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UNREDUCED ZERO–DIMENSIONAL SCHEMES DEFINED OVER $\mathbb{F}_q$ WITH GOOD POSTULATION

E. Ballico
Department of Mathematics
University of Trento
380 50 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Here we prove the existence of many zero-dimensional schemes $Z \subset \mathbf{P}^n$ with a small number of connected components, all of them defined over $\mathbb{F}_q$, and with good postulation (even if length$(Z) \gg \sharp(\mathbf{P}^n(\mathbb{F}_q))$).

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Fix homogeneous coordinates $x_0, \dots, x_n$ on $\mathbf{P}^n$. Notice that all points of $\mathbf{P}^n(\mathbb{F}_q)$ satisfies the degree $q+1$ homogeneous equations $x_i^q x_j - x_j^q x_i = 0$ for all $i \neq j$. Since $\sharp(\mathbf{P}^n(\mathbb{F}_q)) = (q^{n-1}-1)/(q-1)$ and $(q^{n-1}-1)/(q-1) > \binom{n+q+1}{n}$ for all integers $n \geq 2$, we get that $h^1(\mathbf{P}^n, \mathcal{I}_Z(q+1)) \neq 0$ and $h^0(\mathbf{P}^n, \mathcal{I}_Z(q+1)) \neq 0$ if $Z = \mathbf{P}^n(\mathbb{F}_q)$ or if Z is any subset of $\mathbf{P}^n(\mathbb{F}_q)$ with cardinality at least $\binom{n+q+1}{n}+1$. Here we want to find unreduced zero-dimensional schemes Z such that each of their connected component is defined over $\mathbb{F}_q$ and Z has good postulation, i.e. for every $t \in \mathbb{N}$ either $h^1(\mathbf{P}^n, \mathcal{I}_Z(d)) = 0$ or $h^0(\mathbf{P}^n, \mathcal{I}_Z(d)) = 0$. For any integer $m > 0$ and any $P \in \mathbf{P}^n$ let mP denote the closed subscheme of $\mathbf{P}^n$ with $(\mathcal{I}_P)^m$ as its ideal sheaf. Hence $(mP)_{red} = \{P\}$ and length$(mP) = \binom{n+m-1}{n}$.

Theorem 1. *Fix a prime power q, positive integers n, d, c such that $\binom{n+d-2}{n} < c \leq \binom{n+d-1}{n}$ and $P \in \mathbf{P}^n(\mathbb{F}_q)$. Then there exists a zero-dimensional scheme $Z \subset \mathbf{P}^n$ defined over $\mathbb{F}_q$ such that $h^0(\mathbf{P}^n, \mathcal{I}_Z(d-2)) = 0$ and $h^1(\mathbf{P}^n, \mathcal{I}_Z(t)) = 0$ for all $t \geq d-1$.*

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Theorem 2. *Fix a prime power q and positive integers n, d, c, z such that $\binom{n+d-2}{n} < c \le \binom{n+d-1}{n}$, $z \le c$ and $P \in \mathbf{P}^n(\mathbb{F}_q)$. If $n = 1$ assume $z \le q+1$. If $n \ge 2$ let e be the maximal integer such that $\binom{n+e-1}{n} \le c-z+1$. if $n = 2$ assume $z-1 \le (q+1)(\min\{q, d-e-1\})-\eta$, where $\eta := \min\{q+1, c-z+1-(e+1)e/2\}$. For all integers x, y define the integer $u_2(x,y)$ by the formulas $u_2(x,y) = 0$ if $x \ge y-1$, $u_2(x,y) = u_2(0,y)$ if $x < 0$ and $u_2(x,y) = (q+1)(\min\{q-1, y-x-2\})$ if $y-2 \ge x \ge 0$. For all integers n, x, y such that $n \ge 3$ define inductively the integers $u_n(x,y)$ by the formulas $u_n(x,y) = 0$ if $x \ge y-1$, $u_n(x,y) = u_n(0,y)$ if $y < 0$ and $u_n(x,y) = (q+1)(\min\{(q^n - q^{(n-1)})/(q-1), (q+1)u_{n-1}(x,y)\}$ if $y-2 \ge x \ge 0$. If $n \ge 3$ assume $z-1 \le \min\{c-1, u_n(e,d)\}$. Then there exists a zero-dimensional scheme $Z \subset \mathbf{P}^n$ defined over $\mathbb{F}_q$ such that* $\mathrm{length}(Z) = c$*, $\sharp(Z_{red}) = z$, each connected component of Z is defined over $\mathbb{F}_q$, $h^0(\mathbf{P}^n, \mathcal{I}_Z(d-2))) = 0$, $h^1(\mathbf{P}^n, \mathcal{I}_Z(t)) = 0$ for all $t \ge d-1$, $c-z+1$ connected of Z are reduces, while the other one, A, haa P as its support and $eP \subseteq A \subsetneq (e+1)P$, where e is the maximal non-negative integer such that $\binom{n+e-1}{n} \le c-z+1$.*

Remark 1. Take $n = 1$. Using [1] it is easy to construct non-complete linear systems on $\mathbf{P}^1$ with non-classical Hermite invariants, i.e. for which a statement similar to Theorem 1 is not true for any $P \in \mathbf{P}^1(\overline{\mathbb{F}}_q)$.

Proof of Theorem 1. Consider the exact sequence

$$0 \to (\mathcal{I}_P)^d \to (\mathcal{I}_P)^{(d-1)} \to 0(\mathcal{I}_P)^{(d-1)}/(\mathcal{I}_P)^d \to 0 \tag{1}$$

of coherent sheaves on $\mathbf{P}^n$. Since $P \in \mathbf{P}^n(\mathbb{F}_q)$, the exact sequence (1) is defined over $\mathbb{F}_q$. The sheaf $(\mathcal{I}_P)^{(d-1)}/(\mathcal{I}_P)^d$ is a skyscraper sheaf supported by P, defined over $\mathbb{F}_q$, and with length $\binom{d+n-1}{n-1}$. Notice that $c - \binom{n+d-2}{n} \le \binom{d+n-1}{n-1}$. For any $c - \binom{n+d-2}{n}$ linear subspace W of the vector space $(\mathcal{I}_P)^{(d-1)}/(\mathcal{I}_P)^d$ the exact sequence (1) gives a length c zero-dimensional scheme Z such that $(d-1)P \subsetneq Z \subseteq dP$, and $\mathrm{length}(Z) = c$. If V is defined over $\mathbb{F}_q$, then Z is defined over $\mathbb{F}_q$. Since $h^1(\mathbf{P}^n, \mathcal{I}_{dP}(t)) = 0$ for all $t \ge d-1$, $h^1(\mathbf{P}^n, \mathcal{I}_W(t)) = 0$ for all $t \ge d-1$ and all zero-dimensional schemes $W \subseteq dP$. Since $h^1(\mathbf{P}^n, \mathcal{I}_{(d-1)P}(d-2)) = 0$, any scheme W containing $(d-1)P$ satisfies $h^0(\mathbf{P}^n, \mathcal{I}_W(d-2)) = 0$. $\square$

Proof of Theorem 2. Since the case $z = 1$ we may assume $z \ge 2$. Since the case $n = 1$ is obvious, we may assume $n \ge 2$ and use induction on n. First assume $n = 2$. Let L_i, $1 \le i \le q+1$, be the lines of $\mathbf{P}^2$ containing P and defined over $\mathbb{F}_q$. We may do the construction in the proof of Theorem 1 for the integer $c' := c-z+1$ (with either $d' = e$ (case $(e+1)e/2 < c-z+1$) or $d' := e+1$ (case $(e+1)e/2 = c-z+1$) (call Z' the associated zero-dimensional scheme) in such a way that $\mathrm{length}(Z' \cap L_i) = e+1$ if $0 \le i \le$

$\min\{q+1, c-z+1-(e+1)e/2\}$ and $\text{length}(Z' \cap L_i) = e$ for all $q+1 \geq i > c-z+1$. Set $f := \min\{q+1, c-z+1-(e+1)e/2\}$. Hence $0 \leq f \leq q+1$. Define the integers e_i, $1 \leq i \leq q+1$, by the formulas $e_i := \min\{q, d-i-e\}$, if $i \leq f$, and $e_i := \min\{q, d-i-e+1\}$ if $f+1 \leq i \leq q+1$. Take $A_i \in L_i(\mathbb{F}_q) \backslash \{P\}$, $1 \leq i \leq q+1$, such that $\sharp(A_i) = e_i$ and set $Z := Z' \cup \cup_{i=1}^{q+1} A_i$. It is easy to check using Horace Lemma that Z has all the cohomological properties we want to prove Theorem 2 when $n = 2$. If $n \geq 3$ we use Horace Lemma and induction on n and d. The functions $u_n(x, y)$ where defined exactly to make easy the inductive steps. Just note that the case $n = 2$ of Theorem 2 implies the case $z \leq 1 + u_2(e, d)$ used in the inductive steps when $n = 3$. □

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