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STABILIZATION PROBLEM FOR DELAY STOCHASTIC SYSTEMS

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Abstract: In this work, the stability and the stabilizability for delay stochastic systems are considered. For the stabilization problem, we propose the stabilizability by feedback on the state if the system has a complete observation or by feedback on a compensator if the system is partially observed. In the second case, by using the spectrum decomposition, we give sufficient conditions for the existence of the finite-dimensional stabilizing compensator. To illustrate this work, two examples are given.

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1. Introduction

By delay system, we shall mean a dynamical system whose state at time t depends on the past state in some time interval. Mathematical models of delay systems have been used in many areas of science and engineering, to describe dynamics of technological, biological, social and economical systems. Examples of such models will be given in [8].

However in most of studies available, several parameters are neglected and hence their applicability is severely limited. Especially, the stochastic case is

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not considered.

For that, we consider in this work the stability and the stabilizability for delay stochastic systems. This problem is accosted by transforming the initial system to a system of differential stochastic equations on a product space and using the semigroup approach. After defining the conception of the stability, we give a solution of the stabilization problem by feedback on the state if the system has a complete observation or by feedback on a compensator if the system is partially observed. In the second case, by using the spectrum decomposition, we give sufficient conditions for the existence of the finite-dimensional stabilizing compensator. Notice that the proof of the existence theorem is constructive.

Our work is composed by the following sections: In Section 2, we summarize the most important facts about the delay stochastic systems. More details are given in [1] and [3]. After we formulate the concept of stochastic stability and we give sufficient conditions of the stabilization for complete observation systems. In Section 3, we formulate the stabilization problem for the partially observed systems and we give sufficient conditions for the stabilization by feedback on an finite dimensional compensator. To illustrate this work two examples are given.

2. Stabilization for Completely Observed System

In recent years, the subject of stability and stabilizability for infinite dimensional systems has received considerable attention and many different approaches has been used, see [1], [3], [4], [5], [6], [7], [8], [10] and [11]. In this work, we consider the case of delay stochastic systems. Such systems are described by:

$$\begin{cases} d\xi(t) = \displaystyle\sum_{i=0}^{q} A_i \xi(t-h_i)dt \\ + \displaystyle\int_{-h}^{0} A_{01}(\theta)\xi(t+\theta)d\theta dt + P_0 u(t)dt + B_0 dw(t)\,, \\ \xi(0) = \phi^0 \quad \text{and} \quad \xi(\theta) = \phi^1(\theta) \quad \text{for} \quad \theta \in [-h, 0[\,, \end{cases} \tag{1}$$

where $0 = h_0 < ... < h_q = h < +\infty$, ξ, u takes values in $\mathbb{R}^n$, $\mathbb{R}^m$ respectively, $w(t)$ is a $\mathbb{R}^n$-Weiner process independent of (ϕ_1, ϕ_2) with incremental covariance matrix W_0, A_i, P_0, B_0 are matrices of suitable dimensions and $A_{01}(.)$ is an $n \times n$ matrix of L^2-functions.

We assume the initial condition (ϕ^0, ϕ^1) to be square integrable and all stochastic processes to have the same underlying probability space $(\Omega; P)$.

As in [1] and [3], the corresponding M^2-version of (1) is:

$$\begin{cases} dx(t) = Ax(t)dt + Pu(t)dt + Bdw(t)\,, \\ x(0) = (\phi^0, \phi^1)\,, \end{cases} \tag{2}$$

where $x(t) = (\xi(t), \xi_t) \in L^2(\Omega, P; M^2)$ with $M^2 = \mathbb{R}^n \times L^2(-h, 0; \mathbb{R}^n)$, $\xi_t(\theta) = \xi(t+\theta)$ for $\theta \in [-h, 0]$, $P : \mathbb{R}^m \to M^2$ and $B : \mathbb{R}^n \to M^2$ are bounded linear operators that $Pu = (P_0 u, 0)$ and $Bw = (B_0 w, 0)$.

$A : \mathcal{D}(A) \subset M^2 \to M^2$ is the generator of strongly continuous semigroup $\{S(t)\}_{t \geq 0}$ with discrete spectrum $\sigma(A)$ and $\lambda \in \sigma(A)$ if and only if $det(\Delta(\lambda)) = 0$, where:

$$\Delta(\lambda) = \sum_{i=0}^{q} A_i e^{\lambda h_i} + \int_{-h}^{0} A_{01}(\theta) e^{\lambda\theta} d\theta - \lambda I \qquad (3)\,.$$

More details are given in [1], [3], [4] and [9].

The stabilization problem is the following: if the system is not stable, is it possible to find an admissible control $u(.)$ in $L^2([0, +\infty[; L^2(\Omega, P; \mathbb{R}^m))$ such that the solution $\xi(t)$ of (1) corresponding to $u(.)$ is stable in the sense:

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T E[||\xi(t)||^2] dt \;<\; +\infty \;\; \forall\, (\phi^0, \phi^1) \;\in L^2(\Omega, P; M^2). \tag{4}$$

Remark 2.1. 1. The system (1) is stable if and only if the system (2) is stable.

2. $S(t)$ is exponentially stable $\Leftrightarrow \forall\lambda \in \sigma(A);\ \mathrm{Re}\,(\lambda) < 0$.

3. If $S(t)$ is exponentially stable then the system (1) with $u(.) = 0$ is stable.

Now suppose that the system (1) with $u(.) \equiv 0$ is not stable and consider Λ given by:

$$\Lambda = \{\lambda \in \sigma(A) \;: \mathrm{Re}\,(\lambda) \geq 0\} = \{\lambda_1, ..., \lambda_p\}\,.$$

Then the important characterization of the stabilization for complete observed system is as follows:

Proposition 2.1. *If rank* $[\Delta(\lambda)|P_0] = n$ *for all* $\lambda \in \Lambda$, *then the system (1) is stabilizable.*

Proof. Let $\{\lambda_1', ..., \lambda_p'\} \in \mathbb{C}$ with $\mathrm{Re}\,(\lambda_i) < 0,\ 1 \leq i \leq p$. From [10] there exists a bounded operator $L : M^2 \longrightarrow \mathbb{R}^m$ such that the spectrum

$$\sigma(A + PL) = \{\lambda_1', ..., \lambda_p'\} \cup \{\lambda \in \sigma(A) : \mathrm{Re}\,(\lambda) < 0\}.$$

As $\sigma(A + PL)$ is discrete and for $\lambda \in \sigma(A + PL)$ we have $\mathrm{Re}\,(\lambda) < 0$, then the semigroup generated by $A + PL$ is exponentially stable.

From Remark 2.1, the system (1) with $u(t) = Lx(t)$ is stable. □

Example 1. Let the following delay stochastic system

$$\begin{cases} d\xi(t) = \dfrac{-\pi}{2}\xi(t-1)dt + u(t)dt + dw(t)\,, \\ \xi(0) \ \text{ and } \ \xi(\theta) = \mu(\theta) \quad \text{for} \quad \theta \in [-1, 0]\,, \end{cases} \tag{5}$$

$\xi(t) \in \mathbb{R}$, $u(t) \in \mathbb{R}$, $w(t)$ is $\mathbb{R}$-Wiener process and $\mu \in L^2(-1, 0, \mathbb{R})$.

Let us $x(t) = (\xi(t), \xi_t)$, where $\xi_t(\theta) = \xi(t+\theta)$ for $\theta \in [-1, 0]$. We get

$$\begin{cases} dx(t) = Ax(t)dt + Pu(t)dt + Bdw(t)\,, \\ x(0) = (\xi(0), \mu)\,, \end{cases} \tag{6}$$

where $x(t) \in M_2 = \mathbb{R} \times L^2(-1, 0, \mathbb{R})$, $Pu = (u, 0)$, $Bw = (w, 0)$ and

$$A(\phi(0), \phi) = (-\frac{\pi}{2}\phi(-1), \frac{\partial}{\partial\,\theta}\phi).$$

For more details (see [1], [3], [4], and [5]).

Let us consider $\Lambda = \{\lambda \in \sigma(A) : \mathrm{Re}\,(\lambda) \geq 0\} = \{-i\frac{\pi}{2},\ i\frac{\pi}{2}\}$ (see [5]). As $\sigma(A)$ is discrete, A verify the spectrum decomposition hypothesis. Therefore one notes $\mathcal{M}_\lambda = \mathrm{Ker}\,(\lambda I - A)^k$ and $Q_\lambda = \mathrm{Im}\,(\lambda I - A)^k$ with k = order of multiplicity of λ, M_2 is decomposed according to Λ as follows:

$$M_2 = M_\Lambda \oplus Q_\Lambda \text{ with } M_\Lambda = \mathcal{M}_{-i\frac{\pi}{2}} \oplus \mathcal{M}_{i\frac{\pi}{2}} \text{ and } Q_\Lambda = Q_{-i\frac{\pi}{2}} \cap Q_{i\frac{\pi}{2}}\,.$$

As $-i\frac{\pi}{2}$ and $i\frac{\pi}{2}$ are the simple eigenvalues of A (see [5]), we have $\dim M_\Lambda = 2$ and $M_\Lambda = \mathrm{span}\,\{\phi_1, \phi_2\}$ with $\phi_1 = (0, \sin\dfrac{\pi}{2}(.))$ and $\phi_2 = (1, \cos\frac{\pi}{2}(.))$.

We consider

$$\psi_1 = \frac{8}{(4+\pi^2)}(-\phi_1 + \frac{\pi}{2}\phi_2) \text{ and } \psi_2 = \frac{8}{(4+\pi^2)}(\frac{\pi}{2}\phi_1 + \phi_2),$$

then one has: $\quad < \psi_i, \phi_j >= \delta_{ij} \quad$ for $\quad 1 \leq i, j \leq 2$.

If one notes P_Λ the projection of M_2 under M_Λ, then one can write $x(t)$ solution of (S) as follows:

$$x(t) = x_N(t) + x_r(t), \text{ with } x_r(t) = (I_{M_2} - P_\Lambda)x(t)$$

and

$$x_N(t) = P_\Lambda x(t) =< \psi_1, x(t) > \phi_1 + < \psi_2, x(t) > \phi_2 = a_1(t)\phi_1 + a_2(t)\phi_2.$$

$a(t) = \begin{pmatrix} a_1(t) \\ a_2(t) \end{pmatrix}$ verifies the following equation:

$$\begin{cases} da(t) = E_\Lambda a(t)dt + \psi^0 u(t)dt + \psi^0 dw(t)\,, \\ a(0)\,. \end{cases} \tag{7}$$

Here:

$$E_\Lambda = \begin{pmatrix} 0 & -\dfrac{\pi}{2} \\ \dfrac{\pi}{2} & 0 \end{pmatrix} \quad \text{and} \quad \psi^0 = \begin{pmatrix} \dfrac{4\pi}{4+\pi^2} \\ \dfrac{8}{4+\pi^2} \end{pmatrix}.$$

While calculating the matrix F_0 of order 1×2 such that $E_\Lambda + \psi^0 F_0$ admits $-\frac{1}{2} + i\frac{\pi}{2}$ and $-\frac{1}{2} - i\frac{\pi}{2}$ as eigenvalues, we get $F_0 = \begin{pmatrix} \frac{1-\pi^2}{4\pi} & -\frac{5}{8} \end{pmatrix}$.

Let the operator

$$\begin{array}{rcl} L : M_2 & \longrightarrow & \mathbb{R}\,, \\ \phi = (\phi_0, \phi_1) & \longrightarrow & L\phi\,, \end{array}$$

where

$$\begin{aligned} L\phi &= F_0 \begin{pmatrix} < \psi_1, \phi > \\ < \psi_2, \phi > \end{pmatrix} \\ &= -\phi^0 + \frac{\pi}{2}\int_{-1}^{0} \cos(\frac{\pi}{2}s)\phi^1(-1-s)ds + \frac{1}{4}\int_{-1}^{0} \sin(\frac{\pi}{2}s)\phi^1(-1-s)ds\,. \end{aligned}$$

If we put $u(t) = Lx(t) = Lx_N(t) = F_0 a(t)$, we get

$$\begin{cases} da(t) = (E_\Lambda + \psi^0 F_0)a(t) + \psi^0 dw(t)\,, \\ a(0)\,, \end{cases}$$

where $a(t) = e^{(E_\Lambda+\psi^0 F_0)t}a(o) + \int_0^t e^{(E_\Lambda+\psi^0 F_0)(t-s)}\psi^0 dw(s)$. But $-\frac{1}{2} + i\frac{\pi}{2}$ and $-\frac{1}{2} - i\frac{\pi}{2}$ are the only eigenvalues of $E_\Lambda + \psi^0 F_0$. Then there exists $M \geq 1$ and $\alpha > 0$ such that

$$\|e^{(E_\Lambda+\psi^0 F_0)t}\| \leq Me^{-\alpha\ t}\,, \quad \forall t \geq 0\,.$$

As $w(t)$ is a Wiener-process and from [3], we have $E[a(t)] = e^{(E_\Lambda+\psi^0 F_0)t}E[a(o)]$,

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T E[\|a(t)\|^2]dt < +\infty\,.$$

Therefore

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T E[\|x_N(t)\|^2]dt < +\infty.$$

While noting $A_r = A/Q_\Lambda$, $x_r(t)$ verifies the equation

$$\begin{cases} dx_r(t) = A_r x_r(t)dt + (I_{M^2} - P_\Lambda)dw(t)\,, \\ x_r(0)\,. \end{cases}$$

Therefore

$$x_r(t) = S_{A_r}(t)x_r(0) + \int_0^t S_{A_r}(t-s)(I_{M^2} - P_\Lambda)dw(s),$$

with $S_{A_r}(t)$ is the semi-group generated by A_r. As $w(t)$ is a process Wiener and from [3] we have

$$E[x_r(t)] = S_{A_r}(t)E[x_r(0)].$$

But $\sigma(A_r)$ is discrete and $\sigma(A_r) \subset \{\lambda \in \mathbb{C}/\mathrm{Re}\,(\lambda) < 0\}$, there exists $K_r \geq 1$ and $\alpha_r > 0$ such that

$$\|S_{A_r}(t)\phi_r\| \leq K_r e^{-\alpha_r\ t}\|\phi_r\| \qquad \forall \phi_r \in Q_\Lambda.$$

This shows that

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T E[||x_r(t)||^2]dt \ < \ +\infty\,.$$

Therefore

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T E[||x(t)||^2]dt \ < \ +\infty\,,$$

which shows that the control $u(t) = Lx(t)$ stabilize the system (5). □

3. Stabilization for Partially Observed Systems

We consider the system (1) with the observation process given by:

$$dy(t) \ = \ C_0\xi(t)dt + Ddv(t), \tag{8}$$

where $y(t)$ takes values in $\mathbb{R}^r$, $v(t)$ is a $\mathbb{R}^r$-Wiener process; C_0 and D are matrices of suitable dimensions. Using equation (2) the observation process takes the form:

$$dy(t) = Cx(t)dt + Ddv(t)\,, \tag{9}$$

where $C : M^2 \to \mathbb{R}^r$ is given by $C_0(\phi^0, \phi^1) \ = \ C_0\phi^0$. For control function, we shall look the controls of the form:

$$u(t) = \mu(t) + \int_0^t L(t,s)dy(s)\,, \tag{10}$$

where deterministic $\mu \in L^2(0,+\infty;\mathbb{R}^m)$ and $L(.,.) \in \mathcal{L}(\nabla;\mathcal{L}(\mathbb{R}^r,\mathbb{R}^m))$ such that $\|L(t,s)\| \leq g(t-s)$ for all $(t,s) \in \nabla$ with $g \in L^2(0,+\infty;\mathbb{R}) \cup\ L^1(0,+\infty;\mathbb{R})$ and $\nabla\ = \{(t,s) \in \mathbb{R}^2/0 \leq s \leq t\}$.

Remark 3.1. 1. The controls (10) are admissible in the sense that the solution of (2) is weaker, unique and mean square continuous (see [3]).

2. If $S(t)$ is exponentially stable then the system (1) controlled with $u(t)$ defined by (10) is stable.

Now we suppose that the system (1) is not stable and the problem is to find an admissible control of the form (10) such that the system (1) is stable.

As in [11], we shall concentrate on stabilization by feedback on a compensator of the form:

$$\begin{cases} dz(t) = Kz(t)dt + Gdy(t)\,, \quad z(o) = z_0 \ \in H\,, \\ u(t) = -Lz(t)\,. \end{cases} \tag{11}$$

Here K is the generator of strongly continuous semigroup in the Hilbert space H, $G \in \mathcal{L}(\mathbb{R}^r, H)$ and $L \in \mathcal{L}(H, \mathbb{R}^m)$.

We combine (2), (9) and (11) and we obtain:

$$\begin{cases} dx(t) = Ax(t)dt + Bdw(t) - PLz(t)dt\,, \quad x(0) = x_0\,, \\ dz(t) = Kz(t)dt + GCx(t)dt + GDdv(t)\,, \quad z(o) = z_0\,. \end{cases} \tag{12}$$

The equation (12) may also write in the extended state space $M^2 \times H$ with linear product as:

$$\begin{cases} d\bar{x}(t) = \bar{A}\bar{x}(t)dt + \bar{B}dw(t) + \overline{GD}dv(t)\,, \\ \bar{x}(0) = (x(0), z(0))^T\,, \end{cases} \tag{13}$$

where:

$$\bar{A} = \begin{pmatrix} A & -PL \\ GC & K \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} B \\ 0 \end{pmatrix}, \quad \overline{GD} = \begin{pmatrix} 0 \\ GD \end{pmatrix},$$

and

$$\bar{x}(t) = \begin{pmatrix} x(t) \\ z(t) \end{pmatrix} \in M^2 \times H.$$

Definition 3.1. The system (11) is a stabilizing compensator for (1)-(8) if

1. The equation (13) has a weaker and unique solution in $M^2 \times H$.
2. The solution $\bar{x}(t)$ of (13) is stable in the sense

$$\lim_{t\to+\infty} \frac{1}{T}\int_0^T E[\|\bar{x}(t)\|^2_{M^2\times H}]dt < +\infty.$$

Proposition 3.1. *If rank* $[\Delta(\lambda)|P_0] =$ *rank* $\begin{pmatrix} \Delta(\lambda)G_0 \end{pmatrix} = n$ *for all* $\lambda \in \sigma(A)$ *with* $Re\,(\lambda) \geq 0$, *then there exists a stabilizing compensator for the system (1)-(8).*

Proof. If rank $[\Delta(\lambda)|P_0] =$ rank $\begin{pmatrix} \Delta(\lambda) \\ G_0 \end{pmatrix} = n$ for all $\lambda \in \sigma(A)$ with $\mathrm{Re}\,(\lambda) \geq 0$, the pair (A, P) is stabilizable and the pair (C, A) is detectable. From [11], we deduce the result. □

The conditions of Proposition 3.1 guarantee only an infinite dimensional stabilizing compensator. For that the stabilization problem can now be formulated as follows: Given systems (1)-(8) find a finite-dimensional stabilizing compensator (11) (i.e dim $H < +\infty$) such that the extended system (13) is stable. For this, we shall state an analogue Proposition as in [11], Theorem 5.1 for the delay stochastic systems.

Proposition 3.2. *Assume that there exists bounded operator $L : M_2 \longrightarrow \mathbb{R}^m$ and $G : \mathbb{R}^r \longrightarrow \mathcal{V} \in M_2$ with $\mathcal{V} \subset \mathcal{D}(A)$ and its dimension is finite, such that the following holds:*

1. *$A + PL$ generates a stable semigroup.*
2. *$A + GC$ generates a stable semigroup.*
3. *$(A + PL)x \in \mathcal{V} \quad \forall x \in \mathcal{V}$.*
4. *$\mathrm{Im}\, G \subset \mathcal{V}$.*

Then there exists a stabilizing compensator of order N, where $N = \dim \mathcal{V}$.

Poof. We can choose $\bar{A}$ of the extended system (13) such that $\bar{A}$ generates a stable semigroup. By Theorem 5.1 of [10] and Proposition 3.1 shows that the same must hold for $\bar{A}$. □

Lemma 3.1. *Suppose that:*

1. *$\det A_q \neq 0$;*
2. *$\mathrm{rank}\,[\Delta(\lambda)|P_0] = n$ for all $\lambda \in \sigma(A)$ with $Re\,(\lambda) \geq 0$.*

Then there exists bounded mapping $L : M_2 \longrightarrow \mathbb{R}^m$ such that $A + PL$ generates a stable semigroup, the spectrum of $A + PL$ is discrete and the eigenvectors of $A + PL$ are complete in M_2.

Proof. Since $\sigma(A)$ is discrete, there exist $\alpha > 0$ such that:

$$\forall \lambda \in \{\lambda \in \sigma(A) : \mathrm{Re}\,(\lambda) < 0\}, \quad \text{we have} \quad \mathrm{Re}\,(\lambda) < -\alpha.$$

Let $\{\lambda_1', ..., \lambda_p'\} \subset \mathbb{C}$ such that $-\alpha < \mathrm{Re}\,(\lambda_i') < 0,\ 1 \leq i \leq p$. From [10], there exists $L : M_2 \longrightarrow \mathbb{R}^m$ such that $A + PL$ generates a semigroup with

$$\sigma(A + PL) = \{\lambda_1', ..., \lambda_p'\} \cup \{\lambda \in \sigma(A) : \mathrm{Re}\,(\lambda) < 0\}.$$

We see that $\sigma(A + PL)$ is discrete, then $A + PL$ satisfies the spectrum decomposition assumption (see [3], pp. 75).

Since $\det A_q \neq 0$, the system of generalized eigenvectors of A is M_2-complete (see [9], Theorem 5.4). By [11], Lemma 5.4, we have the result. □

Lemma 3.2. *Let $\{\lambda_1', ..., \lambda_p'\} \subset \mathbb{C}$, if rank $\begin{pmatrix} \Delta(\lambda) \\ C_0 \end{pmatrix} = n, \forall \lambda \in \Lambda$, where*

$$\Lambda = \{\lambda \in \sigma(A) \ : \ Re\,(\lambda) \geq \ 0\} = \{\lambda_1, ..., \lambda_p\}.$$

Then there exists a bounded mapping $G : \mathbb{R}^r \longrightarrow M_2$ such that:

$$\sigma(A + GC) = \{\lambda_1', ..., \lambda_p'\} \cup \{\lambda \in \sigma(A) : Re\,(\lambda) < 0\} \text{ and } Im\,(G) \subset \mathcal{M},$$

where

$$\mathcal{M} = \{x \in M_2/\ \exists\lambda \in \mathbb{C}, \exists n \in \mathbb{N} : (\lambda I - A)^n x = 0\}\,.$$

Proof. As rank $\begin{pmatrix} \Delta(\lambda) \\ C_0 \end{pmatrix} = n$ for all $\lambda \in \Lambda$, then the pair (C, A) is detectable. Using the spectrum decomposition satisfied by A and similar arguments as in [10] or as in the proof of [2], Proposition 3.2], we get the desired result. □

Proposition 3.3. *If the following holds:*

1. $detA_q \neq 0$,

2. rank $\begin{pmatrix} \Delta(\lambda) \\ C_0 \end{pmatrix} = rank\,[\Delta(\lambda)|P_0] = n \qquad \forall\lambda \in \Lambda.$

Then there exists a stabilizing compensator of finite order for (1)-(8).

Proof. As $\sigma(A)$ is discrete, there exists $\beta > 0$ such that for all $\lambda \in \{\lambda \in \sigma(A) : \mathrm{Re}\,(\lambda) < 0\}$, $\mathrm{Re}\,(\lambda) < -\beta$ and let $\{\lambda'_1, ..., \lambda'_p\} \subset \mathbb{C}$ with $-\beta < \mathrm{Re}\,(\lambda'_i) < -\frac{\beta}{2}$, $(1 \leq i \leq p)$.

From Lemma 3.1, there exists $G : \mathbb{R}^r \to M_2$ such that $A + GC$ generates a stable semigroup $S(t)$ and $\mathrm{Im}\, G \subset \mathcal{M}$.

Let $b \geq 1$ and $\alpha > 0$ such that $\|S(t)\| \leq be^{-\alpha t} \quad \forall t \geq 0$ and $\widetilde{G} : \mathbb{R}^r \longrightarrow M_2$:

$$\|G - \widetilde{G}\| \leq \frac{\alpha}{2b}\|C\|\,. \tag{14}$$

The operator $A + \widetilde{G}C$ will generate a semi group $\tilde{S}(t)$ with $\|\tilde{S}(t)\| \leq be^{-\frac{\alpha}{2}t}$.

From Lemma 3.1 and [10], there exists $L : M_2 \longrightarrow \mathbb{R}^r$ such that $A + PL$ generates a stable semigroup with

$$\sigma(A + PL) = \{\lambda'_1, ..., \lambda'_p\} \cup \{\lambda \in \sigma(A) : \mathrm{Re}\,(\lambda) < 0\} \text{ and } \overline{\mathcal{M}'} = M_2,$$

where $\overline{\mathcal{M}'} = \{x \in M_2/\exists\lambda \in \mathbb{C},\ \exists\ n \in \mathbb{N} : [\lambda I - (A + PL)]^n x = 0\}$.

Let $\{y_1, ..., y_r\}$ be an orthogonal basis of $\mathbb{R}^r$, defined by $g_i = Gy_i$, $1 \leq i \leq r$. Since $\mathrm{Im}\, G \subset \mathcal{M} \subset M_2$, there exists for every $i \in \{1, ..., r\}$, a finite set $\{\phi_{i1}, ..., \phi_{iN_i}\} \subset \mathcal{M}'$ and suitable numbers α_{ij} $(i = 1, ...r;\ j = 1, ..., N_i)$ such that

$$\|g_i - \sum_{i=1}^{N_i} \alpha_{ij}\phi_{ij}\| \leq \frac{\alpha}{2b}\|C\|.$$

But $\phi_{ij} \in \mathcal{M}'$ for every indices (i, j), there exists $\lambda_{ij} \in \mathbb{C}$ and $n_{ij} \in \mathbb{N}^*$ such that

$$[\lambda_{ij} I - (A + PL)]^{n_{ij}} \phi_{ij} = 0.$$

Now define the subspace of M_2 as follows

$$\mathcal{V} = \text{span}\,\{[\lambda_{ij} I - (A + PL)]^k \phi_{ij} / i = 1, ..., r; j = 1, ..., N_i \text{ and } k = 0, ..., n_{ij} - 1\}.$$

Then it is clear that $\mathcal{V}$ is a finite-dimensional subspace, such that

$$\mathcal{V} \in \mathcal{D}(A + PL) = \mathcal{D}(A) \text{ and } (A + PL)\mathcal{V} \subset \mathcal{V}.$$

Let us write $\tilde{g}_i = \sum_{i=1}^{N_i} \alpha_{ij}\phi_{ij}$ and define $\tilde{G} : \mathbb{R}^r \longrightarrow M_2$ by $\tilde{G}y_i = \tilde{g}_i$. We then have $\text{Im}\,\tilde{G} \subset \mathcal{V}$.

Moreover, (14) holds and so $A + \bar{G}C$ generates a stable semigroup. We can apply Proposition 3.2, using the subspace $\mathcal{V}$ and mappings L and $\tilde{G}$, to conclude that there exists a stabilizing compensator of finite order (equal to $\dim \mathcal{V}$). □

The condition (i) of Proposition 3.3 is solely sufficient as shown by the following example.

Example 2. Let us consider the retarded stochastic system:

$$\begin{cases} d\xi(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \xi(t)dt + \begin{pmatrix} -\frac{\pi}{2} & 0 \\ 0 & 0 \end{pmatrix} \xi(t-1)dt + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t)dt + dw(t)\,, \\ \xi(0) = \phi^0 \quad \text{and} \quad \xi(s) = \phi^1(s) \quad \text{for} \quad s \in [-1, 0[\,, \end{cases} \tag{15}$$

and the observation

$$dy(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} \xi(t)dt + dv(t), \tag{16}$$

where $\xi(t) \in \mathbb{R}^2$, $y(t) \in \mathbb{R}$, $u(t) \in \mathbb{R}$, $w(t)$ and $v(t)$ are respectively Wiener processes to values in $\mathbb{R}^2$ and $\mathbb{R}$. First we notice that the set of the generalized eigenfunctions of A is not complete because $det \begin{pmatrix} -(\pi/2) & 0 \\ 0 & 0 \end{pmatrix} = 0$ but it is F-complete (see [9]).

Let us take the space of state $M_2 = M_2' \oplus \mathbb{R}$ with $M_2' = \mathbb{R} \times L^2([-1, 0], \mathbb{R})$, (15) becomes

$$\begin{cases} dx(t) = Ax(t)dt + Pu(t)dt + Bdw(t)\,, \\ x(0) = \begin{pmatrix} (\phi_1^0,\ \phi_1^1) \\ \phi_2^0 \end{pmatrix}, \end{cases} \tag{17}$$

with

$$\mathcal{D}(A) = \left\{ \begin{pmatrix} (\phi_0,\ \phi) \\ \alpha \end{pmatrix} : \phi_0 \in \mathbb{R}, \phi\ \in w^{1,2}([-1, 0])\ \alpha \in \mathbb{R} \text{ and } \phi(0) = \phi_0 \right\},$$

$A\begin{pmatrix} (\phi_0,\ \phi) \\ \alpha \end{pmatrix} = \begin{pmatrix} (-\frac{\pi}{2}\phi(-1) \quad +\alpha, \phi') \\ 0 \end{pmatrix}$, $Pu = \begin{pmatrix} 0 \\ u \end{pmatrix}$ and

$$C\begin{pmatrix} (\phi_0,\ \phi) \\ \alpha \end{pmatrix} = \phi_0.$$

Here $x(t) = \begin{pmatrix} (\xi_1(t), \xi_{1t}(.)) \\ \xi_2(t) \end{pmatrix}$ with $\xi_{1t}(s) = \xi_1(t+s)$ for $s \in [-1, 0[$.

The characteristic function $det[\Delta(\lambda)]$ is given by

$$det[\Delta(\lambda)] = \begin{pmatrix} \lambda + \frac{\pi}{2}e^{-\lambda} & -1 \\ 0 & \lambda \end{pmatrix} = \lambda(\lambda + \frac{\pi}{4}e^{-\lambda}).$$

If we take $\alpha = 1$, we have $\Lambda = \{\lambda \in \sigma(A),\ \mathrm{Re}\,(\lambda) \leq -1\} = \{-i\frac{\pi}{2}, 0, i\frac{\pi}{2}\}$. M_2 is decomposed according to Λ of the following way: $M_2 = M_\Lambda \oplus Q_\Lambda$, with M_Λ eigenspace of A generated by $\{\phi_1, \phi_2, \phi_3\}$ where $\phi_1 = \begin{pmatrix} (1,\ \cos(\frac{\pi}{2}(.)) \\ 0 \end{pmatrix}$, $\phi_2 = \begin{pmatrix} (0,\ \sin(\frac{\pi}{2}(.)) \\ 0 \end{pmatrix}$ and $\phi_3 = \begin{pmatrix} (\frac{2}{\pi}, \frac{2}{\pi}) \\ 1 \end{pmatrix}$. (A, P) is stabilizable and (C, A) is detectable because

$$\text{rank } = \begin{pmatrix} \lambda + \frac{\pi}{2}e^{-\lambda} & -1 & 0 \\ 0 & \lambda & 1 \end{pmatrix} = \text{rank} \begin{pmatrix} \lambda + \frac{\pi}{2}e^{-\lambda} & -1 \\ 0 & \lambda \\ 1 & 0 \end{pmatrix} = 2 \quad (see[10]).$$

Let $L : M_2 \longrightarrow \mathbb{R}$ be defined by

$$L = \begin{pmatrix} (\phi_0, \phi) \\ \alpha \end{pmatrix} = -3\phi_0 + \int_{-1}^{0} [(\frac{\pi}{2} - 1)\cos(\frac{\pi}{2}s) - 3\frac{\pi}{2}\sin(\frac{\pi}{2}s)]\phi(s)ds - 3\alpha.$$

$\lambda \in \sigma(A+PL)$ if and only if λ is root of $(\lambda+1)[(\lambda+1)^2+\frac{\pi^2}{4}](\lambda+\frac{\pi}{2}e^{-\lambda})(\lambda^2+\frac{\pi^2}{4})^{-1}$.

We notice that $\sigma(A + PL) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}\,(\lambda) \leq -1\}$, therefore $A + PL$ generates a stable semi group. The eigenvalues $\{-i\frac{\pi}{2}, 0\ , i\frac{\pi}{2}\}$ are replaced by $\{-1 - i\frac{\pi}{2},\ -1,\ -1 + i\frac{\pi}{2}\}$, the another are not changed (see [11]).

Let $\mathcal{V}$ the eigenspace of $A+PL$ associated to eigenvalues $\{-1-i\frac{\pi}{2}, -1, -1+i\frac{\pi}{2}\}$. $\mathcal{V}$ is generated by $\{w_1, w_2, w_3\}$, with: $w_1 = \begin{pmatrix} (1,\ e^{-(.)}\cos(\frac{\pi}{2}(.)) \\ -1 \end{pmatrix}$, $w_2 = \begin{pmatrix} (0,\ e^{-(.)}\sin(\frac{\pi}{2}(.)) \\ -1 \end{pmatrix}$ and $w_3 = \begin{pmatrix} (1,\ e^{-(.)} \\ -1 + \frac{\pi}{2}\theta \end{pmatrix}$. We see that $\forall x \in \mathcal{V} \quad (A + PL)x \in \mathcal{V}$. If we calculate $G : \mathbb{R} \longrightarrow M_2$ such that $\mathrm{Im}\, G \subset \mathcal{V}$ and $A + GC$ generates a stable semi group, we find (to see [11]):

$$G_\alpha = \alpha g \quad with\ g = -\pi(-0.66w_1 + 2.89w_2 + 2.66w_3).$$

Now we are going to look for a stabilizing compensator for $(15) - (16)$ of order

3.

Let the isomorphism $R : \mathcal{V} \longrightarrow \mathbb{R}^3$ defined by $R(a_1w_1 + a_2w_2 + a_3w_3) = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $z(t)$ and $u(t)$ takes values respectively in $\mathbb{R}^3$ and $\mathbb{R}$ and given by:

$$dz(t) = R(A + PL + GC)R^{-1}z(t)dt - RGdy(t), \quad z(0) = z_0\,, \tag{18}$$

$$u(t) = LR^{-1}z(t)\,. \tag{19}$$

While calculating $R(A + PL + GC)R^{-1}$, RG, LR^{-1}, we get

$$R(A + PL + GC)R^{-1} = \begin{pmatrix} 1.08 & 1.57 & 2.08 \\ -10.65 & -1 & -9.08 \\ -8.36 & 0 & -9.36 \end{pmatrix}, \quad RG = \begin{pmatrix} 2.08 \\ -9.08 \\ 8.36 \end{pmatrix},$$

and $LR^{-1} = (5.24 \quad 1.13 \quad -3.27)$. Therefore (18) and (19) are a stabilizing compensator of (15)-(16) it is necessary that:

$$\bar{A} = \begin{pmatrix} A & PLR^{-1} \\ -RGC & R(A + PL + GC)R^{-1} \end{pmatrix}$$

generate a stable semi group on the space $M_2 \oplus \mathbb{R}^3$.

Let us put

$$\mathcal{M}' = \left\{ \begin{pmatrix} x \\ Rx \end{pmatrix} : x \in \mathcal{V} \right\} = \left\{ \begin{pmatrix} R^{-1}w \\ w \end{pmatrix} : w \in \mathbb{R}^3 \right\}.$$

There exists an isomorphism between $\mathbb{R}^6$ and $\mathcal{M}'$ given by:

$$T : w \longrightarrow \begin{pmatrix} R^{-1}w \\ w \end{pmatrix}.$$

Hence $\mathcal{H} : M_2 \oplus \mathbb{R}^6 \longrightarrow M_2 \oplus \mathcal{M}'$ given by:

$$\mathcal{H} \begin{pmatrix} x \\ w \end{pmatrix} = \begin{pmatrix} x - R^{-1}w \\ Tw \end{pmatrix}$$

then $\mathcal{H}$ is an isomorphism and $\mathcal{H}^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + R^{-1}T^{-1}y \\ T^{-1}y \end{pmatrix}$.

Therefore $\mathcal{H}\bar{A}\mathcal{H}^{-1} = \begin{pmatrix} A + GC & 0 \\ TRGC & TR(A + PL)R^{-1}T^{-1} \end{pmatrix}$.

As $A + GC$ and $A + PL$ generate some stable semi groups then $\mathcal{H}\bar{A}\mathcal{H}^{-1}$ generates a stable semi group on $M_2 \oplus \mathcal{M}'$ and therefore $\bar{A}$ generates a stable semi group $M_2 \oplus \mathbb{R}^6$. Therefore (18) and (19) is a stabilizing compensator of finite order for (15)-(16).

4. Conclusion

In this paper, we have considered the stochastic stability for delay systems. Using the semigroup approach, we have stabilized the system by feedback on its state if it has a complete observation or by feedback on a compensator if it is partially observed.

By spectrum decomposition, we have designed some results for the existence of stabilizing compensator of finite order.

Other properties of interests, such as, the stabilization for the retarded stochastic systems with delay in control and observation, the stochastic control using the theory of compensators or estimators, are under investigation.

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