

SPECTRAL SOLUTIONS OF THE MHD EQUATIONS  
IN CYLINDRICAL GEOMETRY

Rainer Hollerbach

Department of Applied Mathematics

University of Leeds

Leeds, LS2 9JT, U.K.

e-mail: rh@maths.leeds.ac.uk

**Abstract:** 2D and 3D numerical solutions of the momentum, induction and temperature equations are presented in cylindrical geometry. Poloidal-toroidal expansions are implemented for the fluid flow and the magnetic field. These potential functions, as well as the temperature, are then expanded in terms of Chebyshev polynomials in  $s$ , and Fourier modes in  $\phi$  and  $z$ , where  $(s, \phi, z)$  are standard cylindrical coordinates.

**AMS Subject Classification:** 65M70, 65Z05, 76M22

**Key Words:** computational fluid dynamics

### 1. Introduction

The flow between concentric, differentially rotating cylinders, the so-called Taylor-Couette problem, is one of the most widely studied systems in classical fluid dynamics. The magnetohydrodynamic extension, in which the fluid is taken to be electrically conducting, and a magnetic field is imposed, is also of considerable interest, with applications including the astrophysically important magnetorotational instability. Convection between differentially heated cylinders is equally fundamental in studies of heat and mass transfer. In this work we present a numerical solution of the relevant equations. We begin by developing an axisymmetric code, in which all quantities are independent of

the azimuthal angle  $\phi$ . We next show how the axisymmetric code may be extended to a fully three-dimensional one. The development of both the 2D and 3D codes closely follows the spherical code [2]; see also the cylindrical code [3] for a similar approach.

The momentum and induction equations are given by

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} = -\nabla p + \nabla^2 \mathbf{U} + (\nabla \times \mathbf{B}) \times \mathbf{B}, \quad (1)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla^2 \mathbf{B} + \nabla \times (\mathbf{U} \times \mathbf{B}), \quad (2)$$

to be solved in conjunction with the solenoidal conditions  $\nabla \cdot \mathbf{U} = \nabla \cdot \mathbf{B} = 0$ . The temperature equation will be considered in Section 4. The geometry in which we wish to solve these equations is a concentric cylindrical shell with inner and outer radii  $s_i$  and  $s_o$ , and periodic in  $z$ , with some prescribed basic periodicity  $z_0$ .

Note also that nondimensional parameters such as Reynolds or Hartmann numbers are not included here. The details of how best to nondimensionalize vary too much from one problem to the next for there to be a single ‘standard’ form of the equations. Once the code is in place it is obviously a trivial matter to multiply certain terms by various parameters as appropriate. In this paper we therefore concentrate on outlining just the basic numerical methods, but do not consider any specific applications.

## 2. A 2-Dimensional Code

For axisymmetric solutions, we may decompose  $\mathbf{U}$  and  $\mathbf{B}$  as

$$\mathbf{U} = \nabla \times (\psi \hat{\mathbf{e}}_\phi) + v \hat{\mathbf{e}}_\phi, \quad \mathbf{B} = \nabla \times (a \hat{\mathbf{e}}_\phi) + b \hat{\mathbf{e}}_\phi, \quad (3)$$

thereby automatically satisfying  $\nabla \cdot \mathbf{U} = \nabla \cdot \mathbf{B} = 0$ . The four scalar quantities  $\psi$ ,  $v$ ,  $a$  and  $b$  are then expanded in Fourier modes in  $z$ , e.g.

$$\psi(s, z, t) = \sum_{k=0}^{K_U} \psi_k(s, t) e^{i\kappa z}, \quad a(s, z, t) = \sum_{k=0}^{K_B} a_k(s, t) e^{i\kappa z},$$

with  $\kappa = 2\pi k/z_0$ . The truncations,  $K_U$  for  $\psi$  and  $v$ , and  $K_B$  for  $a$  and  $b$ , may be different. Note also that complex notation is used here purely for convenience; in the actual code all variables are further separated into  $\sin(\kappa z)$  and  $\cos(\kappa z)$  terms.

The  $\phi$ -components of (1) and  $\nabla \times (1)$  then yield

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_k\right)v_k = \hat{\mathbf{e}}_\phi \cdot \mathbf{F}|_k, \quad -\left(\frac{\partial}{\partial t} - \mathcal{L}_k\right)\mathcal{L}_k\psi_k = \hat{\mathbf{e}}_\phi \cdot \nabla \times \mathbf{F}|_k, \quad (4)$$

where  $\mathbf{F} = -\mathbf{U} \cdot \nabla \mathbf{U} + (\nabla \times \mathbf{B}) \times \mathbf{B}$ . Similarly, (2) yields

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_k\right)a_k = \hat{\mathbf{e}}_\phi \cdot \mathbf{F}'|_k, \quad \left(\frac{\partial}{\partial t} - \mathcal{L}_k\right)b_k = \hat{\mathbf{e}}_\phi \cdot \nabla \times \mathbf{F}'|_k, \quad (5)$$

where  $\mathbf{F}' = \mathbf{U} \times \mathbf{B}$ . The subscript  $k$  on the right sides is a reminder that these terms must also be separated into individual Fourier modes  $k$  (using standard pseudo-spectral techniques). The operator  $\mathcal{L}_k$  is defined by

$$\mathcal{L}_k = \nabla^2 - \frac{1}{s^2} = \frac{\partial^2}{\partial s^2} + \frac{1}{s} \frac{\partial}{\partial s} - \frac{1}{s^2} - \kappa^2.$$

The no-slip boundary conditions associated with (4) are

$$\begin{aligned} v_0 = \Omega_i s_i, \quad v_1 = v_2 = \dots = 0, \quad \psi_k = \frac{\partial}{\partial s} \psi_k = 0, \quad \text{at } s = s_i, \\ v_0 = \Omega_o s_o, \quad v_1 = v_2 = \dots = 0, \quad \psi_k = \frac{\partial}{\partial s} \psi_k = 0, \quad \text{at } s = s_o, \end{aligned} \quad (6)$$

where  $\Omega_i$  and  $\Omega_o$  are the externally imposed rotation rates of the inner and outer cylinders.

The insulating boundary conditions associated with (5) are

$$\begin{aligned} \frac{\partial}{\partial s} a_k - \left[\frac{\kappa I_1'(\kappa s_i)}{I_1(\kappa s_i)}\right] a_k = 0, \quad b_k = 0, \quad \text{at } s = s_i, \\ \frac{\partial}{\partial s} a_k - \left[\frac{\kappa K_1'(\kappa s_o)}{K_1(\kappa s_o)}\right] a_k = 0, \quad b_k = 0, \quad \text{at } s = s_o, \end{aligned} \quad (7)$$

where  $I_1$  and  $K_1$  are the modified Bessel functions, see [1].

The next step is to expand  $v_k$ ,  $\psi_k$ ,  $a_k$  and  $b_k$  in terms of Chebyshev polynomials,

$$v_k(s, t) = \sum_{n=1}^{N_U+2} v_{nk}(t) T_{n-1}(x), \quad \psi_k(s, t) = \sum_{n=1}^{N_U+4} \psi_{nk}(t) T_{n-1}(x), \quad (8)$$

$$a_k(s, t) = \sum_{n=1}^{N_B+2} a_{nk}(t) T_{n-1}(x), \quad b_k(s, t) = \sum_{n=1}^{N_B+2} b_{nk}(t) T_{n-1}(x), \quad (9)$$

where

$$s = \frac{s_o + s_i}{2} + \frac{s_o - s_i}{2} x$$

determines  $x$ , the radial coordinate normalized to  $(-1, 1)$  across the gap.

The spectral expansion coefficients  $v_{nk}$ ,  $\psi_{nk}$ ,  $a_{nk}$  and  $b_{nk}$  are then time-

stepped just as in the spherical case [2]. Comparing for example (4) here with (6) in [2], and (5) here with (33) in [2], we note that the details are obviously different, but the general structure is exactly the same. So, equations (4) are collocated at the  $N_U$  zeros of  $T_{N_U}$ , with the extra coefficients, two for  $v_k$  in (8) and four for  $\psi_k$  in (8), needed to satisfy the boundary conditions (6). Similarly, equations (5) are collocated at the  $N_B$  zeros of  $T_{N_B}$ , with the boundary conditions (7) implemented by having two extra coefficients in (9) for each of  $a_k$  and  $b_k$ .

### 3. A 3-Dimensional Code

For three-dimensional solutions, the expansions (3) no longer work; in 3D they do not satisfy  $\nabla \cdot \mathbf{U} = \nabla \cdot \mathbf{B} = 0$ . Instead, we now decompose as

$$\mathbf{U} = \nabla \times (e \hat{\mathbf{e}}_s) + \nabla \times \nabla \times (f \hat{\mathbf{e}}_s), \quad \mathbf{B} = \nabla \times (g \hat{\mathbf{e}}_s) + \nabla \times \nabla \times (h \hat{\mathbf{e}}_s), \quad (10)$$

which enforces the solenoidal conditions even in 3D.

Now, we might be tempted to use these expansions (10) for all parts of  $\mathbf{U}$  and  $\mathbf{B}$ , even the axisymmetric parts we've already dealt with. That is, simply develop a totally new code, completely independent of the previous one. However, there is a slight problem with this, namely those parts of the solution that are independent not only of  $\phi$ , but of  $z$  as well. These cannot be represented by expansions of the form (10), and would have to be treated separately. See for example [3], who follow this approach. We here adopt the alternative approach of using the previous expansion (3) for all the axisymmetric parts of the solution, including the  $z$ -dependent parts that could be expanded as (10), and using (10) only for the non-axisymmetric parts of the solution (including the  $z$ -independent parts). The computational effort is much the same either way, but if certain parts of the solution have to be treated separately anyway, we might as well recycle all of the existing 2D code.

The subsequent development of the equations is very similar to what it was in the 2D case. First expand  $e$ ,  $f$ ,  $g$  and  $h$  as Fourier modes in both  $z$  and  $\phi$ , e.g.

$$e(s, \phi, z, t) = \sum_{k=0}^{K_U} \sum_{m=1}^{M_U} e_{mk}(s, t) e^{i\kappa z} e^{im\phi},$$

where again complex notation is used purely for convenience; the actual code has four separate coefficients corresponding to products of  $\sin(\kappa z)$  and  $\cos(\kappa z)$  with  $\sin(m\phi)$  and  $\cos(m\phi)$ .

The  $s$ -components of  $\nabla \times (1)$  and  $\nabla \times \nabla \times (1)$  then yield

$$\begin{aligned} \frac{\partial}{\partial t} (\mathcal{C}_2 e_{mk} + \mathcal{C}_3 f_{mk}) + (\mathcal{C}_4 e_{mk} + \mathcal{C}_5 f_{mk}) &= \hat{\mathbf{e}}_s \cdot \nabla \times \mathbf{F}|_{mk}, \\ \frac{\partial}{\partial t} (\mathcal{C}_3 e_{mk} + \mathcal{C}_4 f_{mk}) + (\mathcal{C}_5 e_{mk} + \mathcal{C}_6 f_{mk}) &= \hat{\mathbf{e}}_s \cdot \nabla \times \nabla \times \mathbf{F}|_{mk}, \end{aligned} \quad (11)$$

and the  $s$ -components of (2) and  $\nabla \times (2)$  yield

$$\begin{aligned} \frac{\partial}{\partial t} (\mathcal{C}_1 g_{mk} + \mathcal{C}_2 h_{mk}) + (\mathcal{C}_3 g_{mk} + \mathcal{C}_4 h_{mk}) &= \hat{\mathbf{e}}_s \cdot \nabla \times \mathbf{F}'|_{mk}, \\ \frac{\partial}{\partial t} (\mathcal{C}_2 g_{mk} + \mathcal{C}_3 h_{mk}) + (\mathcal{C}_4 g_{mk} + \mathcal{C}_5 h_{mk}) &= \hat{\mathbf{e}}_s \cdot \nabla \times \nabla \times \mathbf{F}'|_{mk}, \end{aligned} \quad (12)$$

where  $\mathbf{F}$  and  $\mathbf{F}'$  are as before. The operators  $\mathcal{C}_l$  are defined by

$$\mathcal{C}_1 = 0, \quad \mathcal{C}_2 = \Delta, \quad \mathcal{C}_3 = -2m\kappa s^{-2},$$

$$\mathcal{C}_4 = -\Delta \partial_s^2 + (m^2 s^{-2} - \kappa^2)(s^{-1} \partial_s - s^{-2}) + \Delta^2,$$

$$\mathcal{C}_5 = 4m\kappa(s^{-2} \partial_s^2 - s^{-3} \partial_s + (1 - m^2)s^{-4} - \kappa^2 s^{-2}),$$

$$\begin{aligned} \mathcal{C}_6 &= \Delta \partial_s^4 - 2(m^2 s^{-2} - \kappa^2) s^{-1} \partial_s^3 + (5m^2 s^{-4} - 3\kappa^2 s^{-2} - 2\Delta^2) \partial_s^2 \\ &\quad + (3m^2(2m^2 - 3)s^{-4} + (4m^2 + 3)\kappa^2 s^{-2} - 2\kappa^4) s^{-1} \partial_s \\ &\quad + m^2(9 - 10m^2) s^{-6} - 3\kappa^2 s^{-4} + 2\kappa^4 s^{-2} + \Delta^3, \end{aligned}$$

where  $\Delta = m^2 s^{-2} + \kappa^2$ , and  $\partial_s$  denotes  $\frac{\partial}{\partial s}$ .

The no-slip boundary conditions associated with (11) are

$$e_{mk} = 0, \quad f_{mk} = \frac{\partial}{\partial s} f_{mk} = 0, \quad \text{at } s = s_i, s_o.$$

Note how (11) does indeed involve two radial derivatives on  $e_{mk}$ , and four on  $f_{mk}$ .

The insulating boundary conditions associated with (12) are

$$\begin{aligned} \Delta g_{mk} &= 2m\kappa s^{-2} h_{mk}, & \text{at } s = s_i, s_o, \\ \Delta h_{mk} &= \frac{I'_m(\kappa s_i)}{I_m(\kappa s_i)} \left[ \kappa \left( \frac{\partial}{\partial s} h_{mk} + s^{-1} h_{mk} \right) - m s^{-1} g_{mk} \right], & \text{at } s = s_i, \\ \Delta h_{mk} &= \frac{K'_m(\kappa s_o)}{K_m(\kappa s_o)} \left[ \kappa \left( \frac{\partial}{\partial s} h_{mk} + s^{-1} h_{mk} \right) - m s^{-1} g_{mk} \right], & \text{at } s = s_o. \end{aligned}$$

And once again, the number of boundary conditions precisely matches the number of radial derivatives.

Comparing (11) with (4) then, the main difference to note is how  $v_k$  and

$\psi_k$  are nicely decoupled in (4), whereas in (11)  $e_{mk}$  and  $f_{mk}$  are not. This causes no problems though; it merely means that in contrast to time-stepping  $v_k$  with one set of  $(N_U + 2)^2$  matrices, and  $\psi_k$  with another set of  $(N_U + 4)^2$  matrices,  $e_{mk}$  and  $f_{mk}$  are time-stepped together with a single set of  $(2N_U + 6)^2$  matrices. And similarly,  $g_{mk}$  and  $h_{mk}$  are time-stepped together with a single set of  $(2N_B + 4)^2$  matrices.

There is one feature of these  $(2N_U + 6)^2$  and  $(2N_B + 4)^2$  time-stepping matrices that is worth pointing out though: because the operators  $\mathcal{C}_l$  depend on  $m$  and  $k$  (through  $\kappa$ ), these matrices also depend on  $m$  and  $k$ . That is, one needs to define a separate set of time-stepping matrices for each  $m$  and  $k$ . This is in sharp contrast to the spherical case, see [2], where the corresponding matrices turn out to be independent of  $m$ . As a result, for the cylindrical code the memory requirements scale as  $L^4$ , where  $L$  is a typical truncation in each of the three dimensions, whereas for the spherical code the memory requirements scale only as  $L^3$ . The computational effort scales as  $L^4$  in both cases though.

Finally, we note once again how the forcings  $\mathbf{F}$  and  $\mathbf{F}'$  on the right sides of (4) and (11), and (5) and (12), are conveniently the same. This means that the pseudo-spectral evaluation of all these terms is quite straightforward: after calculating  $\mathbf{F}$  and  $\mathbf{F}'$  at a set of collocation points in real space, one transforms back to spectral space, as usual in pseudo-spectral methods. The axisymmetric parts are then further processed to produce the right sides of (4) and (5), and the non-axisymmetric parts to produce the right sides of (11) and (12).

#### 4. The Temperature Equation

If desired, a temperature equation of the form

$$\frac{\partial T}{\partial t} + \mathbf{U} \cdot \nabla T = \nabla^2 T \quad (13)$$

can easily be added to the code (along with a buoyancy force added to  $\mathbf{F}$ ). Because  $T$  is already a scalar, it does not need any decompositions such as (3) or (10). Instead, it is directly expanded as

$$T(s, \phi, z, t) = \sum_{k=0}^{K_T} \sum_{m=0}^{M_T} T_{mk}(s, t) e^{i\kappa z} e^{im\phi}.$$

Equation (13) then becomes

$$\left( \frac{\partial}{\partial t} - \mathcal{L}'_{mk} \right) T_{mk} = -\mathbf{U} \cdot \nabla T|_{mk}, \quad (14)$$

where

$$\mathcal{L}'_{mk} = \nabla^2 = \frac{\partial^2}{\partial s^2} + \frac{1}{s} \frac{\partial}{\partial s} - \frac{m^2}{s^2} - \kappa^2.$$

Together with two boundary conditions, typically on either  $T$  or  $\partial_s T$ , (14) is then time-stepped in exactly the same way as equations (5), for example.

### References

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