International Journal of Pure and Applied Mathematics

Volume 41 No. 7 2007, 957-967

A PRODUCTION INVENTORY MODEL WITH UNCERTAIN DEMAND IN AN IMPERFECT PRODUCTION PROCESS: A SYNERGISTIC APPROACH

Soumen Bag

Department of Mathematics
Indian Institute of Technology
Kharagpur, 721302, INDIA
e-mail: soumen@maths.iitkgp.ernet.in

Abstract: The classical inventory control models assume that items are produced by perfectly reliable production process with a fixed set-up cost. In this paper, a production inventory model has been proposed in an imprecise and uncertain mixed environment in a synergistic way. The aim of this paper is to introduce demand as a fuzzy random variable in an imperfect production process. Here expected profit of the production model is a fuzzy quantity. Using Graded Mean Integration Value (GMIV), optimal time has been determined at which profit is maximum. A numerical example has been considered to illustrate the model.

AMS Subject Classification: 26E50
Key Words: EPQ model, fuzzy random variable, reliability, graded mean integration value

1. Introduction

After introduction of EOQ model by Harris (1915) [11], it has been modified and extended by several researchers changing the assumptions with the objective to make it more and more realistic [9], [18]. Inventory models with crisp, stochastic and fuzzy parameters have been studied by several authors. Some models are developed where few parameters are crisp [11], [9], [18], few are stochastic [1], [10] and others are fuzzy in nature [5]. But, there are very few research

Received: October 8, 2007

paper available which considered inventory parameters as fuzzy stochastic in nature. First, Dutta et al [7] incorporated demand as fuzzy random variable in a simple newsboy problem with fuzzy random variable demand. Recently, a fuzzy mixture inventory model involving fuzzy random lead time demand has been developed by Chang et al [2]. But, demand as a fuzzy random variable in production inventory model is yet to be considered.

It is often observed that from past data one retailer can estimate, the demand of a commodity follows a particular distribution. But it is very difficult to estimate the exact value of the parameters of the distribution. In this case, these parameters are vaguely defined and these can be estimated as a fuzzy number. Then the distribution is a fuzzy random distribution and we say the demand is fuzzy random.

A basic assumption in the inventory management system is that set-up cost for production is fixed. In addition, the models also implicitly assume that items produced are of perfect quality. However, in reality, products are not always perfect but directly affected by the reliability of the production process employed to manufacture the product. In the recent paper, Cheng [3] proposed a general equation to model the relationship between production set-up cost and process reliability and flexibility. Recently Leung [14], Maiti and Maiti [16] also established the same relationship for their model.

Then concept of fuzzy random variable and its fuzzy expectation has been presented by H. Kwakernaak [13] and later by Puri and Ralescue [19]. Further, recently the notion of a fuzzy random variable has also been considered by Kim and Ghil [12], Feng et al [8], Lopez et al [15].

In this research work, an EPQ model is considered where demand of the item is a discrete fuzzy random variable with known probability distribution and production process is not 100% perfect, i.e., a fraction of the produced items are defective. Defective items are sold at a reduced price. Selling price of fresh units is a mark-up over the unit production cost. Model is formulated to maximize the expected average profit. Since demand is fuzzy random variable, expected profit is a fuzzy number. For this reason α-cut of the expected profit is obtained. Then its Graded Mean Integration Value (GMIV) is obtained and is optimized. Here production period is evaluated to maximize the GMIV of the objective function. The model is illustrated with a numerical example.

2. Preliminaries

2.1. Fuzzy Number

A fuzzy number is a fuzzy set $\tilde{A} : R \to [0,1]$ if it satisfies:

(i) The core of $\tilde{A}$, i.e. $\{x \in R : \tilde{A}(x) = 1\}$ is non-empty.

(ii) $\alpha-$ cuts of $\tilde{A}$, i.e. $\{x \in R : \tilde{A}(x) \geq \alpha\}$ are all closed, bounded intervals; and

(iii) The support of $\tilde{A}$, i.e. $\{x \in R : \tilde{A}(x) > 0\}$ is bounded.

2.2. LR Fuzzy Number

A fuzzy number $\tilde{A} \subseteq R$ is said to be a LR-fuzzy number if its membership function $\mu_{\tilde{A}}$ is given by :

$$\mu_{\tilde{A}} = \begin{cases} L(x) & a \leq x \leq b\,, \\ 1 & x = b\,, \\ R(x) & b \leq x \leq c\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

where $L(x)$ is continuous and strictly monotonic increasing within $[a,b]$, $R(x)$ is continuous and strictly monotonic decreasing within $[b,c]$.

2.3. Interval Arithmetic

Let $* \in \{+,-,.,/\}$ be a binary operation on the set of positive real numbers. If A and B are closed intervals then $A * B = \{a * b : a \in A, b \in B\}$ defines a binary operation on the set of closed intervals (cf. Moore [17]). In the case of division, it is assumed that $0 \notin B$. The operations on intervals used here may be explicitly calculated from the above definition as

$$\begin{aligned} A + B &= [a_L, a_R] + [b_L, b_R] = [a_L + b_L, a_R + b_R]\,, \\ A - B &= [a_L, a_R] - [b_L, b_R] = [a_L - b_R, a_R - b_L]\,, \\ A.B &= [a_L, a_R].[b_L, b_R] \\ &= [\min\{a_Lb_L, a_Lb_R, a_Rb_L, a_Rb_R\}, \max\{a_Lb_L, a_Lb_R, a_Rb_L, a_Rb_R\}]\,, \\ \frac{A}{B} &= \frac{[a_L, a_R]}{[b_L, b_R]} = [a_L, a_R].[\frac{1}{b_R}, \frac{1}{b_L}], \;\text{ where } 0 \notin B\,, \end{aligned}$$

$$kA = \begin{cases} [ka_L, ka_R], \text{ for } k \geq 0\,, \\ [ka_R, ka_L], \text{ for } k < 0\ , \text{ where } k \text{ is a real number.} \end{cases}$$

2.4. Triangular Fuzzy Number (TFN)

A TFN $\tilde{a} = (a_1, a_2, a_3)$ has three parameters a_1, a_2, a_3, where $a_1 < a_2 < a_3$ and is characterized by the membership function $\mu_{\tilde{a}}$, given by

$$\mu_{\tilde{a}}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1} & \text{for } a_1 \leq x \leq a_2\,, \\ \frac{a_3-x}{a_3-a_2} & \text{for } a_2 \leq x \leq a_3\,, \\ 0 & \text{otherwise.} \end{cases}$$

2.5. α-Cut of a Fuzzy Number

α-cut of a fuzzy number $\tilde{A}$ in $\Re$ ($\Re$ represents set of real numbers) with membership function $\mu_{\tilde{A}}$ is denoted by $A[\alpha]$ and is defined as the following crisp set:

$$A[\alpha] = \{x : \mu_{\tilde{A}}(x) \geq \alpha, x \in \Re\}, \text{ where } \alpha \in [0, 1]\,.$$

$A[\alpha]$ is a non-empty bounded closed interval contained in $\Re$ and it can be denoted by $A[\alpha] = [A_L(\alpha), A_R(\alpha)]$.

2.6. Fuzzy Random Variable (f.r.v.)

Let F be the set of all fuzzy numbers. A fuzzy random variable (f.r.v.) is a measurable function from a probability space to collection of fuzzy variables i.e., $\tilde{X} : \Omega \longrightarrow F$.

2.7. Expectation of f.r.v.

If $\tilde{X}$ is a discrete f.r.v., such that $P(\tilde{X} = \tilde{x}_i) = \tilde{p}_i$; $i = 1, 2, 3, ...$ then its fuzzy expectation is given by $\mathrm{E}\tilde{X} = \sum_{i=1}^{\infty} \tilde{x}_i \tilde{p}_i$

2.8. Graded Mean Integration Value of Fuzzy Number

Chen and Hsieh [15] introduced Graded Mean Integration Representation method based on the integral value of graded mean α-level of LR-fuzzy number for defuzzifing LR-fuzzy numbers. Suppose $\tilde{A}$ is a LR-fuzzy number. Then according to Chen et al [4], GMIV of $\tilde{A}$ is denoted by P($\tilde{A}$) and is defined as:

$$P(\tilde{A}) = \int_0^1 (x/2)\{L^{-1}(x) + R^{-1}(x)\}dx \Big/ \int_0^1 x dx$$

$$= \int_0^1 x\{L^{-1}(x) + R^{-1}(x)\}dx\,.$$

3. Model Formulation

The following notations and assumptions are used in developing the model.

3.1. Notations

(i) K – production rate for one day and it is constant.
(ii) D – demand rate for one day, $K > D$.
(iii) P – production cost per unit time.
(iv) h – holding cost per unit quantity per one day.
(v) c_0 – set-up cost per cycle.
(vi) s_1 – selling price of fresh units.
(vii) s_2 – selling price of defective units.
(viii) r – the reliability of the production process.
(ix) T – duration of each cycle.
(x) Z – expected average profit.
(xi) $F(T)$ – total profit function of T.

3.2. Assumptions

(i) No excess stock is carried and no backorders and lost sales are allowed.
(ii) Preparation time is negligible.
(iii) T is decision variable.
(iv) Production is instantaneous.
(v) Defective items are sold immidiately with a low price.

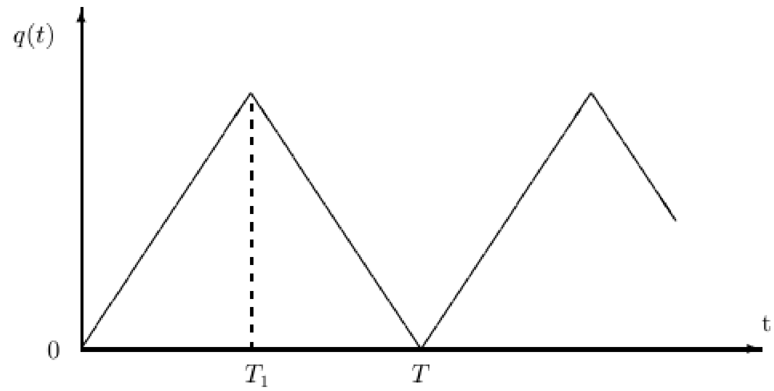

Figure 1: Inventory level over time

(vi) The fresh units are greater than the total demand, i.e. $rK > D$, where $0 < r < 1$.

(vii) Selling price s_1 of fresh units is mark-up(m) of production cost, i.e. $s_1 = mP$, $m > 1$.

(viii) Selling price s_2 of defective units is mark-up (m_1) of production cost, i.e. $s_2 = m_1 P$, $0 < m_1 < 1$.

4. Mathematical Formulation

According to assumptions inventory level at time t, $q(t)$ is given by:

$$\frac{dq(t)}{dt} = \begin{cases} rK - D & \text{for } 0 < t \leq T_1\,, \\ -D & \text{for } T_1 < t \leq T\,, \end{cases} \tag{1}$$

where $q(0) = q(T) = 0$.

Clearly

$$T_1 = \frac{DT}{rK}\,. \tag{2}$$

Total holding cost per cycle, H, is given by

$$H = h\left[\int_0^{T_1} q(t)dt + \int_{T_1}^{T} q(t)dt\right] = \frac{hDT^2}{2}\left(1 - \frac{D}{rK}\right). \tag{3}$$

The total profit incurred in each production cycle =(selling price of fresh units)+(selling price of defective units)-(production cost)-(holding cost)-(set-up cost). Thus we get, the total average expected profit per cycle,

$$Z = \frac{1}{T}\left[mPDT + \frac{(1-r)m_1PDT}{r} - \frac{PDT}{r} - \frac{hDT^2(1-\frac{D}{rK})}{2} - c_0\right]. \qquad (4)$$

Now, we consider the demand D as fuzzy random variable $\tilde{D}$ with the given set of data

$$(\tilde{d_1}, \tilde{p_1}), (\tilde{d_2}, \tilde{p_2}), ..., (\tilde{d_n}, \tilde{p_n}), \qquad (5)$$

then

$$E\tilde{Z} = \sum_{i=1}^{n}\left[mP\tilde{d}_i + \frac{(1-r)m_1P\tilde{d}_i}{r} - \frac{P\tilde{d}_i}{r} - \frac{hT\tilde{d}_i(1-\frac{\tilde{d}_i}{Kr})}{2} - \frac{c_0}{T}\right]\tilde{p_i}$$

$$= \sum_{i=1}^{n}\left[l_1\tilde{d}_i\tilde{p}_i - \frac{l_2\tilde{d}_i\tilde{p}_i}{r} + \frac{l_3\tilde{d}_i^{\,2}\tilde{p}_i}{r} - l_4\tilde{p}_i\right], \qquad (6)$$

where $l_1 = mP - m_1P - \frac{hT}{2}$, $l_2 = P - m_1P$, $l_3 = \frac{hT}{2K}$, $l_4 = \frac{c_0}{T}$.

Taking, α-cut we get,

$$E\tilde{Z}_\alpha = \sum_{i=1}^{n}\left[l_1\tilde{d_{i\alpha}}\tilde{p_{i\alpha}} - \frac{l_2\tilde{d_{i\alpha}}\tilde{p_{i\alpha}}}{r} + \frac{l_3\tilde{d_{i\alpha}^2}\tilde{p_{i\alpha}}}{r} - l_4\tilde{p_{i\alpha}}\right]. \qquad (7)$$

Let $\tilde{d}_i = (\underline{d_i}, d_i, \overline{d_i})_{LR}$, $\tilde{p_i} = (\underline{p_i}, p_i, \overline{p_i})_{LR}$, $E\tilde{Z} = (\underline{EZ}, EZ, \overline{EZ})_{LR}$, for $i = 1, 2, 3, ..., n$. Then $\tilde{d}_{i\alpha} = [\underline{d_i} + \alpha(d_i - \underline{d_i}), \overline{d_i} - \alpha(\overline{d_i} - d_i)]$, $\tilde{p}_{i\alpha} = [\underline{p_i} + \alpha(p_i - \underline{p_i}), \overline{p_i} - \alpha(\overline{p_i} - p_i)]$.

Using these and interval arithmetic we get,

$$E\tilde{Z}_\alpha =$$

$$\Bigg[\sum_{i=1}^{n}\Bigg[l_1\{\underline{d_i} + \alpha(d_i - \underline{d_i})\}\{\underline{p_i} + \alpha(p_i - \underline{p_i})\} - \frac{l_2}{r}\{\overline{d_i} - \alpha(\overline{d_i} - d_i)\}\{\overline{p_i} - \alpha(\overline{p_i} - p_i)\}$$

$$+ \frac{l_3}{r}\{\underline{d_i} + \alpha(d_i - \underline{d_i})\}^2\{\underline{p_i} + \alpha(p_i - \underline{p_i})\} - l_4\{\overline{p_i} - \alpha(\overline{p_i} - p_i)\}\Bigg],$$

$$\sum_{i=1}^{n}\Bigg[l_1\{\overline{d_i} - \alpha(\overline{d_i} - d_i)\}\{\overline{p_i} - \alpha(\overline{p_i} - p_i)\} - \frac{l_2}{r}\{\underline{d_i} + \alpha(d_i - \underline{d_i})\}\{\underline{p_i} + \alpha(p_i - \underline{p_i})\}$$

$$+ \frac{l_3}{r}\{\overline{d_i} - \alpha(\overline{d_i} - d_i)\}^2\{\overline{p_i} - \alpha(\overline{p_i} - p_i)\} - l_4\{\underline{p_i} + \alpha(p_i - \underline{p_i})\}\Bigg]\Bigg]. \qquad (8)$$

This gives,

$$L^{-1}(\alpha) =$$

$$\sum_{i=1}^{n}\Bigg[l_1\{\underline{d_i}+\alpha(d_i-\underline{d_i})\}\{\underline{p_i}+\alpha(p_i-\underline{p_i})\}-\frac{l_2}{r}\{\overline{d_i}-\alpha(\overline{d_i}-d_i)\}\{\overline{p_i}-\alpha(\overline{p_i}-p_i)\}$$
$$+\frac{l_3}{r}\{\underline{d_i}+\alpha(d_i-\underline{d_i})\}^2\{\underline{p_i}+\alpha(p_i-\underline{p_i})\}-l_4\{\overline{p_i}-\alpha(\overline{p_i}-p_i)\}\Bigg],$$

$$R^{-1}(\alpha)=$$
$$\sum_{i=1}^{n}\Bigg[l_1\{\overline{d_i}-\alpha(\overline{d_i}-d_i)\}\{\overline{p_i}-\alpha(\overline{p_i}-p_i)\}-\frac{l_2}{r}\{\underline{d_i}+\alpha(d_i-\underline{d_i})\}\{\underline{p_i}+\alpha(p_i-\underline{p_i})\}$$
$$+\frac{l_3}{r}\{\overline{d_i}-\alpha(\overline{d_i}-d_i)\}^2\{\overline{p_i}-\alpha(\overline{p_i}-p_i)\}-l_4\{\underline{p_i}+\alpha(p_i-\underline{p_i})\}\Bigg].$$

Now, using the method of representation of generalized fuzzy number based on the integral values of graded mean α-level, we find a defuzzified representation of the unique fuzzy number $E\tilde{Z}$ as

$$G(E\tilde{Z})=\frac{\displaystyle\int_0^1\alpha\left\{\frac{L^{-1}(\alpha)+R^{-1}(\alpha)}{2}\right\}d\alpha}{\displaystyle\int_0^1\alpha d\alpha}. \tag{9}$$

Substituting the above results of $L^{-1}(\alpha)$ and $R^{-1}(\alpha)$ and then after simplification we obtained

$$G(E\tilde{Z})=E_1(mP-m_1P-\frac{hT}{2})-\frac{E_2(P-m_1P)}{r}+\frac{E_3hT}{2Kr}-\frac{E_4c_0}{T}$$
$$=F(T)\qquad(\text{say}),$$

where,

$$E_1=\sum_{i=1}^{n}\left[\frac{1}{12}\{\underline{d_i}\,\underline{p_i}+\overline{d_i}\overline{p_i}+\underline{d_i}p_i+d_i\underline{p_i}+\overline{d_i}p_i+d_i\overline{p_i}\}+\frac{1}{2}(d_ip_i)\right],$$
$$E_2=\sum_{i=1}^{n}\left[\frac{1}{12}\{\underline{d_i}\,\underline{p_i}+\overline{d_i}\overline{p_i}+\underline{d_i}p_i+d_i\underline{p_i}+\overline{d_i}p_i+d_i\overline{p_i}\}+\frac{1}{2}(d_ip_i)\right],$$

$$E_3=\sum_{i=1}^{n}\Bigg[\frac{\overline{d_i}^2\overline{p_i}}{2r}+\frac{\underline{d_i}^2\underline{p_i}}{2r}-\frac{\overline{d_i}^2(\overline{p_i}-p_i)}{3r}-\frac{2\overline{d_i}\overline{p_i}(\overline{d_i}-d_i)}{3r}+\frac{2\underline{d_i}\underline{p_i}(d_i-\underline{d_i})}{3r}$$
$$+\frac{\underline{d_i}^2(p_i-\underline{p_i})}{3r}+\frac{\overline{d_i}(\overline{d_i}-d_i)(\overline{p_i}-p_i)}{2r}+\frac{\overline{p_i}(\overline{d_i}-d_i)^2}{4r}+\frac{\underline{p_i}(d_i-\underline{d_i})^2}{4r}$$
$$+\frac{\underline{d_i}(d_i-\underline{d_i})(p_i-\underline{p_i})}{2r}-\frac{(\overline{d_i}-d_i)^2(\overline{p_i}-p_i)}{5r}+\frac{(d_i-\underline{d_i})^2(p_i-\underline{p_i})}{5r}\Bigg],$$

$$E_4 = \sum_{i=1}^{n} \left[\frac{\overline{p_i}}{6} + \frac{\underline{p_i}}{6} + \frac{2p_i}{3} \right].$$

Now to find optimal T we have $F'(T) = 0$ which gives, $T = \sqrt{\frac{2c_0KE_4}{h(KE_1-E_3)}}$, $F''(T) = -\frac{2c_0E_4}{T^3} < 0$, at $T = \sqrt{\frac{2c_0KE_4}{h(KE_1-E_3)}}$. So at $T = \sqrt{\frac{2c_0KE_4}{h(KE_1-E_3)}}$, $F(T)$ is maximum.

5. Numerical Illustration

To illustrate the model a particular EPQ problem is considered. Suppose for a particular EPQ problem, $K = 60$ units; $h = 2$ units; $c_0 = 200$ units; $m = 1.5$; $m_1 = 0.8$; $P = 20$ units; $r = 0.98$ and the demand data and associated probabilities are given by

Demand	Probability
$(18, 20, 22)_{LR}$	$(.045, .05, .055)_{LR}$
$(23, 25, 27)_{LR}$	$(.143, .15, .157)_{LR}$
$(28, 30, 32)_{LR}$	$(.292, .30, .308)_{LR}$
$(33, 35, 37)_{LR}$	$(.192, .20, .208)_{LR}$
$(38, 40, 42)_{LR}$	$(.092, .10, .108)_{LR}$
$(43, 45, 47)_{LR}$	$(.093, .10, .107)_{LR}$
$(48, 50, 52)_{LR}$	$(.094, .10, .106)_{LR}$

Using these data we get the optimal time $T = 3.903696$ units and the maximum profit is $Z = 237.3991$ units.

6. Conclusion

For the first time an imperfect production model is developed in the presence of imprecision and uncertainty. The reason for adaptation of this model is threefold:

1. The implementation of fuzzy random variable as demand gives a more realistic information where the variable values are imprecise.

2. Incorporation of imprecision and uncertainty in imperfect production process.

3. This amalgamation may be extended for other inventory models.

References

[1] S.P. Aggarwal, A review of current inventory theory and its applications, *International Journal of Production Research*, **12** (1974), 443-482.

[2] H.C. Chang, J.S. Yao, L.Y. Ouyang, Fuzzy mixture inventory model involving fuzzy random variable demand and fuzzy total demand, **169** (2004), 65-80.

[3] T.C.E. Cheng, An economic production quantity model with flexibility and reliability consideration, *European Journal of Operation Research*, **39** (1989), 174-179.

[4] S.H. Chen, C.H. Hsieh, Graded mean integration representation of generalized fuzzy numbers, *Journal of Chinese Fuzzy Systems*, **5**, No. 2 (1999), 1-7.

[5] K. Das, T.K. Roy, Multi-item stochastic and fuzzy stochastic inventory models under two restrictions, **31** (2004), 1793-1806.

[6] D. Dubois, H. Prade, *Fuzzy Sets and Systems*, Theory and Applications, Academic Press (1980).

[7] P. Dutta, D. Chakraborty , A.R. Roy, A single period inventory model with fuzzy random variable demand, *Mathematical and Computer Modelling*, **41** (2005), 915-922.

[8] Y. Feng, L. Hu, H. Shu, The variance and covariance of fuzzy random variables and their applications, *Fuzzy Sets and Systems*, **120** (2001), 487-497.

[9] G. Hadley, T.M. Whiten, *Analysis of Inventory Systems*, Prentice-Hall, Clifs, NJ (1963).

[10] G. Hadley, T. M. Whitin, *An Optimal Final Inventory Model*, Prentice Hall (1975).

[11] F. Harris, *Operations and Cost (Factory Management Series)*, Chicago: A. W. Shaw Co. (1915).

[12] Y.K. Kim, B.M. Ghil, Integrals of fuzzy number valued functions, *Fuzzy Sets and Systems*, **86** (1997), 213-222.

[13] H. Kwakernaak, Fuzzy random variables:Definition and theorems, *Information Sciences*, **15** (1978), 1-29.

[14] K.F. Leung, A generalized geometric programming solution to-An economic production quantity model with flexibility and reliability considerations, *European Journal of Operation Research*, **176** (2007), 240-251.

[15] M. Lopez-Diaz, M.A. Gil, The λ average value of the random variable, *Fuzzy Sets and Systems*, **99** (1998), 347-391.

[16] M.K. Maiti, M. Maiti, Production policy of damageable items with variable cost function in an imperfect production process via genetic algorithm, *Mathematical and Computer Modelling*, **42** (2005), 977-990.

[17] R. Moore, *Interval Analysis*, Prentice-Hall, Englewood Cliffs, NJ(1966).

[18] E. Nadoor, *Inventory Systems*, John Wiley and Sons, Inc., New York (1965).

[19] M.L. Puri, D.A. Ralescu, Fuzzy random variables, *J. Mathematical Analysis and Applications*, **114** (1986), 409-422.