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ON THE COMPLEMENTARY PAIRS OF SOME POLYNOMIALS

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Abstract: In this paper we first consider a class of polynomial pair whose coefficients lie on the unit circle $\mathbf{T}$ and then we show some properties of their complementary pairs.

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1. Introduction

We define the class $\mathcal{T}_n$ as a collection of all (complex) polynomials of degree $n \geq 1$ so that if $p \in \mathcal{T}_n$, then $p(z) = \sum_0^n c_k z^k$ with each c_k a complex number and $|c_k| = 1$. One can easily check that by Parseval's formula, $\int_0^{2\pi} |p_n(e^{it})|^2 dt = 2\pi(n+1)$ and so $\min |p(z)| \leq \sqrt{n+1} \leq \max |p(z)|$, where both min and max are taken over all z with $|z| = 1$. Let $\mathcal{S}_n \subset \mathcal{T}_n$ be such that if if $p \in \mathcal{S}_n$ with $p(z) = \sum_0^n c_k z^k$, then $c_k = \pm 1$ for all k. For $0 < q < \infty$ and $f \in \mathcal{S}_n$ put

$$||f||_q = (\frac{1}{2\pi} \int_0^{2\pi} |f(e^{it})|^q dt)^{1/q}.$$

If $q \geq 1$, this is the usual L_q norm of f on the unit circle. We also define $||f||_\infty$ denote the suprimum norm of f,

$$||f||_\infty = \lim_{q \to \infty} ||f||_q = \sup_{|z|=1} |f(z)|,$$

and we let $||f||_0$ denote its geometric mean on the unit circle,

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$$||f||_0 = \lim_{q\to 0+} ||f||_q = \exp\left(\int_0^1 \log|f(e^{2\pi it})|dt\right).$$

This is *Mahler's measure* of the polynomial. We recall that if $p < q$ are positive real numbers and f is not monomial, then

$$||f||_0 < ||f||_q < ||f||_q < ||f||_\infty.$$

Let $p \in \mathcal{S}_n$ and write

$$p(z) = \sum_{k=0}^{n} a_k z^k,$$

where $a_k = \pm 1$ for all k. Define *the conjugate reciprocal polynomial of* p by $p^*(z) := z^n \bar{p}(1/z)$. One can easily verify that $p^*(z) = \sum_{k=0}^{n} \bar{a}_{n-k} z^k$ and moreover

$$\int_{\mathbf{T}} |p(z) - p^*(z)|^2 |dz| = 2n + \circ(n),$$

where $\circ(n)$ denotes a quality for which $\lim_{n\to\infty} \circ(n)/n = 0$. Next note that if $\check{p}(z) = p(-z)$, then $(\check{p})^* = (-1)^{deg(p)} (\check{p^*})$.

We now define the classical Rudin-Shapiro polynomials p_n and q_n in $\mathcal{S}_n$ inductively as follows: Setting $p_0 = q_0 = 1$, put

$$p_n = p_{n-1} + e^{i2^{n-1}t} q_{n-1}, \quad q_n = p_{n-1} - e^{i2^{n-1}t} q_{n-1} \quad (n \in \mathbf{N}). \tag{1}$$

Thus

$$p_n(t) = \sum_{k=0}^{2^n-1} \epsilon_k e^{ikt}, \quad q_n(t) = \sum_{k=0}^{2^n-1} \delta_k e^{ikt},$$

where ϵ_k and δ_k only take the values of $+1$ and -1. To see the Rudin-Shapiro polynomials satisfy complementary pair (1), first note that both $p_n(t)$ and $q_n(t)$ have length 2^n. For $n = 0$, we have $|p_0(t)|^2 + |q_0(t)|^2 = 2$ and so p_0 and q_0 form a complementary pair. Suppose that for some $n \geq 0$, $p_{n-1}(t)$ and $q_{n-1}(t)$ form a complementary pair, that is $|p_{n-1}(t)|^2 + |q_{n-1}(t)|^2 = 2 \times 2^{n-1} = 2^n$. Then by (1), we have $|p_n(t)|^2 + |q_n(t)|^2 = 2(|p_{n-1}(t)|^2 + |q_{n-1}(t)|^2)$. Hence

$$|p_n(t)|^2 + |q_n(t)|^2 = 2 \times 2^n = 2^{n+1}.$$

2. The Main Results

Letting f_n to be one of the Rudin-Shapiro polynomials p_n or q_n, if we define

$$(f_n)^\wedge(k) = \frac{1}{2\pi}\int_0^{2\pi} f_n(e^{it})e^{-ikt}dt,$$

then $(f_n)^\wedge(k) = (f_{n+1})^\wedge(k)$ for every $|k| < 2^n$. To see this; let $|k| < 2^n$, then $f_{n+1}(e^{it}) = p_n(e^{it}) \pm e^{i2^n t}q_n(e^{it})$ and $deg(e^{i2^n t}q_n(e^{it})) \geq 2^n$. So for all k such that $|k| < 2^n$

$$(f_{n+1})^\wedge(k) = (p_n)^\wedge(k) \pm (e^{i2^n t}q_n(e^{it}))^\wedge(k) = (f_n)^\wedge(k).$$

Moreover $||p_n||_q = ||q_n||_q$ holds for all $q \in \mathbf{N}$. To see this we only need to show that for every $n \geq 0$,

$$q_n(e^{it}) = (-1)^n e^{(2^n-1)it}p_n(-e^{-it}). \tag{3}$$

Clearly (3) holds for $n = 0$ and $n = 1$. Assume the relation holds for any $n \geq 1$. Then

$$\begin{aligned}
&(-1)^{n+1}e^{(2^{n+1}-1)it}p_{n+1}(-e^{-it}) \\
&\quad = (-1)^{n+1}e^{(2^{n+1}-1)it}[p_n(-e^{-it}) + (-e^{-it})^{2^n}q_n(-e^{-it})] \\
&\quad\quad = (-1)^{n+1}e^{(2^{n+1}-1)it}[(-1)^n q_n(e^{it})/e^{(2^n-1)it} \\
&\quad\quad\quad + e^{(-2^n)it}(-1)^n(-e^{-it})^{2^n-1}p_n(e^{it})] \\
&\quad\quad\quad\quad = p_n(e^{it}) - e^{(2^n)it}q_n(e^{it}) = q_{n+1}(e^{it}).
\end{aligned}$$

Therefore, by induction we have (3).

Next let $q \geq 2$ be a fixed integer and consider a polynomial of the form

$$p(z) = A_1(z)A_2(z)\cdots A_q(z),$$

where $A_k \in \{p_n, \check{p_n}, p_n^*, \check{p_n^*}\}$ and where p_n is the n^{th} Rudin-Shapiro polynomial of degree $2^n - 1$. Hence

$$p(z) = \sum_{k=0}^{q(2^n-1)} b_k z^k$$

is a unimodular polynomial. Moreover $\|p\|_1 \leq cL^{\frac{q}{2}}$ (the proof is similar to that $\|p_n\|_{\mathbf{C}} \leq \sqrt{2}L^{\frac{1}{2}}$ shown in [2]).

Case 1. Suppose that:

(i) *q is an even integer*, and

(ii) $A \in \{p_n, \check{p_n}, p_n^*, \check{p_n^*}\}$ implies that $A^* \in \{p_n, \check{p_n}, p_n^*, \check{p_n^*}\}$.

The central coefficient of p here is the same order as $\|p\|$ and for every k there exist $\delta_q > 0$ and c_q both depended only on q so that

$$|b_k| \leq c_q 2^{n(\frac{q}{2}-\delta_q)}). \tag{4}$$

Of course the central coefficient $k = 0$ case excluded in (4).

Case 2. Suppose that niehter (i) nor (ii) in case 1 hold. Then again for every k there exist $\delta_q > 0$ and c_q both depended only on q so that

$$\max_{0 \leq k \leq q(2^n-1)} |b_k| \leq c_q 2^{n(\frac{q}{2}-\delta_q)}.$$

It is surprising that the set of polynomials that their coefficients taken from unit circle, namely *Unimodular polynomials*, has been subject to extensive investigations throughout the past 50 years. Looking difficult conjectures has been solved by simple methods while simple questions still remains unanswered. Marcel Goley [1] introduced the complementary sequences in the context of infrared spectrometry. Two unimodular polynomials $p(z)$ and $q(z)$ of the same length N are said to form *complementary pair* if they satisfy, for all z in the unit circle (that is $|z| = 1$),

$$|p(z)|^2 + |p(z)|^2 = 2N.$$

The interest in the complementary polynomials pair begun when applications was found for Ising spin systems in physics, orthogonal designs and Hadamard matrices in Combinatorics and most of all channel-measurement in Telecommunications. Digital communication engineers have sought to identify them for which the absolute values of the aperiodic autocorrelation are collectively small, for application in radar synchronization. The construction of Unimodular polynomials dates back to the beginning of 20-th century. At that time the purpose was not to apply them to Physics, designing signals or Telecommunications. The incitement then was rather a mathematical interest in certain interesting trigonometric series. Of course research on the polynomials on the unit circle has continued largely independently in the mathematical community for many years. A remarkable type of the complementary pair Unimodular polynomials was discovered by Shapiro in his 1951 Master thesis [4] on which according to his memos, he accidently made the discovery as he had many stimulating conversations with fellow student D.J. Newman about the Fejer-Riesz Theorem on non negative trigonometric polynomials. A short summary of Shapiro's appears in 1953 and followed by an extremal work by W. Rudin in [3]. Since 1959, the course of those studies was led to the Rudin-Shapiro polynomials.

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