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THE SCHUR GEOMETRICAL CONVEXITY OF INTEGRAL ARITHMETIC MEAN

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Abstract: Suppose that $f : [a,b] \subseteq (0,\infty) \to (0,\infty)$ is a second order differentiable function, and

$$G(x,y) = \begin{cases} \dfrac{1}{y-x}\displaystyle\int_x^y f(t)dt, & x,y \in [a,b], x \neq y, \\ f(x), & x = y \in [a,b]. \end{cases}$$

If $3f'(x) + xf''(x) \geq 0$ (or ≤ 0, resp.) for all $x \in [a,b]$, then $G(x,y)$ is a Schur geometrical convex (or concave, resp.) function on $[a,b]$.

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1. Introduction

For the convenience of the readers, we recall the notations and definitions as follows: Let $R_+^2 = \{(x_1,x_2) : x_1 > 0, x_2 > 0\}$, for $x = (x_1,x_2) \in R_+^2$ and $\alpha \geq 0$, denote $\log x = (\log x_1, \log x_2)$ and $x^\alpha = (x_1^\alpha, x_2^\alpha)$. For $x = (x_1,x_2), y = (y_1,y_2) \in R^2$, denote $xy = (x_1y_1, x_2y_2)$ and $e^x = (e^{x_1}, e^{x_2})$.

Definition 1. A set $E_1 \subseteq R^2$ is called a convex set if $\frac{x+y}{2} \in E_1$ whenever

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$x, y \in E_1$. And a set $E_2 \subseteq R_+^2$ is called a geometrical convex set if $x^{\frac{1}{2}}y^{\frac{1}{2}} \in E_2$ whenever $x, y \in E_2$.

It is obviously that $E_1 \subseteq R_+^2$ is a geometrical convex set if and only if $\log E_1 = \{\log x : x \in E_1\}$ is a convex set, and $E_2 \subseteq R^2$ is a convex set if and only if $e^{E_2} = \{e^x : x \in E_2\}$ is a geometrical convex set.

Definition 2. Let $E \subseteq R^2$ be a convex set. A function $f : E \to R$ is called a convex function on E if $f(\frac{x+y}{2}) \le \frac{f(x)+f(y)}{2}$ for all $x, y \in E$. And f is called a concave function if $-f$ is a convex function.

Definition 3. Let $E \subseteq R_+^2$ be a geometrical convex set. A function $f : E \to (0, \infty)$ is called a geometrical convex function on E if $f(x^{\frac{1}{2}}y^{\frac{1}{2}}) \le f^{\frac{1}{2}}(x)f^{\frac{1}{2}}(y)$ for all $x, y \in E$. And f is called a geometrical concave function if $\frac{1}{f}$ is a geometrical convex function.

From Definition 2 and Definition 3, the following Theorem A is obvious.

Theorem A. *Suppose that $E_1 \subseteq R_+^2$ is a geometrical convex set, and $f : E_1 \to (0, \infty)$ is a geometrical convex function, then*

$$F(x) = \log f(e^x) : \log E_1 \to R$$

is a convex function. Conversely, if E_2 is a convex set and $F : E_2 \to R$ is a convex function, then

$$f(x) = e^{F(\log x)} : e^{E_2} \to (0, \infty)$$

is a geometrical convex function.

Definition 4. Let $E \subseteq R^2$ be a set. A function $F : E \to R$ is called a Schur convex function on E if

$$F(x_1, x_2) \le F(y_1, y_2)$$

for each two 2-tuples $x = (x_1, x_2), y = (y_1, y_2)$ in E, such that $x \prec y$ holds, i.e.

$$x_{[1]} \le y_{[1]},$$

and

$$x_{[1]} + x_{[2]} = y_{[1]} + y_{[2]},$$

where $x_{[i]}$ denotes the i-th largest component in x. And F is called a Schur concave function if $-F$ is a Schur convex function.

Definition 5. Let $E \subseteq R_+^2$ be a set. A function $F : E \to (0, \infty)$ is called a Schur geometrical convex function on E if

$$F(x_1, x_2) \le F(y_1, y_2)$$

for each two 2-tuples $x = (x_1, x_2), y = (y_1, y_2)$ in E, such that $\log x \prec \log y$, i,e.

$$x_{[1]} \le y_{[1]}$$

and

$$x_{[1]}x_{[2]} = y_{[1]}y_{[2]},$$

and F is called a Schur geometrical concave function if $\frac{1}{F}$ is a Schur geometrical convex funciton.

From Definition 4 and Definition 5, the following Theorem B is obvious.

Theorem B. *Let $E \subseteq R^2_+$ and $H = \log E = \{\log x : x \in E\}$. Then $f : E \to (0, \infty)$ is a Schur geometrical convex (or concave, resp.) function on E if and only if $\log f(e^x)$ is a Schur convex (or concave, resp.) function on H.*

The following well-known result for Schur convexity or concavity was proved by A.W. Marshall and I. Olkin in [11].

Theorem C. *Let $E \subseteq R^2$ be a symmetric convex set with nonempty interior E^*, $\varphi(x, y) : E \to R$ is a continuous symmetric function on E. If φ is differentiable on E^*, then φ is Schur convex (or concave, resp.) on E if and only if*

$$(x - y)(\frac{\partial \varphi}{\partial x} - \frac{\partial \varphi}{\partial y}) \geq 0 \qquad (\text{or } \leq 0, \text{ resp.})$$

for all $(x, y) \in E^*$.

Following Theorem D can be derived from Theorem B and Theorem C, immediately.

Theorem D. *Let $E \subseteq R^2_+$ be a symmetric geometrical convex set with nonempty interior E^*, $\varphi(x, y) : E \to (0, \infty)$ is a continuous symmetric function on E. If φ is differentiable on E^*, then φ is Schur geometrical convex (or concave, resp.) on E if and only if*

$$(\log x - \log y)(x\frac{\partial \varphi}{\partial x} - y\frac{\partial \varphi}{\partial y}) \geq 0 \qquad (\text{or } \leq 0, \text{ resp.})$$

for all $(x, y) \in E^*$.

The theory of convex functions and Schur convex functions is one of the most important research fields in modern analysis and geometry. It can be used extensively in global Riemannian geometry [7], [8], operator inequalities [1], nonlinear PDE of elliptic type [10], combinatorial optimization [9], isoperimetric problem for polytopes [16], linear regression [14], graphs and matices [3], improperly posed problems [15], inequalities and extremum problems [4], nilpotent groups [6], global surface theory [13], and other related fields.

Geometrical convex function was first researched by P. Montel [12], in a beautiful paper discussing the analogues of the notion of convex function in n variables. In a long time, the subject of geometrical convexity seems to be even forgotten, which is a pity because of its richness. Recently, C.P. Niculescu [2] discussed the beautiful class of inequalities, which arise from the notion of geometrical convexity for functions. Niculescu's contribution is not only to call the attention to the beautiful zoo of inequalities falling in the realm of geometrical convexity, but also to prove that many classical inequalities can benefit of a better understanding via the geometrical approach of convexity.

One of the important problems is how to distinguish the Schur convexity of functions.

The following result was obtained by N. Elezović and J. Pečarić in [9].

Theorem E. *Let $I \subset R$ be an open interval, $f : I \to R$ is a continuous function, if*

$$F(x,y) = \begin{cases} \frac{1}{y-x}\int_x^y f(t)dt, & x, y \in I, x \neq y, \\ f(x), & x = y \in I, \end{cases}$$

then F is a Schur convex function on I^2 if and only if f is a convex function on I.

The main purpose of this paper is to prove the following theorem.

Theorem 1. *Suppose that $f : [a,b] \subseteq (0,\infty) \to (0,\infty)$ is a second differentiable function, and*

$$G(x,y) = \begin{cases} \frac{1}{y-x}\int_x^y f(t)dt, & x, y \in [a,b], x \neq y, \\ f(x), & x = y \in [a,b]. \end{cases}$$

If $3f'(x) + xf''(x) \geq 0$ (or ≤ 0, resp.) for all $x \in [a,b]$, then $G(x,y)$ is a Schur geometrical convex (or concave, resp.) function on $[a,b] \times [a,b]$.

2. Proof of Theorem 1

Proof of Theorem 1. We shall prove that G is Schur geometrical convex on $[a,b] \times [a,b]$ if $3f'(x) + xf''(x) \geq 0$ for $x \in [a,b]$. The parallel method can prove that G is Schur geometrical concave if $3f'(x) + xf''(x) \leq 0$ for $x \in [a,b]$.

From the definition of $G(x,y)$ we can get

$$\frac{\partial G(x,y)}{\partial x} = \frac{\partial G(x,y)}{\partial y} = \frac{1}{2}f'(x)$$

for $x = y \in [a,b]$. Hence by the symmetry of $G(x,y)$ with respect to x and y, and Theorem D, we need only to prove that

$$(\log y - \log x)(y\frac{\partial G}{\partial y} - x\frac{\partial G}{\partial x}) \geq 0 \tag{1}$$

for all $(x,y) \in [a,b]$ and $y > x$.

For $x, y \in [a,b]$ and $y > x$, we have

$$(\log y - \log x)(y\frac{\partial G}{\partial y} - x\frac{\partial G}{\partial x})$$

$$= \frac{\log y - \log x}{(y-x)^2}[(y-x)(yf(y) + xf(x)) - (y+x)\int_x^y f(t)dt]. \tag{2}$$

Taking

$$g(x,y) = (y-x)[yf(y) + xf(x)] - (y+x)\int_x^y f(t)dt, \tag{3}$$

$$h(x,y) = \frac{g(x,y)}{x+y} \tag{4}$$

and

$$p(x,y) = (x+y)^2 h'_y(x,y)\,, \tag{5}$$

then

$$h(x,y) = \frac{y-x}{y+x}[yf(y) + xf(x)] - \int_x^y f(t)dt, \tag{7}$$

$$h(x,x) = 0,$$

$$h'_y(x,y) = \frac{2x}{(y+x)^2}[yf(y) + xf(x)] + \frac{y-x}{y+x}[f(y) + yf'(y)] - f(y), \tag{8}$$

$$h'_y(x,x) = 0, \tag{9}$$

$$p(x,y) = 2x^2 f(x) - 2x^2 f(y) + y^3 f'(y) - yx^2 f'(y), \tag{10}$$

$$p(x,x) = 0$$

and

$$p'_y(x,y) = (y^2 - x^2)[3f'(y) + yf''(y)]. \tag{11}$$

$y > x$, $3f'(y) + yf''(y) \geq 0$ for $y \in [a,b]$ and (11) lead to

$$p'_y(x,y) \geq 0 \tag{12}$$

for $x, y \in [a,b]$ and $y > x$.

Then (1) follows from (12) and (2)-(10). □

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