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INTERPOLATION THEOREMS FOR SUBLINEAR OPERATORS

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Abstract: We prove the appropriate sublinear operator interpolation theorems between the weak type $(1,1)$ estimate and the strong estimate from L^∞ to BMO.

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1. Introduction

We know that the major tools to study strong estimate are a variety of interpolation theorems. We also know BMO, the space of functions of bounded mean oscillation, was first introduced and used in different contexts by John and Nirenberg [6]. Fefferman and Stein proved that the dual of H^1 is BMO [5]. Since then, in several instances, BMO has served as a substitution for L^∞, see [3], [5], [8], [9], [10], [11]. The significance for us is that this substitution provides the link between BMO and L^p.

Now, let us start with some basic definitions.

Definition 1.1. The definition of $\wedge f_{Q_0}$.

$$\wedge f_{Q_0}(\mathbf{x}) = \sup_{Q\subset Q_0, x\in Q} \frac{1}{|Q|}\int |f(\mathbf{y}) - f_Q| d\mathbf{y} \quad x \in Q_0,$$

where Q_0 and Q are cubes with sides parallel to the axes in $\mathbb{R}^d$ and $f_Q = \frac{1}{|Q|}\int\limits_Q f(\mathbf{x})d\mathbf{x}$.

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Definition 1.2. *Bounded Mean Oscillation on* Q_0 are the collection of functions on Q_0 with $\wedge f_{Q_0} \in L^\infty(Q_0)$. We write as $BMO(Q_0)$.

The Bounded Mean Oscillation on whole space $\mathbb{R}^d$, we write as BMO. We also can find very detailed definitions and results about BMO in [4] and [5].

Definition 1.3. T is a *sublinear operator*, if T is an operator from a linear space of functions on $(\mathbb{R}^d, \mu)$ to another linear space of functions on $(\mathbb{R}^d, \mu)$ and satisfies the following two conditions:

1. $T(f+g) \le T(f) + T(g)$,
2. $T(\lambda f) = |\lambda| T(f)$,

where f and g are functions on $(\mathbb{R}^d, \mu)$ in the domain of T, where $\mathbb{R}^d$ is $d-$dimensional Euclidean space and μ is a measure on $\mathbb{R}^d$.

Definition 1.4. Let $\mathcal{M}_0(\mathbb{R}^d, \mu)$ be the family of all μ measurable functions with finite values almost everywhere.

Definition 1.5. *The distribution function of* f is defined as follows: for $\lambda \ge 0$ and $f(x) \in \mathcal{M}_0(\mathbb{R}^d, \mu)$,

$$\mu_f(\lambda) = \mu\{\mathbf{x} \in \mathbb{R}^d : |f(\mathbf{x})| > \lambda\}.$$

Definition 1.6. *The decreasing rearrangement* of f is denoted as f^*. So f^* is defined on $[0, \infty)$ and if f belongs to $\mathcal{M}_0(\mathbb{R}^d, \mu)$, then

$$\begin{aligned} f^*(t) &= \inf\{\lambda : \mu_f(\lambda) \le t\} \quad t \ge 0 \\ &= \sup\{\lambda : \mu_f(\lambda) > t\}. \end{aligned}$$

Note. $f^*(t)$ is a decreasing function.

2. Interpolation Theorems

In this section, we are going to prove an interpolation result that shows sublinear operators are strong type (p,p) for $1 < p < \infty$, if these operators are weak type $(1,1)$ and bounded operators from L^∞ to BMO. We use the notations and definitions as in [2], [1], and [3].

Definition 2.1. If $f \in \mathcal{M}_0(\mathbb{R}^d, \mu)$, $f^{**}(t)$ will denote the maximal function of f^* defined by

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds, \qquad t > 0.$$

Proposition 2.2. *If $f \in \mathcal{M}_0(\mathbb{R}^d, \mu)$ and $0 < p < \infty$, then*

$$\int_{\mathbb{R}^d} |f|^p d\mu = p \int_0^\infty \lambda^{p-1} \mu_f(\lambda) d\lambda = \int_0^\infty |f^*(t)|^p dt. \tag{2.1}$$

Furthermore, in the case $p = \infty$

$$\operatorname{ess} \sup_{\mathbf{x} \in \mathbb{R}^d} |f(\mathbf{x})| = inf\{\lambda : \mu_f(\lambda) = 0\} = f^*(0) \geq f^*(t) \ t \geq 0. \tag{2.2}$$

Proof. This proof is on p. 43 of [2]. ☐

Definition 2.3. *The Lorentz space $L^{p,q} = L^{p,q}(\mathbb{R}^d, \mu)$* consists of all f in $\mathcal{M}_0(\mathbb{R}^d, \mu)$ for which the quantity

$$\|f\|_{pq} = \begin{cases} \{\int_0^\infty [t^{\frac{1}{p}} f^*(t)]^q \frac{dt}{t}\}^{\frac{1}{q}} & \text{if } 0 < q < \infty \ , \\ \sup_{0<t<\infty} t^{\frac{1}{p}} f^*(t) & \text{if } q = \infty \, , \end{cases}$$

is finite.

Note. $L^{p,p} = L^p$ since $\|f\|_{p,p} = \|f\|_p$ by Proposition 2.2.

Definition 2.4. If there exists an M such that

$$t(T\chi_E)^*(t) \leq M\mu(E) \quad \text{for } t > 0,$$

we say that T is of *restricted weak type* $(1,1)$.

Proposition 2.5. *If T is weak type $(1,1)$, then T is of restricted weak type $(1,1)$.*

Proof. Let E be a measurable set in $\mathbb{R}^d$ and χ_E be the characterizing function of E. By the definition weak type (1,1), we have

$$\lambda\mu\{\mathbf{x} : |(T\chi_E)(\mathbf{x})| > \lambda\} \leq M\|\chi_E\|_1. \tag{2.3}$$

According to the definition of the decreasing rearrangement

$$(T\chi_E)^*(t) = \sup\{\lambda : \mu_{T\chi_E}(\lambda) > t\}.$$

In (2.3) replace λ by $T\chi_E^*(t)$ and $\mu_{T\chi_E}(\lambda)$ by t. We have

$$t(T\chi_E)^*(t) \leq M\|\chi_E\|_1.$$

But

$$\|\chi_E\|_1 = \int_E 1 d\mu = \mu(E).$$

So, we have

$$t(T\chi_E)^*(t) \leq M\mu(E).$$

Therefore, T is of restricted weak type (1,1). □

Definition 2.6. If there is a constant C such that

$$\|T\chi_E\|_W = \sup_{0<t}\{(T\chi_E)^{**}(t) - (T\chi_E)^*(t)\} \leq C\|\chi_E\|_\infty$$

on $\mathbb{R}^d$, we say that T is of restricted type (∞,∞).

Proposition 2.7. *If f is an integrable function in Q_0, then*

$$f^{**}(t) - f^*(t) \leq C(\wedge f_{Q_0})^*(t), \quad 0 < t < \frac{|q_0|}{6}.$$

Proof. The proof of this proposition is in [2], p. 377. □

Proposition 2.8. *If T is a bounded operator from L^∞ to BMO, then T is of restricted weak type (∞,∞).*

Proof. For any measurable set $E \subset \mathbf{R}^d$ with $\mu(E) < \infty$, we obviously have $\chi_E \in L^\infty$. Since T is a bounded operator from L^∞ to BMO, we have

$$\|T\chi_E\|_{BMO} \leq C\|\chi_E\|_\infty.$$

For any cube Q with sides parallel to the axes in $\mathbb{R}^d$

$$\|T\chi_E\|_{BMO} = \sup_Q \frac{1}{|Q|}\int_Q |T\chi_E(\mathbf{x}) - T\chi_{EQ}|d\mathbf{x}.$$

For any $0 < t < \infty$, choose Q_0 with sides parallel to the axes such that $0 < t < \frac{|Q_0|}{6}$. χ_E is an integrable function with support on E. By Proposition 2.7

$$T\chi_E^{**}(t) - T\chi_E^*(t) \leq C(T\chi_{EQ_0})^*(t).$$

Since the decreasing rearrangement is decreasing, we have

$$(T\chi_{EQ_0})^*(t) \leq (T\chi_{EQ_0})^*(0)$$

and the following estimate

$$\begin{aligned} T\chi_E^{**}(t) - T\chi_E^*(t) &\leq C(T\chi_{EQ_0})^*(0) = C \operatorname{ess\,sup}_{\mathbf{R}^n} |\wedge T\chi_{EQ_0}| \\ &\leq C\|\wedge T\chi_{EQ_0}\|_\infty \leq C\|T\chi_E\|_{BMO} \leq C\|\chi_E\|_\infty. \end{aligned}$$

Therefore, T is of restricted weak type (∞,∞). □

Theorem 2.9. *If T is a sublinear operator and is of restricted type $(1,1)$ and of restricted type (∞,∞), for any simple function f, we have*

$$\|Tf\|_p \leq C_p\|f\|_p \quad 1 < p < \infty,$$

where C_p are dependent only on T and p. In particular, T has a unique extension to a bounded sublinear operator on L^p for $1 < p < \infty$.

Proof. The proof of this theorem is on p. 386 of [2] and in [1]. □

Theorem 2.10. *If T is a sublinear operator and weak type $(1,1)$, and a bounded operator from L^∞ to BMO, then T is strong type (p,p) for $1 < p < \infty$*

$$\|Tf\|_p \le C_p \|f\|_p \,.$$

Proof. Since T is weak type $(1,1)$ and a bounded operator from L^∞ to BMO, by Proposition 2.5 and Proposition 2.8, we have that T is of restricted weak type $(1,1)$ and of restricted weak type (∞,∞). We know that T is also a sublinear operator. By Theorem 2.9, we have that for all simple functions f,

$$\|Tf\|_p \le C_p \|f\|_p \quad 1 < p < \infty \,.$$

By Theorem 2.9 again, T has a unique extension to a bounded sublinear operator on L^p. Therefore, we have

$$\|Tf\|_p \le C_p \|f\|_p \quad f \in L^p \,.$$

□

3. Applications

We should notice that square functions, oscillation operators, and variation operators are only sublinear operators. In the paper [7], we proved these operators are strong bounded from L^∞ to BMO. Therefore, we have these operators are strong type (p,p) for $1 < p < \infty$.

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