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ON A CERTAIN SUBCLASS OF HARMONIC UNIVALENT FUNCTIONS BY APPLYING AN OPERATOR

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Abstract: In the present paper, we study various properties of the harmonic univalent functions with analytic nature by applying the operator $\mathcal{I}_{\sigma}^{\delta}$. We have obtained coefficient estimate for a function to be in the class $W_{\mathcal{H}CV}(k,\beta,\lambda,\delta,\sigma)$ of harmonic univalent functions. We have also obtained convolution condition, convex combination, distortion bounds and extreme points for harmonic univalent functions.

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1. Introduction

A continuous function $f = u + iv$ is a complex-valued harmonic function in a domain $D \subset \mathbb{C}$ if both u and v are real harmonic in D. We can write $f = h+\overline{g}$, where h and g are analytic in D.

Definition 1.1. Let $f = u+iv$ be harmonic in D, then there exist analytic functions G, H such that $u = \operatorname{Re} G$ and $v = \operatorname{Im} H$, therefore $f = u+iv = h+\overline{g}$, where $h = \frac{G+H}{2}, \overline{g} = \frac{\overline{G}-\overline{H}}{2}$ and we call h and g analytic part and co-analytic part of f respectively.

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The Jacobian of f is given by $\mathcal{J}_f(z) = |h'(z)|^2 - |g'(z)|^2$, also we show by $w(z)$ the dilatation function for f and define $w(z) = \frac{g'(z)}{h'(z)}$. Lewy [11], Clunie and Sheil-Small [3] have shown that the mapping $z \to f(z)$ is sense preserving and locally injective in D if and only if $\mathcal{J}_f > 0$ in D.

Definition 1.2. Let f be a harmonic function ($f = h + \overline{g}$), then f is said to be univalent in D if the mapping $z \to f(z)$ is sense preserving and injective in D. We denote by $\mathcal{H}$ the class of all harmonic functions $f = h + \overline{g}$ that are univalent and sense preserving in the open unit disk $U = \{z : |z| < 1\}$, where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1 \tag{1.1}$$

with the normalization conditions $f(0) = 0, f_z(0) = 1$, where $f_z(0)$ denote the partial derivative of $f(z)$ at $z = 0$.

In case $g \equiv 0$ this class reduces to the class T of analytic univalent functions, this class with famous subclass $S^*(\beta)$ (starlike of order β) and $C(\beta)$ (convex of order β) were studied by Robertson [13], MacGregor [12], Jack [4] and others.

Also denote by $\overline{\mathcal{H}}$ the subclass of $\mathcal{H}$ consisting of the functions $f = h + \overline{g}$, where h and g take the form

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad g(z) = \sum_{n=1}^{\infty} |b_n| z^n, \quad |b_1| < 1. \tag{1.2}$$

We denote by $k - UCV, 0 \leq k < \infty$, the class of k-uniformly convex univalent functions and we say that $h(z) \in k - UCV$ if and only if

$$\operatorname{Re}\left\{1 + (1 + ke^{i\theta})\left(\frac{zh''(z)}{h'(z)}\right)\right\} \geq \beta, \quad \text{for real} \quad \theta$$

and $0 \leq \beta < 1$ (see Kanas and Wiśniowska [9], and Kanas and Srivastava [8]). The class $\mathcal{H}$ with some different subclasses are investigated by many authors, for example Jahangiri [5], [6], Jahangiri and Silverman [7], Rosy et al [14], Clunie and Sheil-Small [3] and others.

Definition 1.3. The operator of $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$ for $\delta \geq 0$, $\sigma \geq 0$ is denoted by $\mathcal{I}_\sigma^\delta$ and defined as following:

$$\mathcal{I}_\sigma^\delta f(z) = z + \sum_{n=2}^{\infty} \left(\frac{n+\delta}{1+\delta}\right)^\sigma a_n z^n \qquad (z \in U). \tag{1.3}$$

The operator $\mathcal{I}_\sigma^\delta$ was studied recently by Cho and Srivastava [2] and Cho and Kim [1], see also [16] and [17].

Also we define the operator $\mathcal{I}_\sigma^\delta$ of harmonic functions $f = h + \overline{g}$ in $\mathcal{H}$ by

$$\mathcal{I}_\sigma^\delta f(z) = \mathcal{I}_\sigma^\delta h(z) + \overline{\mathcal{I}_\sigma^\delta g(z)}\,, \tag{1.4}$$

where $\mathcal{I}_\sigma^\delta f(z)$ is given by (1.3).

Definition 1.4. Let $f = h+\overline{g}$, be the harmonic function of the form (1.1), then $f \in W_{\mathcal{H}CV}(k, \beta, \lambda, \delta, \sigma)$ if and only if

$$\begin{aligned}\mathrm{Re}\,\{(1+ke^{i\theta})[z(\mathcal{I}_\sigma^\delta h(z))' + \lambda z^2(\mathcal{I}_\sigma^\delta h(z))'' \\ + \overline{(2\lambda-1)z(\mathcal{I}_\sigma^\delta g(z))' + \lambda z^2(\mathcal{I}_\sigma^\delta g(z))''}]/[(1-\lambda)(\mathcal{I}_\sigma^\delta h(z) \\ + \overline{\mathcal{I}_\sigma^\delta g(z)}) + \lambda(z(\mathcal{I}_\sigma^\delta h(z))' - \overline{z(\mathcal{I}_\sigma^\delta g(z))'})] - ke^{i\theta}\} \geq \beta\,,\end{aligned} \tag{1.5}$$

where $0 \leq \theta \leq 2\pi, 0 \leq \lambda \leq 1, k \geq 0, 0 \leq \beta < 1, \delta \geq 0$ and $\sigma \geq 0$. We note that when $\lambda = 1, \sigma = 0$, the family $W_{\mathcal{H}CV}(k, \beta, \lambda, \delta, \sigma)$ reduces to the class $W_{\mathcal{H}CV}(k, \beta)$, the class of uniformly convex analytic functions defined by [10]. Also when $k = 1, \lambda = 1$ and $\sigma = 0$, we obtain the class of harmonic convex function $GK_{\mathcal{H}}(\beta)$, which is defined by [15], when $k = 1, \lambda = 0$ and $\sigma = 0$, we get the class of harmonic starlike function $G_{\mathcal{H}}(\beta)$, studied by [14].

2. Main Results

We need the following lemma due to [6].

Lemma 2.1. *Let f be of the form (1.1). If*

$$\sum_{n=1}^{\infty}[(n-\beta)|a_n| + (n+\beta)|b_n|] \leq 1-\beta,$$

then f is harmonic, orientation preserving univalent in U.

We begin by proving a sufficient condition for functions to be in $W_{\mathcal{H}CV}(k,$ $\beta,\ \lambda, \delta, \sigma)$.

Theorem 2.1. *Let $f = h + \overline{g}$ be given by (1.1) and*

$$\sum_{n=2}^{\infty} \frac{(1+n\lambda-\lambda)\left(\frac{n+\delta}{1+\delta}\right)^\sigma (n+nk-k-\beta)}{1-\beta}|a_n| + \sum_{n=1}^{\infty} \frac{(1+n\lambda-\lambda)\left(\frac{n+\delta}{1+\delta}\right)^\sigma (n+nk+k+\beta)}{1-\beta}|b_n| \leq 1\,, \tag{2.1}$$

where $a_1 = 1, k \geq 0, 0 \leq \lambda \leq 1, 0 \leq \beta < 1, \delta \geq 0$ and $\sigma \geq 0$. Then f is harmonic univalent in U and $f \in W_{\mathcal{H}CV}(k, \beta, \lambda, \delta, \sigma)$.

Proof. We have to confirm that f is locally univalent and sense-preserving in U. Since $(1+n\lambda-\lambda)\geq 1, \frac{n+\delta}{1+\delta}\geq 1, n-\beta\leq(n+nk-k-\beta)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}(1+n\lambda-\lambda)$ and $n+\beta\leq(1+n\lambda-\lambda)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}(n+nk+k+\beta)$ for $k\geq 0, 0\leq\lambda\leq 1$, it follows from Lemma 2.1 that f is harmonic, sense-preserving, univalent.

On the other hand, it is sufficient to show that if (2.1) holds, then:

$$\begin{aligned}\mathrm{Re}\,\{&(1+ke^{i\theta})[z(\mathcal{I}_{\sigma}^{\delta}h(z))'+\lambda z^2(\mathcal{I}_{\sigma}^{\delta}h(z))''\\&+\overline{(2\lambda-1)z(\mathcal{I}_{\sigma}^{\delta}g(z))'+\lambda z^2(\mathcal{I}_{\sigma}^{\delta}g(z))''}]/[(1-\lambda)(\mathcal{I}_{\sigma}^{\delta}h(z)\\&+\overline{\mathcal{I}_{\sigma}^{\delta}g(z))}+\lambda(z(\mathcal{I}_{\sigma}^{\delta}h(z))'-\overline{z(\mathcal{I}_{\sigma}^{\delta}g(z))'})]-ke^{i\theta}\}\geq\beta\,,\end{aligned}$$

where $z=re^{i\theta}, 0\leq\theta\leq 2\pi, 0\leq r<1, k\geq 0, 0\leq\beta<1, 0\leq\lambda\leq 1, \delta\geq 0$ and $\sigma\geq 0$. Let

$$\begin{aligned}E(z)=&[z(\mathcal{I}_{\sigma}^{\delta}h(z))'+\lambda z^2(\mathcal{I}_{\sigma}^{\delta}h(z))''\\&+\overline{(2\lambda-1)z(\mathcal{I}_{\sigma}^{\delta}g(z))'+\lambda z^2(\mathcal{I}_{\sigma}^{\delta}g(z))''}](1+ke^{i\theta})-ke^{i\theta}[(1-\lambda)\\&\times(\mathcal{I}_{\sigma}^{\delta}h(z)+\overline{\mathcal{I}_{\sigma}^{\delta}g(z)})+\lambda(z(\mathcal{I}_{\sigma}^{\delta}h(z))'-\overline{z(\mathcal{I}_{\sigma}^{\delta}g(z))'})]\end{aligned}$$

and

$$F(z)=(1-\lambda)(\mathcal{I}_{\sigma}^{\delta}h(z)+\overline{\mathcal{I}_{\sigma}^{\delta}g(z)})+\lambda(z(\mathcal{I}_{\sigma}^{\delta}h(z))'-\overline{z(\mathcal{I}\delta_{\sigma}g(z))'}).$$

Using the fact that $\mathrm{Re}\,(w)\geq\beta$ if and only if $|w+1-\beta|\geq|w-(1+\beta)|$. We want to show that

$$|E(z)+(1-\beta)F(z)|-|(1+\beta)F(z)-E(z)|\geq 0. \tag{2.2}$$

Substituting for $E(z)$ and $F(z)$ in (2.2), we get

$$\begin{aligned}&|E(z)+(1-\beta)F(z)|-|(1+\beta)F(z)-E(z)|\\&=|(2-\beta)z+\sum_{n=2}^{\infty}(1+n\lambda-\lambda)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}[n+1-\beta\\&+ke^{i\theta}(n-1)]a_nz^n+\sum_{n=1}^{\infty}n\lambda\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}[(n-1)+\beta\\&+ke^{i\theta}(n+1)]\overline{b}_n\overline{z}^n-\sum_{n=1}^{\infty}(1-\lambda)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}[n-1+\beta\\&+ke^{i\theta}(n+1)]\overline{b}_n\overline{z}^n-|\beta z-\sum_{n=2}^{\infty}(1+n\lambda-\lambda)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}[n-1-\beta\\&+ke^{i\theta}(n-1)]a_nz^n-\sum_{n=1}^{\infty}n\lambda\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}\end{aligned}$$

$$\times [n+1+\beta+ke^{i\theta}(n+1)]\overline{b}_n\overline{z}^n - \sum_{n=1}^{\infty}(1-\lambda)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}[n+1+\beta$$
$$+ke^{i\theta}(n+1)]\overline{b}_n\overline{z}^n| \geq (2-\beta)|z| - \sum_{n=2}^{\infty}(n\lambda+1-\lambda)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}$$
$$\times (n+nk-\beta-k-1)|a_n||z|^n - \sum_{n=1}^{\infty}(n\lambda+1-\lambda)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}$$
$$\times (n+nk+k+\beta-1)|b_n||z|^n - \beta z - \sum_{n=2}^{\infty}(n\lambda+1-\lambda)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}$$
$$\times (n+nk-k-\beta+1)|a_n|z|^n - \sum_{n=1}^{\infty}(n\lambda+1-\lambda)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}$$
$$\times (n+nk+k+\beta+1)|b_n||z|^n \geq 2(1-\beta)|z| - 2\sum_{n=2}^{\infty}(1+n\lambda-\lambda)$$
$$\times \left(\frac{n+\delta}{1+\delta}\right)^{\sigma}(n+nk-k-\beta)|a_n||z|^n - 2\sum_{n=1}^{\infty}(1+n\lambda-\lambda)$$
$$\times \left(\frac{n+\delta}{1+\delta}\right)^{\sigma}(n+nk+k+\beta)|b_n||z|^n \geq 0 \qquad \text{(by (2.1))}.$$

To confirm the sharpness of the theorem, we have the harmonic function

$$f(z) = z + \sum_{n=2}^{\infty}\frac{(1-\beta)M_n z^n}{(1+n\lambda-\lambda)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}(n+nk-k-\beta)}$$
$$+\sum_{n=1}^{\infty}\frac{(1-\beta)N_n\overline{z}^n}{(1+n\lambda-\lambda)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}(n+nk+k+\beta)}, \quad (2.3)$$

where $\sum_{n=2}^{\infty}|M_n|+\sum_{n=1}^{\infty}|N_n|=1$, show that the coefficient bound given in Theorem 2.1 is sharp. Now

$$\sum_{n=2}^{\infty}\frac{(1+n\lambda-\lambda)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}(n+nk-k-\beta)}{1-\beta}|a_n|$$
$$+\sum_{n=1}^{\infty}\frac{(1+n\lambda-\lambda)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}(n+nk+k+\beta)}{1-\beta}|b_n| = \sum_{n=2}^{\infty}|M_n|+\sum_{n=1}^{\infty}|N_n| = 1.$$

Therefore the function of the form (2.3) are in $W_{\mathcal{H}CV}(k,\beta,\lambda,\delta,\sigma)$. □

Theorem 2.2. *Let $f = h + \overline{g}$ with h and g of the form (1.2). Then $f \in W_{\overline{\mathcal{H}CV}}(k,\beta,\lambda,\delta,\sigma)$ if and only if*

$$\sum_{n=2}^{\infty} \frac{(1+n\lambda-\lambda)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}(n+nk-k-\beta)}{1-\beta}|a_n| + \sum_{n=1}^{\infty} \frac{(1+n\lambda-\lambda)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}(n+nk+k+\beta)}{1-\beta}|b_n| \leq 1\,, \quad (2.4)$$

where $a_1 = 1, k \geq 0, 0 \leq \lambda \leq 1, n \in I\!N, 0 \leq \beta < 1, \delta \geq 0$ and $\sigma \geq 0$.

Proof. Clearly, by Theorem 2.1, if $f(z)$ given by (1.2) satisfies (2.4), that is in $W_{\mathcal{H}CV}(k,\beta,\lambda,\delta,\sigma)$, then $f(z) \in W_{\overline{\mathcal{H}CV}}(k,\beta,\lambda,\delta,\sigma)$. Conversely it is sufficient to show that $f \notin W_{\overline{\mathcal{H}CV}}(k,\beta,\lambda,\delta,\sigma)$ if the condition (2.4) does not hold.

The necessary and sufficient condition for $f(z) = h(z) + \overline{g}(z)$ defined by (1.2), to be in $W_{\overline{\mathcal{H}CV}}(k,\beta,\lambda,\delta,\sigma)$, is that the condition (1.3) holds true, thus we must show that

$$\operatorname{Re}\{(1+ke^{i\theta})[z(\mathcal{I}_\sigma^\delta h(z))' + \lambda z^2(\mathcal{I}_\sigma^\delta h(z))'' + \overline{(2\lambda-1)z(\mathcal{I}_\sigma^\delta g(z))' + \lambda z^2(\mathcal{I}_\sigma^\delta g(z))''}] \\ /[(1-\lambda)(\mathcal{I}_\sigma^\delta h(z) + \overline{\mathcal{I}_\sigma^\delta g(z))} + \lambda(z(\mathcal{I}_\sigma^\delta h(z))' - \overline{z(\mathcal{I}_\sigma^\delta g(z))'})] \\ -(ke^{i\theta}+\beta)\} = \operatorname{Re}\left\{\frac{I(z)}{J(z)}\right\} \geq 0,$$

where

$$I(z) = (1-\beta)z - [\sum_{n=2}^{\infty}(1+n\lambda-\lambda)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}((n-\beta) \\ +(n-1)ke^{i\theta})|a_n|z^n + \sum_{n=1}^{\infty} n\lambda\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}((n+\beta) \\ +(n+1)ke^{i\theta})|b_n|\overline{z}^n - \sum_{n=1}^{\infty}(1-\lambda)((n+\beta)+(n+1)ke^{i\theta})\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}|b_n|\overline{z}^n]$$

and

$$J(z) = z - \left[\sum_{n=2}^{\infty}(1+n\lambda-\lambda)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}|a_n|z^n + \sum_{n=1}^{\infty}(n\lambda+\lambda-1)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}|b_n|\overline{z}^n\right].$$

By choosing the values of z on the positive real axis, where $0 \leq z = r < 1$ and since $\operatorname{Re}(-e^{i\theta}) \geq -|e^{i\theta}| = -1$, then we must show that

$$\frac{(1-\beta)-\sum_{n=2}^{\infty}(1+n\lambda-\lambda)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}(n+nk-k-\beta)|a_n|}{1-\left\{\sum_{n=2}^{\infty}(n\lambda+1-\lambda)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}|a_n|+\sum_{n=1}^{\infty}(n\lambda-1+\lambda)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}|b_n|\right\}}$$
$$-\frac{\sum_{n=1}^{\infty}(1+n\lambda-\lambda)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}(n+nk+k+\beta)|b_n|}{1-\left\{\sum_{n=2}^{\infty}(n\lambda+1-\lambda)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}|a_n|+\sum_{n=1}^{\infty}(n\lambda-1+\lambda)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}|b_n|\right\}}$$
$$\geq 0. \quad (2.5)$$

We note that (2.5) is negative for r sufficiently close to 1 when the condition (2.4) does not hold. Therefore there exists a $z_0 = r_0$ in (0, 1) for which the quotient in (2.5) is negative. Then as $f \in W_{\overline{\mathcal{H}}CV}(k,\beta,\lambda,\delta,\sigma)$ we get contradiction. This completes the proof. □

Now, we define the convolution of two harmonic functions.

Let us have harmonic functions

$$f(z) = z-\sum_{n=2}^{\infty}a_{n,1}z^n-\sum_{n=1}^{\infty}b_{n,1}\overline{z}^n \quad \text{and} \quad g(z) = z-\sum_{n=2}^{\infty}a_{n,2}z^n-\sum_{n=1}^{\infty}b_{n,2}\overline{z}^n.$$

Then we define the convolution of $f(z)$ and $g(z)$ as

$$(f*g)(z) = f(z)*g(z) = z-\sum_{n=2}^{\infty}a_{n,1}a_{n,2}z^n-\sum_{n=1}^{\infty}b_{n,1}b_{n,2}\overline{z}^n.$$

Theorem 2.3. *Let $f(z) \in W_{\overline{\mathcal{H}}CV}(k,\beta,\lambda,\delta,\sigma)$ and $g(z) \in W_{\overline{\mathcal{H}}CV}(k,\alpha,\lambda,\delta,\sigma)$, then for $0 \leq \alpha \leq \beta < 1$, we have*

$$(f*g)(z) \in W_{\overline{\mathcal{H}}CV}(k,\beta,\lambda,\delta,\sigma) \subset W_{\overline{\mathcal{H}}CV}(k,\alpha,\lambda,\delta,\sigma)\,.$$

Proof. Since $f(z) \in W_{\overline{\mathcal{H}}CV}(k,\beta,\lambda,\delta,\sigma)$, and $g(z) \in W_{\overline{\mathcal{H}}CV}(k,\alpha,\lambda,\delta,\sigma)$, then $f(z), g(z)$ satisfy the coefficient inequality (2.4). The coefficient of $f*g$ as

$$\sum_{n=1}^{\infty}\Bigg[\frac{(1+n\lambda-\lambda)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}(n+nk-k-\beta)}{1-\beta}|a_{n,1}a_{n,2}|$$
$$+\frac{(n\lambda-\lambda+1)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}(n+nk+k+\beta)}{1-\beta}|b_{n,1}b_{n,2}|\Bigg]$$
$$\leq \sum_{n=1}^{\infty}\Bigg[\frac{(1+n\lambda-\lambda)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}(n+nk-k-\beta)}{1-\beta}|a_{n,1}|$$

$$+\frac{(1+n\lambda-\lambda)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}(n+nk+k+\beta)}{1-\beta}|b_{n,1}|\Bigg] \quad (\text{as} \quad |a_{n,2}|<1, |b_{n,2}|<1)$$

$$\leq 2 \quad (\text{as} \quad f \in W_{\overline{\mathcal{H}CV}}(k,\beta,\lambda,\delta,\sigma)).$$

This completes the proof. □

Theorem 2.4. *The family $W_{\overline{\mathcal{H}CV}}(k,\beta,\lambda,\delta,\sigma)$ is closed under convex combination.*

Proof. Assume $f_i \in W_{\overline{\mathcal{H}CV}}(k,\beta,\lambda,\delta,\sigma)(i = 1,2,3,\cdots)$, are defined by $f_i(z) = z - \sum_{n=2}^{\infty} |a_{i,n}|z^n - \sum_{n=1}^{\infty} |b_{i,n}|\overline{z}^n$, from (2.4), we have

$$\sum_{n=1}^{\infty}\left[\frac{(1+n\lambda-\lambda)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}(n+nk-k-\beta)}{1-\beta}|a_{i,n}| + \frac{(n\lambda-\lambda+1)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}(n+nk+k+\beta)}{1-\beta}|b_{i,n}|\right] \leq 2. \quad (2.6)$$

For $\sum_{i=1}^{\infty} \ell_i = 1, 0 \leq \ell_i \leq 1$, we can write the convex combination of f_i as

$$\sum_{i=1}^{\infty} \ell_i f_i(z) = z - \sum_{n=2}^{\infty}\left(\sum_{i=1}^{\infty}\ell_i|a_{i,n}|\right)z^n - \sum_{n=1}^{\infty}\left(\sum_{i=1}^{\infty}\ell_i|b_{i,n}|\right)\overline{z}^n$$

from (2.6) we have

$$\sum_{n=1}^{\infty}\left[\frac{(1+n\lambda-\lambda)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}(n+nk-k-\beta)}{1-\beta}\sum_{i=1}^{\infty}\ell_i|a_{i,n}| + \frac{(n\lambda-\lambda+1)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}(n+nk+k+\beta)}{1-\beta}\sum_{i=1}^{\infty}\ell_i|b_{i,n}|\right]$$

$$= \sum_{i=1}^{\infty}\ell_i\left\{\sum_{n=1}^{\infty}\left[\frac{(1+n\lambda-\lambda)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}(n+nk-k-\beta)}{1-\beta}|a_{i,n}| + \frac{(n\lambda-\lambda+1)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}(n+nk+k+\beta)}{1-\beta}|b_{i,n}|\right]\right\} \leq 2\sum_{i=1}^{\infty}\ell_i \leq 2\,,$$

then $\sum_{i=1}^{\infty} \ell_i f_i(z) \in W_{\overline{\mathcal{H}CV}}(k,\beta,\lambda,\delta,\sigma)$. This completes the proof. □

Theorem 2.5. *If* $f \in W_{\overline{\mathcal{H}}CV}(k,\beta,\lambda,\delta,\sigma)$, *then*

$$|f(z)| \leq (1+|b_1|)r + \left(\frac{1-\beta}{(1+\lambda)\left(\frac{2+\delta}{1+\delta}\right)^{\sigma}(2+k-\beta)} - \frac{(1+2k+\beta)}{(1+\lambda)\left(\frac{2+\delta}{1+\delta}\right)^{\sigma}(2+k-\beta)}|b_1| \right) r^2,$$

$$|f(z)| \geq (1+|b_1|)r - \left(\frac{1-\beta}{(1+\lambda)\left(\frac{2+\delta}{1+\delta}\right)^{\sigma}(2+k-\beta)} - \frac{(1+2k+\beta)}{(1+\lambda)\left(\frac{2+\delta}{1+\delta}\right)^{\sigma}(2+k-\beta)}|b_1| \right) r^2.$$

Proof. Assume $f \in W_{\overline{\mathcal{H}}CV}(k,\beta,\lambda,\delta,\sigma)$, then we have

$$|f(z)| \leq (1+|b_1|)r + \sum_{n=2}^{\infty}(|a_n|+|b_n|)r^n \leq (1+|b_1|)r + \sum_{n=2}^{\infty}(|a_n|+|b_n|)r^2$$
$$\leq (1+|b_1|)r + \frac{1-\beta}{(1+\lambda)\left(\frac{2+\delta}{1+\delta}\right)^{\sigma}(2+k-\beta)}$$
$$\times \sum_{n=2}^{\infty}\left[\frac{(1+\lambda)\left(\frac{2+\delta}{1+\delta}\right)^{\sigma}(2+k-\beta)}{(1-\beta)}|a_n| + \frac{(1+\lambda)\left(\frac{2+\delta}{1+\delta}\right)^{\sigma}(3k+\beta+2)}{1-\beta}|b_n| \right] r^2$$
$$\leq (1+|b_1|)r + \frac{1-\beta}{(1+\lambda)\left(\frac{2+\delta}{1+\delta}\right)^{\sigma}(2+k-\beta)}\left(1 - \frac{(1+2k+\beta)}{1-\beta}|b_1|\right) r^2$$
$$= (1+|b_1|)r + \left(\frac{1-\beta}{(1+\lambda)\left(\frac{2+\delta}{1+\delta}\right)^{\sigma}(2+k-\beta)} - \frac{(1+2k+\beta)}{(1+\lambda)\left(\frac{2+\delta}{1+\delta}\right)^{\sigma}(2+k-\beta)}|b_1| \right) r^2.$$

The second inequality can be proved by using similar arguments. □

Now we get the extreme points of the closed convex hulls of $W_{\overline{\mathcal{H}}CV}(k,\beta,\lambda,\delta,\sigma)$ denoted by clco $W_{\overline{\mathcal{H}}CV}(k,\beta,\lambda,\delta,\sigma)$.

Theorem 2.6. *Let* f *be of the form (1.2). Then* f *is in clco* $W_{\overline{\mathcal{H}}CV}(k,\beta,$

$\lambda, \delta, \sigma)$ if and only if it can be expressed of the form

$$f(z) = \sum_{n=1}^{\infty}(M_n h_n + N_n g_n)\,, \tag{2.7}$$

where

$$h_1(z) = z, \quad h_n(z) = z - \frac{1-\beta}{(1+n\lambda-\lambda)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}(n+nk-k-\beta)}z^n,$$

$$g_1(z) = z, \quad g_n(z) = z - \frac{1-\beta}{(1+n\lambda-\lambda)(n+nk+k+\beta)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}}\overline{z}^n,$$

$$(n = 2, 3, \cdots); \qquad \sum_{n=1}^{\infty}(M_n + N_n) = 1, \quad M_n \geq 0 \;\text{ and }\; N_n \geq 0.$$

In particular, the extreme points of $W_{\mathcal{H}CV}(k, \beta, \lambda, \delta, \sigma)$ are $\{h_n\}$ and $\{g_n\}$.

Proof. We can write (2.7) as

$$f(z) = \sum_{n=1}^{\infty}(M_n h_n + N_n g_n) = \sum_{n=1}^{\infty}(M_n + N_n)z$$
$$- \sum_{n=2}^{\infty}\frac{1-\beta}{(1+n\lambda-\lambda)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}(n+nk-k-\beta)}M_n z^n$$
$$- \sum_{n=1}^{\infty}\frac{1-\beta}{(1+n\lambda-\lambda)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}(n+nk+k+\beta)}N_n \overline{z}^n.$$

Then by (2.4), we have

$$\sum_{n=2}^{\infty}\frac{(1+n\lambda-\lambda)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}(n+nk-k-\beta)}{1-\beta}$$
$$\times \frac{1-\beta}{(1+n\lambda-\lambda)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}(n+nk-k-\beta)}M_n$$
$$+ \sum_{n=1}^{\infty}\frac{(1+n\lambda-\lambda)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}(n+nk+k+\beta)}{1-\beta}\times$$
$$\frac{1-\beta}{(1+n\lambda-\lambda)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}(n+nk+k+\beta)}N_n = \sum_{n=2}^{\infty}M_n + \sum_{n=1}^{\infty}N_n = 1 - M_1 \leq 1.$$

Then $f \in clco\; W_{\overline{\mathcal{H}}CV}(k, \beta, \lambda, \delta, \sigma)$.

Conversely, assume that $f \in clco\, W_{\overline{\mathcal{H}}CV}(k, \beta, \lambda, \delta, \sigma)$, then by Theorem 2.2, we have

$$|a_n| \leq \frac{1-\beta}{(1+n\lambda-\lambda)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}(n+nk-k-\beta)}$$

and

$$|b_n| \leq \frac{1-\beta}{(1+n\lambda-\lambda)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}(n+nk+k+\beta)}.$$

Set

$$M_n = \frac{(1+n\lambda-\lambda)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}(n+nk-k-\beta)}{1-\beta}|a_n|, \quad n = 2, 3, 4, \dots ,$$

$$N_n = \frac{(1+n\lambda-\lambda)\left(\frac{n+\delta}{1+\delta}\right)^{\sigma}(n+nk+k+\beta)}{1-\beta}|b_n|, \quad n = 1, 2, 3, \dots ,$$

$$M_1 = 1 - \sum_{n=2}^{\infty} M_n - \sum_{n=1}^{\infty} N_n, \quad M_1 \geq 0.$$

By Theorem 2.2, we have $0 \leq M_n \leq 1$ and $0 \leq N_n \leq 1$.

Finally we conclude that $f(z) = \sum_{n=1}^{\infty}(M_n h_n + N_n g_n)$. This completes the proof. □

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