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FEKETE-SZEGÖ LIKE INEQUALITY FOR CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS RELATED TO COMPLEX ORDER

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Abstract: In the present investigation, the authors obtain Fekete-Szegö inequality for certain normalized analytic function $f(z)$ defined on the open unit disk for which

$$1+\frac{1}{b}\left[\left(1+\frac{zf''(z)}{f'(z)}\right)^{\alpha}\left(\frac{zf'(z)}{f(z)}\right)^{1-\alpha}-1\right]\prec\phi(z)$$

$(0\le\alpha\le 1, b\neq 0$, a complex number) lies in a region starlike with respect to 1 and symmetric with respect to the real axis. Also certain application of the main result for a class of functions of complex order defined by convolution is given. As a special case of this result, Fekete-Szegö inequality for a class of functions defined through fractional derivatives is obtained.

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1. Introduction

Let $\mathcal{A}$ denote the class of all *analytic* functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad (z \in \Delta := \{z \in \mathbb{C} : |z| < 1\}) \tag{1}$$

and $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions. Let $\phi(z)$ be an analytic function with positive real part on Δ with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the unit disk Δ onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Let $S^*(\phi)$ be the class of functions in $f \in \mathcal{S}$ for which

$$\frac{zf'(z)}{f(z)} \prec \phi(z) \qquad (z \in \Delta)$$

and $C(\phi)$ be the class of functions in $f \in \mathcal{S}$ for which

$$1 + \frac{zf''(z)}{f'(z)} \prec \phi(z) \qquad (z \in \Delta),$$

where $\prec$ denotes the subordination between analytic functions. These classes were introduced and studied by Ma and Minda [6]. They have obtained the Fekete-Szegö inequality for the functions in the class $C(\phi)$. Since $f(z) \in C(\phi)$ if and only if $zf'(z) \in S^*(\phi)$, we get the Fekete-Szegö problem for functions in the class $S^*(\phi)$. For a brief history of Fekete-Szegö problem for the class of starlike, convex and close-to-convex functions, see the recent paper by Srivastava et al [10].

Very recently Ravichandran et al [9] introduced the following classes of functions involving complex order.

Definition 1.1. Let $b \neq 0$ be a complex number. Let $\phi(z)$ be an analytic function with positive real part on Δ with $\phi(0) = 1, \phi'(0) > 0$ which maps the unit disk Δ onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Then the class $S_b^*(\phi)$ consists of all analytic functions $f \in \mathcal{A}$ satisfying

$$1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \phi(z).$$

The class $C_b(\phi)$ consists of functions $f \in \mathcal{A}$ satisfying

$$1 + \frac{1}{b}\frac{zf''(z)}{f'(z)} \prec \phi(z).$$

They have obtained the Fekete-Szegö inequalities for functions in these classes.

Motivated by the aforementioned works, we obtain the Fekete-Szegö inequality for functions of complex order in a more general class $M_b^{\alpha}(\phi)$ of functions which we define below. Also we give applications of our results to certain functions defined through convolution (or the Hadamard product) and in particular we consider a class $M_b^{\alpha,\lambda}(\phi)$ of functions defined by fractional derivatives. The motivation of this paper is to give a generalization of the Fekete-Szegö inequalities of subclasses of starlike and convex functions of complex order obtained by Ravichandran et al [9].

Definition 1.2. Let $b \neq 0$ be a complex number. Let $\phi(z)$ be an analytic function with positive real part on Δ with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the unit disk Δ onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Then the class $M_b^{\alpha}(\phi)$ consists of all functions $f \in \mathcal{A}$ satisfying

$$1 + \frac{1}{b}\left[\left(1 + \frac{zf''(z)}{f'(z)}\right)^{\alpha}\left(\frac{zf'(z)}{f(z)}\right)^{1-\alpha} - 1\right] \prec \phi(z) \quad (0 \leq \alpha \leq 1).$$

For fixed $g \in \mathcal{A}$, we define the class $M_b^{\alpha,g}(\phi)$ to be the class of functions $f \in \mathcal{A}$ for which $(f * g) \in M_b^{\alpha}(\phi)$.

Our definition of the function class $M_b^{\alpha}(\phi)$ is motivated essentially by the earlier investigation of Darus and Thomas [2], in which a closely related class can be found.

Also, we note that $M_b^0(\phi) = S_b^*(\phi)$ and $M_b^1(\phi) = C_b(\phi)$.

To prove our main result, we need the following:

Lemma 1.1. (see [9]) *If $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is a function with positive real part, then for any complex number μ,*

$$|c_2 - \mu c_1^2| \leq 2\max\{1, |2\mu - 1|\}$$

and the result is sharp for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2}, \qquad p(z) = \frac{1+z}{1-z}.$$

2. Fekete-Szegö Problem

Our main result is the following:

Theorem 2.1. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. If $f(z)$ given by*

(1) belongs to $M_b^{\alpha}(\phi)$*, then*

$$|a_3 - \mu a_2^2| \leq \frac{B_1|b|}{2(1+2\alpha)} \max\left\{1, \left|\frac{B_2}{B_1} + \left(\frac{(2+7\alpha-\alpha^2) - 4\mu(1+2\alpha)}{2(1+\alpha)^2}\right] bB_1\right|\right\}.$$

The result is sharp.

Proof. If $f(z) \in M_b^{\alpha}(\phi)$, then there is a Schwarz function $w(z)$, analytic in Δ with $w(0) = 0$ and $|w(z)| < 1$ in Δ such that

$$1 + \frac{1}{b}\left[\left(1 + \frac{zf''(z)}{f'(z)}\right)^{\alpha}\left(\frac{zf'(z)}{f(z)}\right)^{1-\alpha} - 1\right] = \phi(w(z)). \tag{2}$$

Define the function $p_1(z)$ by

$$p_1(z) := \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + \cdots. \tag{3}$$

Since $w(z)$ is a Schwarz function, we se that $\Re(p_1(z)) > 0$ and $p_1(0) = 1$. Define the function $p(z)$ by

$$p(z) := 1 + \frac{1}{b}\left[\left(1 + \frac{zf''(z)}{f'(z)}\right)^{\alpha}\left(\frac{zf'(z)}{f(z)}\right)^{1-\alpha} - 1\right] = 1 + b_1 z + b_2 z^2 + \cdots. \tag{4}$$

In view of the equations (2), (3), (4), we have

$$p(z) = \phi\left(\frac{p_1(z) - 1}{p_1(z) + 1}\right) \tag{5}$$

and from this equation (5), we obtain

$$b_1 = \frac{1}{2}B_1 c_1 \tag{6}$$

and

$$b_2 = \frac{1}{2}B_1(c_2 - \frac{1}{2}c_1^2) + \frac{1}{4}B_2 c_1^2. \tag{7}$$

From the equation (4), we obtain

$$\begin{aligned}(1+\alpha)a_2 &= bb_1,\\ (2+4\alpha)a_3 &= bb_2 + \left[\frac{2+7\alpha-\alpha^2}{2}\right]a_2^2,\end{aligned}$$

or equivalently we have

$$a_2 = \frac{bb_1}{1+\alpha}, \tag{8}$$

$$a_3 = \frac{1}{2+4\alpha}\left[bb_2 + \left(\frac{2+7\alpha-\alpha^2}{2(1+\alpha)^2}\right)b^2 b_1^2\right]. \tag{9}$$

Applying (6) in (8) and (6), (7) in (9), we have

$$a_2 = \frac{bB_1c_1}{2(1+\alpha)},$$

$$a_3 = \frac{bB_1c_2}{4(1+2\alpha)} + \frac{c_1^2}{8(1+2\alpha)}\left[\frac{b^2B_1^2}{2}\left(\frac{2+7\alpha-\alpha^2}{(1+\alpha)^2}\right) - b(B_1-B_2)\right].$$

Therefore we have

$$a_3 - \mu a_2^2 = \frac{bB_1}{4(1+2\alpha)}\left\{c_2 - vc_1^2\right\}, \tag{10}$$

where

$$v := \frac{1}{2}\left[1 - \frac{B_2}{B_1} + \left(\frac{4\mu(1+2\alpha)-(2+7\alpha-\alpha^2)}{2(1+\alpha)^2}\right)bB_1\right].$$

Our result now follows by an application of Lemma 1.1. The result is sharp for the function defined by

$$1 + \frac{1}{b}\left[\left(1+\frac{zf''(z)}{f'(z)}\right)^{\alpha}\left(\frac{zf'(z)}{f(z)}\right)^{1-\alpha} - 1\right] = \phi(z^2)$$

and

$$1 + \frac{1}{b}\left[\left(1+\frac{zf''(z)}{f'(z)}\right)^{\alpha}\left(\frac{zf'(z)}{f(z)}\right)^{1-\alpha} - 1\right] = \phi(z). \qquad \square$$

For $\alpha = 0$, in Theorem 2.1 we get the result obtained by Ravichandran et al [9].

Corollary 2.1. *Let $\phi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \cdots$. If $f(z)$ given by (1) belongs to $S_b^*(\phi)$, then*

$$|a_3 - \mu a_2^2| \leq \frac{B_1|b|}{2}\max\left\{1; \left|\frac{B_2}{B_1} + (1-2\mu)bB_1\right|\right\}.$$

The result is sharp.

For a special case $\alpha = 1$, Theorem 2.1 reduces to another result obtained by Ravichandran et al [9].

Corollary 2.2. *Let $\phi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \cdots$. If $f(z)$ given by (1) belongs to $C_b(\phi)$, then*

$$|a_3 - \mu a_2^2| \leq \frac{B_1|b|}{6}\max\left\{1; \left|\frac{B_2}{B_1} + \left(1-\frac{3\mu}{2}\right)bB_1\right|\right\}.$$

The result is sharp.

Example 2.1. By taking $\alpha = 0$, $b = (1-\beta)e^{-i\lambda}\cos\lambda$, $\phi(z) = \frac{1+z}{1-z}$, we obtain the following sharp inequality for λ-spirallike function $f(z)$ of order β:

$$|a_3 - \mu a_2^2| \leq \frac{(1-\beta)\cos\lambda}{1+\alpha} \times \max\left\{1, \left|e^{i\lambda} - \left[\frac{4\mu(1+2\alpha) - (2+7\alpha-\alpha^2)}{(1+\alpha)^2}\right](1-\beta)\cos\lambda\right|\right\}.$$

This result was obtained by Keogh and Merkes [5].

3. Application to Functions Defined by Fractional Derivatives

In order to introduce the class $M_b^{\alpha,\lambda}(\phi)$, we need the following:

Definition 3.1. (see [8, 7]; see also [11, 12]) Let the function $f(z)$ be analytic in a simply connected region of the z-plane containing the origin. The fractional derivative of f of order λ is defined by

$$D_z^\lambda f(z) := \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\lambda} d\zeta \qquad (0 \leq \lambda < 1),$$

where the multiplicity of $(z-\zeta)^\lambda$ is removed by requiring $\log(z-\zeta)$ is real for $(z-\zeta) > 0$.

Using the above Definition 3.1 and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [8] introduced the operator $\Omega^\lambda : \mathcal{A} \to \mathcal{A}$ defined by

$$(\Omega^\lambda f)(z) = \Gamma(2-\lambda) z^\lambda D_z^\lambda f(z) \qquad (\lambda \neq 2, 3, 4, \dots).$$

The class $M_b^{\alpha,\lambda}(\phi)$ consists of functions $f \in \mathcal{A}$ for which $\Omega^\lambda f \in M_b^\alpha(\phi)$. Note that $M_b^{0,0}(\phi) = S_b^*(\phi)$ and $M_b^{1,0}(\phi) = C_b(\phi)$. Also $M_b^{\alpha,\lambda}(\phi)$ is the special case of the class $M_b^{\alpha,g}(\phi)$ when

$$g(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} z^n. \tag{11}$$

Let $g(z) = z + \sum_{n=2}^\infty g_n z^n$ ($g_n > 0$). Since $f(z) = z + \sum_{n=2}^\infty a_n z^n \in M_b^{\alpha,g}(\phi)$ if and only if $(f * g)(z) = z + \sum_{n=2}^\infty g_n a_n z^n \in M_b^\alpha(\phi)$, we obtain the coefficient estimate for functions in the class $M_b^{\alpha,g}(\phi)$, from the corresponding estimate for functions in the class $M_b^\alpha(\phi)$.

Applying Theorem 2.1 for the function $(f*g)(z) = z + g_2 a_2 z^2 + g_3 a_3 z^3 + \dots$, we get the following theorem after an obvious change of the parameter μ:

Theorem 3.1. *Let the function $\phi(z)$ be given by $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. If $f(z)$ given by (1) belongs to $M_b^{\alpha,g}(\phi)$, then*

$$|a_3 - \mu a_2^2| \leq \frac{B_1|b|}{2g_3(1+2\alpha)} \times \max\left\{1, \left|\frac{B_2}{B_1} + \left[\frac{(2+7\alpha-\alpha^2)g_2^2 - 4\mu(1+2\alpha)g_3}{2(1+\alpha)^2 g_2^2}\right] bB_1\right|\right\}.$$

The result is sharp.

Since

$$(\Omega^\lambda f)(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n z^n,$$

we have

$$g_2 := \frac{\Gamma(3)\Gamma(2-\lambda)}{\Gamma(3-\lambda)} = \frac{2}{2-\lambda} \tag{12}$$

and

$$g_3 := \frac{\Gamma(4)\Gamma(2-\lambda)}{\Gamma(4-\lambda)} = \frac{6}{(2-\lambda)(3-\lambda)}. \tag{13}$$

For g_2 and g_3 given by (12) and (13), Theorem 3.1 reduces to the following:

Theorem 3.2. *Let the function $\phi(z)$ be given by $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. If $f(z)$ given by (1) belongs to $M_b^{\alpha,\lambda}(\phi)$, then*

$$|a_3 - \mu a_2^2| \leq \frac{(2-\lambda)(3-\lambda)B_1|b|}{12(1+2\alpha)} \times \max\left\{1, \left|\frac{B_2}{B_1} + \left[\frac{(2+7\alpha-\alpha^2)(3-\lambda) - 6\mu(1+2\alpha)(2-\lambda)}{2(1+\alpha)^2(3-\lambda)} bB_1\right]\right|\right\}.$$

The result is sharp.

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