i, \varphi(u_i)_x)_x + f_i, \\ u_i \in L^\infty(\mathbb{R}), b(u_i) \in L^1(\mathbb{R}), \varphi(u_i) \in W^{1,\infty}(\mathbb{R}), \\ \text{for } i = 1, ..., n, \end{cases} \tag{PDE}$$

where

$$\begin{cases} t_0 = 0 < t_1 < ... < t_n \leq T, t_i - t_{i-1} \leq \epsilon, T - t_n \leq \epsilon, \\ f_1, ..., f_n \in L^\infty(\mathbb{R}), \displaystyle\sum_i \int_{t_{i-1}}^{t_i} \|f(t) - f_i\| \, dt \leq \epsilon, \\ u_0 \in L^\infty(\mathbb{R}), \|v_0 - b(u_0)\| \leq \epsilon, \text{ for } \epsilon > 0. \end{cases} \tag{DE}$$

This method is actually the method of nonlinear semigroup theory. Naturally, we are led to give the following definition according to [7] (see also [3], [9]).

Definition 3.1. A mild solution of (EP) is a mesurable function $u : \mathbb{Q} \rightarrow \mathbb{R}$ satisfying $v = b(u) \in C([0,T]; L^1(\mathbb{R})), v(0) = v_0$ and, for any $\epsilon > 0$, there exist $(t_0, ..., t_n, f_1, ..., f_n, u_0, ..., u_n)$ satisfying (DE) and (PDE), such that $\|v(t) - b(u_i)\|_1 \leq \epsilon \ \ \forall t \in]t_{i-1}, t_i], \ i = 1, ..., n$.

Using nonlinear semigroup theory (cf. [3, 5]) by interpreting problem (EP) in the form of evolution equation in $L^1(\mathbb{R})$:

$$\begin{cases} \dfrac{dv}{dt} + A_b v \ni f \quad \text{in} \quad [0,T] \, , \\ v(0) = v_0 \, , \end{cases} \tag{CP}$$

with $v = b(u)$; one deduces immediately from the preceding lemma, the following theorem:

Theorem 3.2. *Under assumption (H_1), for any $v_0 \in L^1(\mathbb{R})$, $f \in L^1(\mathbb{Q})$, there exists a unique mild solution u of (EP) which is characterized by the property:*

$$\begin{cases} b(u(0,.)) = v_0 \text{ and } \forall \underline{u} \in D(A), \xi \in D([0,T]), \xi \geq 0, \\ \exists \alpha \in L^\infty(\mathbb{Q}), \alpha \in \operatorname{sign}(b(u) - b(\underline{u})) \text{ a.e. } (t,x) \in Q \text{ such that} \\ \displaystyle\iint_{\mathbb{Q}} \alpha \left\{ (b(u) - b(\underline{u}))\xi_t + (f - A_b b(\underline{u})\xi) \right\} dxdt \geq 0 \, . \end{cases} \tag{3.1}$$

Moreover, if u_1, u_2 are mild solutions with respect to $(v_{0,1}; f_1)$, $(v_{0,2}; f_2)$, then

$$\begin{aligned} & \max_{t \in [0,T]} \int_{\mathbb{R}} [b(u_1(t)) - b(u_2(t))]^+ dx \\ & \leq \int_{\mathbb{R}} (v_{0,1} - v_{0,2})^+ dx + \iint_{\mathbb{Q}} (f_1 - f_2)^+ dxdt. \end{aligned} \tag{3.2}$$

In particular,

$$v_{0,1} \leq v_{0,2} \text{ a.e. on } \mathbb{R}, \quad f_1 \leq f_2 \text{ on } \mathbb{Q} \Longrightarrow b(u_1) \leq b(u_2) \text{ a.e on } \mathbb{Q}. \tag{3.3}$$

If v_0 and f verify

$$v_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), f \in L^1(Q) \text{ and } \int_0^T \|f(t,.)\|_\infty < \infty, \tag{3.4}$$

then, the mild solution is bounded; more precisely, one has the following estimation

Proposition 3.3. *Let us suppose (3.4) is satisfied and u being the mild* *solution of* (EP)*, then* $u \in L^\infty(Q)$ *and*

$$\|b(u)\|_{L^\infty(Q)} \leq \|v_0\|_{L^\infty(\mathbb{R})} + \int_0^T \|f(t,.)\|_{L^\infty(\mathbb{R})}\, dt. \tag{3.5}$$

Proof of Proposition 3.3. For the proof of this proposition, we interprete the problem (EP) in the form of the evolution equation (CP) in $L^1(\mathbb{R})$ with $v = b(u)$; and, in this case, the result follows immediately by using Propositions 1-4 of [6]. □

4. Continuous Dependence of Entropy Solution of the Stationary Problem

We show in this section a result of continuous dependence of entropy solution of (SP) with respect to the data a and f; this result will allow us in the next section to deduce by using nonlinear semigroup theory (cf. [3]) the continuous dependence of the mild solution of (EP) with respect to data a, f and v_0.

We make the following additional assumption:

$$b + \varphi \text{ is one-to-one.} \tag{$H2$}$$

We have the following result.

Theorem 4.1. *Let* $\{f_n, a_n\}_n$ *be a sequence in* $\big(BV(\mathbb{R}) \cap L^1(\mathbb{R})\big) \times C(\mathbb{R}\times\mathbb{R})$ *and* $f \in BV(\mathbb{R}) \cap L^1(\mathbb{R})$*. Assume* (H_1) *and* (H_2)*. If*

$$f_n \longrightarrow f \text{ in } L^1(\mathbb{R}), \tag{4.1}$$

$$a_n(\eta,.) \text{ is nondecreasing such that } a_n \longrightarrow a \text{ in } C(\mathbb{R}\times\mathbb{R}), \tag{4.2}$$

and

$$\lim_{|\xi|\to\infty} \inf_{|\eta|<R,\ n\in\mathbb{N}} |\, a_n(\eta,\xi)\, | = +\infty \text{ for all } R > 0, \tag{4.3}$$

then the entropy solution u_n *of* (SP) *with respect to* (a_n, f_n) *converges in* $L^1(\mathbb{R})$ *to the entropy solution* u *of* (SP) *with respect to* (a, f)*; consequently, for any* $\lambda > 0$*,* $f \in BV(\mathbb{R}) \cap L^1(\mathbb{R})$

$$(I + \lambda A_b^n)^{-1} f_n \longrightarrow (I + \lambda A_b)^{-1} f,$$

where A_b^n *is the* L^1 *operator associated with respect to* a_n *and* f_n*.*

Proof. We proceed into two steps.

Step 1. In this step, we prove *a priori estimates* results for entropy solutions to the stationary problem (SP).

Lemma 4.2. *Let* $f \in W^{1,1}(\mathbb{R}) \cap L^\infty(\mathbb{R})$*, then we have:*

(i) $\|b(u)\|_r \le \|f\|_r \ \ \forall 1 \le r \le +\infty$;

(ii) $\|\varphi(u)\|_{W^{1,\infty}(\mathbb{R})} \le C$;

(iii) $\operatorname{Var}(b(u)) \le \operatorname{Var} f$*; where* C *is a positive constant and* $\operatorname{Var} f$ *the total variation of the function* f *with bounded variation.*

Proof of Lemma 4.2. One can assumes by approximation that the data a, b, φ are regular.

(i) Let $p : \mathbb{R} \longrightarrow \mathbb{R}$ be Lipschitz with compact support such that $p^{'} \ge 0$, $0 \le p \le 1$ et $0 \le |p'(r)r| \le 1$.

Multiplying (SP) by $p(b(u))$, we have

$$p(b(u))b(u) - p(b(u))a(u, \varphi(u)_x)_x = p(b(u))f. \tag{4.4}$$

By integrating (4.4) on $\mathbb{R}$, we get:

$$\int_{\mathbb{R}} p(b(u))b(u)dx - \int_{\mathbb{R}} p(b(u))a(u, \varphi(u)_x)_x dx = \int_{\mathbb{R}} p(b(u))f dx. \tag{4.5}$$

After integrating by parts the second term of (4.5), we obtain:

$$\int_{\mathbb{R}} p(b(u))b(u)dx + \int_{\mathbb{R}} a(u, \varphi(u)_x)p(b(u))_x dx = \int_{\mathbb{R}} p(b(u))f dx. \tag{4.6}$$

Consider the term

$$A_1 = \int_{\mathbb{R}} a(u, \varphi(u)_x)p(b(u))_x dx = \int_{\mathbb{R}} \left[a(u, \varphi(u)_x) - a(u, 0)\right] p(b(u))_x dx + \int_{\mathbb{R}} a(u, 0)p(b(u))_x dx.$$

By regularization, we can assume that $b'(u) \ge 0$ and $\varphi'(u) \ge 0$ and as a is nondecreasing with respect to the second variable, we have

$$\int_{\mathbb{R}} \left[a(u, \varphi(u)_x) - a(u, 0)\right] p'(b(u))b'(u)u_x dx \ge 0$$

and we get

$$A_1 \ge \int_{\mathbb{R}} p'(b(u))b'(u)u_x a(u, 0)dx.$$

Let us examine then the term

$$A_2 = \int_{\mathbb{R}} p'(b(u))b'(u)u_x H(u)dx\,,$$

$$A_2 = \int_{\mathbb{R}} q(u)_x dx \ \text{ with } q(r) = \int_0^r p'(b(s))b'(s)H(s)ds\,.$$

Consequently

$$A_2 = 0.$$

We deduce then

$$A_1 \geq 0.$$

Thus, we obtain from (4.6)

$$\int_{\mathbb{R}} p(b(u))b(u)dx \leq \int_{\mathbb{R}} p(b(u))f dx.$$

By approximation of p, one can take $p(u) = |u|^{r-2}\, u$ with $1 \leq r < +\infty$. We get from the preceding inequality

$$\int_{\mathbb{R}} |b(u)|^r \, dx \leq \int_{\mathbb{R}} |f| \, |b(u)|^{r-1} \, dx. \tag{4.7}$$

Inequality (4.7) implies that

$$\int_{\mathbb{R}} |b(u)|^r \, dx \leq \left(\int_{\mathbb{R}} |b(u)|^r) dx \right)^{\frac{r-1}{r}} \left(\int_{\mathbb{R}} |f|^r \, dx \right)^{\frac{1}{r}}.$$

From where it follows

$$\|b(u)\|_r \leq \|f\|_r \,, \text{ for all } 1 \leq r < +\infty.$$

As $f \in L^\infty(\mathbb{R})$ then, letting $r \to \infty$ in the preceding inequality, we get also

$$\|b(u)\|_\infty \leq \|f\|_\infty \,.$$

Thus,

$$\|b(u)\|_r \leq \|f\|_r \,, \text{ for all } 1 \leq r \leq +\infty.$$

(ii) Since b is onto, we deduce from (i) that u is uniformly bounded in $L^\infty(\mathbb{R})$ and then that $\varphi(u)$ is uniformly bounded in $L^\infty(\mathbb{R})$. Since $h_x = b(u) - f$ then $\|h_x\|_{L^1(\mathbb{R})} \leq 2 \|f\|_{L^1(\mathbb{R})}$ and then $h \in AC(\mathbb{R})$ uniformly. Therfore $h \in L^\infty(\mathbb{R})$ since $\lim\limits_{x \longrightarrow \pm\infty} h(x) = h_\pm$ exist and is finite. Consequently, using (H_1), we get that $\varphi(u)_x$ is uniformly bounded in $L^\infty(\mathbb{R})$.

(iii) By regularization, we can differentiate the equation (SP) to obtain

$$b(u)_x - a(u, \varphi(u)_x)_{xx} = f_x. \tag{4.8}$$

Next, we multiply (4.8) by $p(\varphi(u)_x)$ to obtain

$$p(\varphi(u)_x) b(u)_x - a(u, \varphi(u)_x)_{xx} p(\varphi(u)_x) = f_x p(\varphi(u)_x). \tag{4.9}$$

By integrating (4.9) over $\mathbb{R}$, we get

$$\int_{\mathbb{R}} p(\varphi(u)_x) b(u)_x dx + \int_{\mathbb{R}} a(u, \varphi(u)_x)_x p(\varphi(u)_x)_x dx = \int_{\mathbb{R}} f_x p(\varphi(u)_x) dx$$
$$\iff I_1 + I_2 = I_3. \tag{4.10}$$

Consider I_2;

$$I_2 = \int_{\mathbb{R}} \frac{\partial}{\partial k} a(u, \varphi(u)_x) u_x p(\varphi(u)_x)_x dx + \int_{\mathbb{R}} \frac{\partial}{\partial \xi} a(u, \varphi(u)_x)(\varphi(u)_{xx})^2 p'(\varphi(u)_x) dx\,,$$

$$I_2 = I_{2-1} + I_{2-2}.$$

As a is nondecreasing with respect to the second variable and $p' \geq 0$, we otain $I_{2-2} \geq 0$. For the term I_{2-1}, we have

$$I_{2-1} = \int_{\mathbb{R}} \frac{\partial}{\partial k} a(u, \varphi(u)_x) u_x p(\varphi(u)_x)_x dx = \int_{\mathbb{R}} \frac{\partial}{\partial k} a(u, \varphi(u)_x) u_x \times p'(\varphi(u)_x)\varphi(u)_{xx} dx = \int_{\mathbb{R}} p'(\varphi(u)_x)\varphi(u)_x \frac{\partial}{\partial k} a(u, \varphi(u)_x) \frac{\varphi(u)_{xx}}{\varphi'(u)} dx.$$

Then

$$|I_{2-1}| \leq \int_{\mathbb{R}} \left|p'(\varphi(u)_x)\varphi(u)_x\right| \left|\frac{\partial}{\partial k} a(u, \varphi(u)_x) \frac{\varphi(u)_{xx}}{\varphi'(u)}\right| dx.$$

By approximation of sign_0 by a sequence (p_n) such that $p_n^{'} \geq 0$, $0 \leq p_n \leq 1$ and $|p_n'(r)r| \leq 1$, rises that

$$\lim_{n \to +\infty} |I_{2-1}| = 0;$$

from where

$$\int_{\mathbb{R}} |b(u)_x|\, dx \leq \int_{\mathbb{R}} f_x \text{sign}_0(\varphi(u)_x) dx. \tag{4.11}$$

The inequality (4.11) implies that

$$\int_{\mathbb{R}} |b(u)_x|\, dx \leq \int_{\mathbb{R}} |f_x|\, dx.$$

We then deduce from the inequality above

$$\text{Var}(b(u)) \leq \text{Var} f.$$

□

Step 2. Since $f_n \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, then there exist $f_{nk} \in W^{1,1}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that $f_{nk} \longrightarrow f_n$ when $k \longrightarrow +\infty$. According to Lemma 4.2, we show that:

$$\begin{cases} b(u_n) \to W_1 \text{ in } L^1_{loc}(\mathbb{R})\,, \\ \varphi(u_n)_x \overset{*}{\rightharpoonup} W_2 \text{ in } L^\infty(\mathbb{R})\,, \\ \varphi(u_n) \to W_3 \text{ in } L^1_{loc}(\mathbb{R})\,. \end{cases} \tag{4.12}$$

By using (4.3), we deduce that $u_n \longrightarrow u$ in $L^1_{loc}(\mathbb{R})$ and then $W_1 = b(u)$,

$W_3 = \varphi(u)$. While recapitulating, we have:

$$\begin{cases} b(u_n) \to b(u) \text{ in } L^1_{loc}(\mathbb{R})\,, \\ \varphi(u_n)_x \overset{*}{\rightharpoonup} \varphi(u)_x \text{ in } L^\infty(\mathbb{R})\,, \\ \varphi(u_n) \to \varphi(u) \text{ in } L^1_{loc}(\mathbb{R})\,, \\ u_n \to u \text{ in } L^1_{loc}(\mathbb{R}). \end{cases} \tag{4.13}$$

As u_n is an entropy solution of (SP) with respect to (a_n, f_n), it checked the following inequalities (cf [6, 15]):

$$\int_{\mathbb{R}} \text{sign}_0^+(b(u_n) - b(s))\,\{\xi_x(H_n(s) - h_n) + \xi(f - b(u_n))\}\,dx \geq 0 \qquad (4.14_n)$$

and

$$\int_{\mathbb{R}} \text{sign}_0^+(b(s) - b(u_n))\,\{\xi_x(H_n(s) - h_n) + \xi(f - b(u_n))\}\,dx \leq 0\,, \qquad (4.15_n)$$

for all $s \in \mathbb{R}, \xi \in D^+(\mathbb{R})$.

We get by passing to the limit in (4.14_n) and (4.15_n) by using (4.13) and a pseudomonotonicity argument (cf. [11]),

$$\int_{\mathbb{R}} \theta_1\,\{\xi_x(H(s) - h) + \xi(f - b(u))\}\,dx \geq 0\,, \tag{4.16}$$

$$\int_{\mathbb{R}} \theta_2\,\{\xi_x(H(s) - h) + \xi(f - b(u))\}\,dx \leq 0\,, \tag{4.17}$$

where $\theta_1 \in \text{sign}^+(b(u) - b(s))$, $\theta_2 \in \text{sign}^+(b(s) - b(u))$.

According to [5], inequalities (4.16) and (4.17) are equivalent to the entropy inequalities; from where the proof of the theorem. □

5. Continuous Dependence of Mild Solution of the Evolution Problem

We show in this section, a result of continuous dependence of the mild solution with respect to the data a, f and v_0. Indeed, according to the nonlinear semi-group theory (cf. [3, 5]), the continuous dependence of the mild solution with respect to the data a, f, and v_0 is a direct consequence of the preceding results.

Let us note A_b^n, the operator of $L^1(\mathbb{R})$ associated to $(SP)_n$, where

$$b(u_n) - a_n(u_n, \varphi(u_n)_x)_x = f_n \text{ on } \mathbb{R}. \tag{SP}$$

$v_n \in A_b^n b(u_n) \Longleftrightarrow b(u_n) \in L^1(\mathbb{R})$, $v_n \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and u_n is the entropy solution of $(SP)_n$ with $f_n = v_n + b(u_n)$.

Let us note

$$\begin{cases} a_n \longrightarrow a \text{ in } C(\mathbb{R}\times\mathbb{R})\,, \\ f_n \longrightarrow f \text{ in } L^1(Q)\,, \\ v_{0_n} \longrightarrow v_0 \text{ in } L^1(\mathbb{R}). \end{cases} \tag{5.1}$$

Now let us state the main result of this section.

Theorem 5.1. *Assume* (H_1), (H_2), *(4.3) and (5.1) with* f, $f_n \in L^1(\mathbb{Q})$ *and* v_0, $v_{0_n} \in L^1(\mathbb{R})$; *let* u_n *and* u *be the mild solutions of* (EP) *with respect to* (a_n, f_n, v_{0_n}) *and* (a, f, v_0), *then*

$$u_n \text{ converges to } u \text{ in } C([0,T], L^1(\mathbb{R})).$$

Proof. We have shown in Section 4 of this paper that for all $f \in BV(\mathbb{R}) \cap L^1(\mathbb{R})$,

$$(I + \lambda A_b^n)^{-1} f_n \longrightarrow (I + \lambda A_b)^{-1} f. \qquad \square$$

Since A_b^n and A_b are two T-accretive operators with dense domains and checking the image condition (cf. Section 2), the proof of Theorem 5.1 is an application of the results of the general theory of evolution equations in Banach spaces (cf. [3, 5]).

Remark 5.2. The result of this paper is specific to the one dimension case since assumption (H_1) is the main assumption of the work which is equivalent to the coerciveness assumption on the function $a(\eta, \xi)$ only in one dimension. For the moment it is open problem how to solve existence of weak solution of (EP) without the structure condition and then the continuous dependence of weak entropy solution without the structure condition (cf. [14, 16]). The several dimension case will be considered later.

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