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VISUALIZATION OF INFINITESIMAL BENDING OF SOME CLASS OF TOROID

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Abstract: The basic aim of the deformation theory is to point out to a class of rigid or non-rigid surfaces. In this paper are considered torus like surfaces obtained by revolution of quadrangular meridian. Infinitesimal bending of generated surfaces is explored using Cohn-Vossen's method. We present visualization of such surfaces and their infinitesimal deformation obtained by our programm. It is written in *C++* and for modelling three dimensional surfaces and curves is used *OpenGL* graphic library.

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1. Introduction

The surface bending theory considers the bending of surfaces, the isometrical deformations, as well as, the infinitesimal bending of surfaces and presents one of the main consisting parts of global differential geometry.

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Under bending surface is included in continuous family of isometrical surfaces, so that the curve preserves its arc length. The angles are also preserved.

Two surfaces are trivially isometrical if we get one from another by rigid motion or by plane symmetry (or by finite number of such transformations). A surface is uniquely defined if there are only trivially isometrical surfaces. Each uniquely defined surface is rigid in a sense of isometrical bending. On the other hand, infinitesimal bending of surfaces is not an isometric deformation, or roughly speaking it is with appropriate precision. Arc length is stationary under infinitesimal bending.

The basic task of flexibility and rigidity (isometrical bending) problems is to find as many surfaces, that represent the class of uniquely defined, rigid, surfaces, as well as, to find those that represent the class of bendable, flexible surface.

The first result of the infinitesimal bending of the non-convex surface belongs to H. Liebman [7], [8]. He has proved that the torus and analytic surfaces containing the convex strip are rigid in a sense of infinitesimal bending. In 1938 A.D. Alexandrov [1] has widened the above mentioned result of Liebman. He considered closed surfaces, divided in finite number of regions by piecewise smooth curves with constant Gaussian curvature. He called this surfaces T-surfaces and proved that analytical T-surfaces are rigid in a sense of the analytic infinitesimal bending.

The results of H. Liebman on the rigidity of the torus and the ovaloid naturally led to the question of the existence of non-rigid closed surfaces. The first to answer this question was S. Cohn-Vossen [3], [6]. He has proved that from each plane curve we can get the meridian of non-rigid surface of revolution of genius 0. This result of S. Cohn-Vossen and his method, indicated many works on infinitesimal bending of non-convex surfaces of revolution. Such surfaces of genius 0 or 1 generated by rotation of broken (polygonal line) were considered by Bublik, K.M. Belov [2], N.G. Perlova [9]. K.M. Belov [2] has pointed out a class of toroids with meridian in shape of special quadrangle (with mutually perpendicular diagonals-one parallel to the axe of rotation), unlike parallelogram. Generalization of the consideration from the paper [2] was given in [11]-[14]. Toroid surfaces non containing plane part, generated by triangular meridian are rigid, see [11].

We shall consider here infinitesimal bending of toroid rotational surfaces generated by quadrangular meridian. We also consider existence of the field of infinitesimal bending. The rigidity condition expressed by the coordinates of vertex of polygon, i.e. by an analytical expression is considered geometrically.

2. Some Basic Facts of the Infinitesimal Bending Theory

We shall give the basic facts of the theory of infinitesimal bending of surfaces according to [4] and [6].

Definition 2.1. Let the surface

$$S : \bar{r} = \bar{r}(u, v), \quad (u, v) \in D, \quad D \subset \Re^2 \tag{2.1}$$

be included in a family of surfaces $S_\varepsilon : \bar{r}_\varepsilon = \bar{r}_\varepsilon(u, v, \varepsilon)$ $(\varepsilon \geq 0, \varepsilon \to 0)$, depending continuously on the parameter ε and we get S for $\varepsilon = 0$. In this way

$$S_\varepsilon : \bar{r}_\varepsilon = \bar{r}(u, v) + \varepsilon \overset{(1)}{\bar{z}}(u, v) + \varepsilon^2 \overset{(2)}{\bar{z}}(u, v) + ... + \varepsilon^m \overset{(m)}{\bar{z}}(u, v), \quad m \geq 1, \tag{2.2}$$

where $\overset{(j)}{\bar{z}}(u, v) \in C^\alpha$ $(\alpha \geq 3)$, $j = 1, ..., m$, are given fields, family S_ε is *infinitesimal deformation of the order m of the surface S.*

Theory considering geometric objects in connection with S_ε up to the precision of the order m with respect to ε $(\varepsilon \to 0)$ is *infinitesimal deformation theory of surfaces of the order m.* Giving different more special conditions we get different kinds of surface deformations.

Let the regular surface S of the class C^α, $\alpha \geq 3$ be given in the vector form with (2.1) included in the family of surfaces

$$S_\varepsilon : \bar{r}_\varepsilon(u, v, \varepsilon) = \bar{r}(u, v) + \varepsilon \bar{z}(u, v), \tag{2.3}$$

where $\varepsilon(\varepsilon \to 0), (u, v) \in D,\ D \subset \Re^2$ and $\bar{r}_0(u, v, 0) = \bar{r}(u, v)$.

Definition 2.2. The surfaces S_ε are *infinitesimal bending of the first order* of the surface S if

$$ds_\varepsilon^2 - ds^2 = o(\varepsilon)\,, \tag{2.4}$$

i.e. if the difference of the squares of the line elements of this surfaces is of the order higher then the first.

The field $\bar{z}(u, v)$ for which

$$\frac{\partial \bar{r}(u, v, \varepsilon)}{\partial \varepsilon} = \bar{z}(u, v) \tag{2.5}$$

is *velocity* or *infinitesimal bending field* of the infinitesimal bending.

According to [4], [6] this definition is equivalent to the next theorem.

Theorem 2.1. *Necessary and sufficient condition for the surface S_ϵ (2.3) to be infinitesimal bending of the surface S (2.1) is*

$$d\bar{r} d\bar{z} = 0, \tag{2.6}$$

where $\bar{z}(u, v)$ is the velocity field at the initial moment of deformation.

The equation (2.6) is equivalent to the next three partial differential equations: $\bar{r}_u \bar{z}_u = 0, \quad \bar{r}_u \bar{z}_v + \bar{r}_v \bar{z}_u = 0, \quad \bar{r}_v \bar{z}_v = 0.$

Under infinitesimal bending of the surfaces each line element gets non-zero addition, which is the infinitesimal value of the second order with respect to ε, i.e. $ds_\varepsilon - ds = o(\varepsilon) \geq 0$.

3. Rigidity Condition of Toroid With Quadrangular Meridian

Let P_4 be the quadrangle with apices $A_i(u_i \rho_i)$ $(i = 1,2,3,4)$ considered at Descartes coordinate system $uO\rho$ with the axe of rotation u. The equations of the sides are:

$$A_i A_{i+1} : \rho_{(i)} = \rho_i + \frac{\rho_{i+1} - \rho_i}{u_{i+1} - u_i}(u - u_i),$$
$$\rho'_{(i)} = \frac{\rho_{i+1} - \rho_i}{u_{i+1} - u_i} = k_i \quad (i = 1,2,3,4; A_5 \equiv A_1)\,, \tag{3.1}$$

where ρ_i is the value of ρ on $A_i A_{i+1}$.

If we denote $\bar{e}$ is unit vector of the axis of rotation, $\bar{a}(v)$ unit vector of the ρ-axis, where v is the angle between the plane of initial position of the meridian and $\bar{a}(v)$ then $\bar{a}'(v) \perp \bar{a}(v)$ and $\bar{a}'(v) \perp \bar{e}$ (see [6], p. 90, or [4] p. 253). Radius vector of a surface of rotation, in the coordinate system with the base $\bar{e}, \bar{a}, \bar{a}'$ is $\bar{r}(u,v) = u\bar{e} + \rho(u)\bar{a}(v)$.

Fundamental field of infinitesimal bending of the surface we try to find in the form

$$\bar{z}(u,v) = \bar{z}_k(u,v) = [\varphi_k(u)e^{ikv} + \tilde{\varphi}_k(u)e^{-ikv}]\bar{e}$$
$$+ [\psi_k(u)e^{ikv} + \tilde{\psi}_k(u)e^{-ikv}]\bar{a}(v) + [\chi_k(u)e^{ikv} + \tilde{\chi}_k(u)e^{-ikv}]\bar{a}'(v),$$

where $\tilde{\varphi}_k(u)$ conjugated complex value for $\varphi_k(u)$. The functions $\varphi_k(u)$, $\psi_k(u)$, $\chi_k(u)$ satisfy the equations

$$\varphi'_k(u) + \rho'(u)\psi'_k(u) = 0, \quad \psi_k(u) + ik\chi'_k(u) = 0,$$
$$ik\psi_k(u) + \rho'(u)[ik\psi_k(u) - \chi_k(u)] + \rho(u)\chi'_k(u) = 0.$$

Functions $\psi_k(u)$, $\chi_k(u)$ satisfy also the equation

$$\rho(u)\lambda''(u) + (k^2 - 1)\rho''(u)\lambda(u) = 0, \tag{3.2}$$

where $\lambda(u)$ is unknown function. We omit index k, and denote with $\psi_{(i)}$ the value of the function ψ on $A_i A_{i+1}$, $\quad i = 1,2,3,4, \quad A_5 \equiv A_1$.

From the equations (3.1) and (3.2) it follows also the linearity of the functions $\psi_{(i)} = M_i u + N_i \quad (i = 1,2,3,4)$. At the points $u = \sigma$ of the meridian, where $\rho(\sigma-0)=\rho(\sigma+0)$, i.e. at the apices of the quadrangle, continuity of the function $\psi_{(i)}(u)$ gives us $\psi_{(i)}(u_i)=\psi_{(i-1)}(u_i)$, $i=1,2,3,4$; $\psi_{(0)}(u_1)=\psi_{(4)}(u_1)$ and from there

$$M_i u_i + N_i = M_{i-1} u_i + N_{i-1} \quad (i=1,2,3,4); \quad M_0 \equiv M_4, \quad N_0 \equiv N_4\,.$$

Considering this system as a system with respect to unknowns N_i, $i=1,2,3,4$ we get system of algebraic homogenous linear equations with determinant of the coefficients A.

Theorem 2.2. *Necessary and sufficient condition for the surface for the existence of the field of infinitesimal bending of a toroid rotational surface with quadrangular meridian with apices* $A_i(u_i,\rho_i)(\rho_i>0, u_{i+1}\neq u_i, i=1,2,3,4)$, *around the* Ou *axis is*

$$\det A = 0.$$

4. Visualization of Infinitesimal Bending of Toroid Generated by a Quadrangular Meridian

Using a computer enables us analyzing non rigidity conditions and determination if generated toroid satisfy them. Graphical representations of deformations is considered in article [5]. It is useful to see rotational surfaces and influence of fields of infinitesimal bending on them. For this purpose we developed *SurfBend*. *SurfBend*, the programm devoted to visualize infinitesimal bending of toroids, developed in *C++* using *OpenGL*, is partialy presented at the ESI Conference Rigidity and Flexibility, Viena, 2006. It takes as input Descartes coordinates of points of quadrangle, then performs non rigidity analysis. If quadrangle satisfy non rigidity conditions we are able to visualize family of bendable surfaces. *SurfBend* is an *ANSI C++* program which uses *OpenGL* standard to display graphics. It should therefore be portable. Underlaying calculations of the geometry model have done in *ANSI C++*, but rising control to interactive level has done using MFC, see [10].

Upon launch, a window is opened with the image of example Belov's toroid with $3Pi/2$ angle of rotation. It is possible then give input points of quadrangle to examine rigidity or to choose one of supplied examples of non rigid toroids to work further. The programm can run in different modes and in all of them are available to rotate the object by pressing left mouse button and moving then, as well as obtaining wire or fill model by pressing “w” or “f”.

When we select quadrangle and appropriate flexible toroid, to examine the shape we can choose View\Property dialog. Visibility and colors of the cones obtained by rotating the quadrangle can be changed from it. Bending parameter, the angle of rotation and number of division points of the grid can be changed via appropriate scroll bars. Program can run in "drag an rotate" mode where one's remembered rotation of the model can be repeated continuously (loop or reverted way). There is a bright spotlight to achieve more realistic picture of the 3D object. It is used perspective projection for creating and displaying the image.

Apices of the selected quadrangle form circles during rotation around z axes. Bending deforms them in curves. Those curves and surfaces formed by them are visible and manageable via cone borders dialog. Its activation pushes program to run in mode which hides cones, and shows only them. Color and visibility properties, number of subdivision points, as well as interval values of bending parameter is adjustable. The picture can consist only curves representing borders or meshes which is more suitable for surfaces.

We are using a kind of free-form deformation (FFD) in modelling infinitesimal bending. FFD is a general method for deforming objects that provides higher and more powerful level of control and is computationally efficient. It enables us to present infinitesimal deformations as animation. We define a starting and ending forms of the model via appropriate dialogs. They give ability to set some model properties like visibility and color of the cones, as well as bending, angle and distance of the point of view for both forms. Checking the drag to animate box will memorize mouse moving and induced rotations. Releasing the left mouse button will finish animation creation and it will be displayed transforming model from starting to ending form.

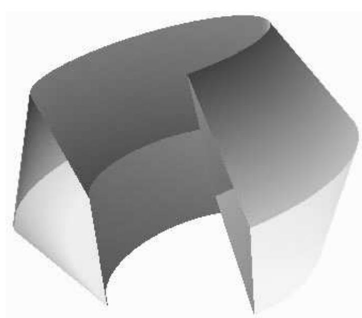

Figure 1:

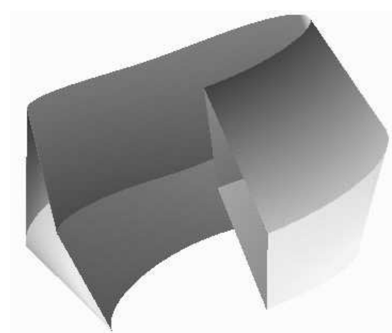

Figure 2:

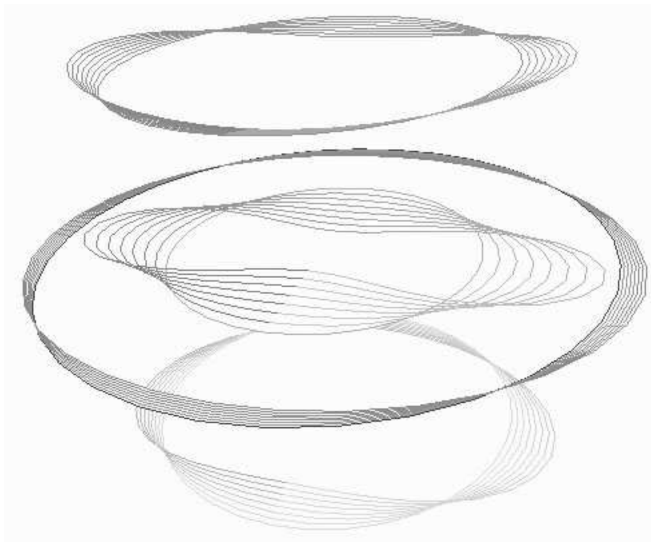

Figure 3:

5. Examples

On the first example on Figure 1 is shown Belov's toroid obtained by rotation of the quadrangle around z-axes without bending and with angle of rotation of $\frac{3\pi}{2}$. On the Figure 2 is shown with bending of 0.12.

On the second example on Figure 3 is shown curves formed by bending circles obtained by rotation of apices of the quadrangle around z-axes.

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