International Journal of Pure and Applied Mathematics
Volume 43 No. 3 2008, 423-434

REMARKS ON THE MORAVA K-THEORIES OF SPECIAL ORTHOGONAL GROUPS

Vidhyānāth K. Rao

Department of Mathematics
The Ohio State University at Newark
Newark, OH 43055, USA

Abstract: We obtain an explicit and canonical system of generators for the v_l-torsion free part connective Morava K-theories of $SO(2n+1)$. Some simple consequences for the algebra structure are derived from that.

AMS Subject Classification: 57T10, 55N22
Key Words: Morava K-theory, special orthogonal groups

1. Introduction

This paper is part of a long-term project of determining the Hopf algebra structure of Morava K-theories of special orthogonal groups. The module structure was determined in [9]. However, the generators found there turned out to be hard to handle in practice. Our aim in this paper to find a more suitable set of generators that can more easily related to ordinary homology. We expect this to be useful because the determination of the algebra structure of $P(l)_*SO(2^{l+1}-1)$ in [11] played off operations in ordinary (co)homology of $SO(2^{l+1}-1)$ and its projective plane against information from the bar spectral sequence, and we hope to use a similar approach in the general case.

Also, there has been a remarkable advance in our understanding of BP and $K(l)$-theories of infinite complexes, in particular that of spaces related to BO, see [2]. While the methods used there do not give us anything new when applied to finite complexes, using maps induced by inclusion can give us some information.

Progress in studying $K(l)$-theories of finite complexes has not seen similar

Received: February 14, 2008

leaps. But, there has been some advance. Mimura and Nishimoto (see [3], and [5]) determined the Hopf algebra structure of $K(l)_*$ of the exceptional Lie groups when l is the smallest integer for which the group in question has no v_l torsion. This corresponds to the case studied in [11] for the special orthogonal groups. Thus it seems to be an opportune time to return to the question of the algebra structure of $K(l)_*SO(2n+1)$.

After setting up our notation in the next section, a new system of generators for the v_l-torsion free part of $k(l)_*SO(2n+1)$ is given in Section 3. These generators are determined by their image in mod-2 homology and the elements they represent in the bar spectral sequence. As part of this, we prove the homology analog of the result of [6] that $k(l)$-theories of $SO(q)$ have no higher v_l-torsion. In the last section, we note some of the basic consequences of our results for the algebra structure of $K(l)_*SO(2n+1)$ for $n > 2^{l+1}$.

For the reason explained in the remark at the end of Section 4 of [8], we concentrate on the case of $SO(2n+1)$: There is an element $\beta_n - \epsilon \in BP_{2n}\Omega SO(2n+2)$ whose homology suspension $\overline{\beta_n - \epsilon}$ survives to the E^∞-term of the bar spectral sequence. Furthermore, $\overline{\beta_n - \epsilon}$ is primitive, has square zero and commutes with $\overline{\beta}_i$'s and ζ_s's among our generators, and $k(l)_*SO(2n+2)$ is a free $k(l)_*SO(2n+2)$-module with basis $\{1, \overline{\beta_n - \epsilon}\}$. Thus to determine the algebra structure of $K(l)_*SO(2n+2)$, it is enough to determine the commutators of $\overline{\beta_n - \epsilon}$ with the elements $\widehat{\gamma}_{ij}$ defined in the next section. This should be doable by methods similar to those used for our generators. For example, if $n < 2^l$, the algebra structure of $P(l)_*SO(2n)$ can be determined by the method used in [11].

2. Preliminaries

Throughout this paper, we work at the prime 2. In particular, BP will denote the 2-primary Brown-Peterson theory and $k(l)$ and $K(l)$ will denote respecively the connective and periodic lth Morava K-theories at the prime 2. Also, H_*X will denote the ordinary homology of X with $\mathbb{Z}/(2)$ coefficients.

Next, we recall some of the eariler results and set up our notation.

The source for this paragraph is [7]. $G_n = SO(n+2)/(SO(2) \times SO(n))$, the generating variety for the homology of $\Omega_0 SO(n+2)$, has torsion-free homology [1]. Note that $G_\infty = \mathbb{C}P^\infty$. Let x be the image of the standard generator of $MU^*\mathbb{C}P^\infty$ in MU^*G_n. Then $MU\mathbb{Q}^*G_{2n-1} = MU\mathbb{Q}^*[x]/(x^{2n})$. Let $\{\beta_0', \beta_1, \ldots \beta_{2n-1}\}$ be the basis of $MU\mathbb{Q}_*G_{2n-1}$ that is dual to $\{1, x, \ldots, x^{2n-1}\}$.

Let $\beta_0 \in \widetilde{MU}_0\Omega SO(2n+1)$ be the unique element such that $\beta_0^2 = 2\beta_0$. Define $\alpha_i = \sum_{j=0}^{i} c_{i-j}\beta_j$, where c_j is the coefficient of t^{j+1} in the MU-[2]-series. If $1 \le i \le 2n-1$ and $1 \le j \le n-1$, then α_i and β_j are in $\widetilde{MU}_*\Omega SO(2n+1)$.

If h is an MU-algbra theory, we will denote the images of the β's and α's in $h_*\Omega SO(2n+1)$ by the same symbols. These elements are independent of n in the sense that if $0 \le i < n < q$ and $j < 2n$, then the map $h_*\Omega SO(2n+1) \to h_*\Omega SO(2q+1)$ induced by inclusion preserves β_i and α_j. The image of u under the homology suspension $\widetilde{h}_*\Omega SO(2n+1) \to h_{*+1}SO(2n+1)$ will be denoted by $\overline{u}$.

For the rest of this paper we will fix an $l > 0$. For $0 \le i < 2^l$, define $p(i)$ by $2^l \le 2^{p(i)}(2i+1) < 2^{l+1}$ (this was denoted by $k(i)$ in [8], [9] and [11]. It has been changed to avoid confusion with connective Morava K-theory).

Fix a ground ring of charateristic 2. Let $\Gamma_k(u)$ denote the divided power algebra of height k on u. This is the dual of the primitively generated truncated polynomial algebra $P(x)/(x^{2^k})$. The j-th divided power of u will be denoted by $\gamma_j(u)$. These are characterized by the relations $\gamma_1(u) = u$, $\gamma_s(u)\gamma_t(u) = (s,t)\gamma_{s+t}(u)$ where (s,t) is the binomial coefficient $(s+t)!/s!t!$, and $\Delta(\gamma_s(u)) = \sum_{j=0}^{s}\gamma_j(u) \otimes \gamma_{s-j}(u)$, with the convention that $\gamma_0(u) = 1$.

We need some facts concerning the bar spectral sequence; for the details, see [8], Section 3. If G is a compact connected Lie group and h a BP-algebra theory, then the bar spectral sequence

$$E^2_{**}(G,h) = \mathrm{Tor}^{h_*\Omega G}_{**}(h_*, h_*) \Rightarrow h_*G$$

is a spectral sequence of commutative algebras. If $E^r_{**}(G,h)$ is free over h_* for all r, then it a spectral sequence of bicommutative, biassociative Hopf algebras.

Proposition 2.1. (see [8], Theorem 1.1) *Let $l \le s \le \infty$. Then*

$$E^{\infty}_{**}\left(SO(2^{l+1}-1), P(s)\right) = \bigotimes_{i=0}^{2^l-2} \Gamma_{p(i)+1}(\overline{\beta}_i)\,.$$

If $s > l$, then there are no Hopf algebra extension problems.

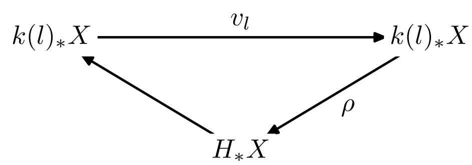
Hence there are unique elements $\gamma_{ij} \in P(l)_{2j(2i+1)}SO(2^{l+1}-1)$ which project to $\gamma_{2j}(\overline{\beta}_i) \in H_{2j(2i+1)}SO(2^{l+1}-1)$ for $0 \le i \le 2^l-2$, $1 \le j \le 2^{p(i)-1}$. For notational convenience, we write $\widehat{\gamma}_{is}$ for $\gamma_{i,2^s}$. We will use the same symbols to denote the images of these elements in $h_*SO(N)$ for $N \ge 2^{l+1}$, where h may be $P(l)$, $k(l)$ or $K(l)$.

Let A be an algebra over a commutative ring R. A is said to be simply generated by $g_1, g_2, \ldots$ over R if A is a free R-module with basis

$$\{1\} \cup \{g_{i_1} g_{i_2} \ldots g_{i_s} \mid i_1 < i_2 < \cdots < i_s\}.$$

As we do not assume that A is commutative, the order of the elements may be important.

We also need some basic facts concerning the Bockstein spectral sequence for Morava K-theories. This spectral sequence, which converges to $(k(l)_*X/v_l\text{-torsion}) \otimes_{k(l)_*} \mathbb{Z}/p$ comes from the exact couple, where ρ is the re-



duction map and v_l denotes multiplication by v_l. In particular, d^1 is the Milnor bockstein Q_l. This spectral sequence is essentially equivalent to the Atiyah-Hizebruch spectral sequence for $K(l)$. For use in the proof of Theorem 3.1 below, we note the fact that $x \in k(l)_*X$ maps to 0 in $H_*(H_*X, Q_l)$ if and only if $x = v_l x_1 + x_2$, where $v_l x_2 = 0$.

The following is standard:

Proposition 2.2. *If* $\operatorname{rank}_{K(l)_*} K(l)_*X = \dim_{\mathbb{Z}/p} H_*(H_*X, Q_l)$, *then Bockstein spectral sequence collapses at* E^2, *and the* v_l*-torsion of* $k(l)_*X$ *is annihilated by* v_l. *Furthermore,* $k(l)_*X$ *injects into* $H_*X \oplus K(l)_*X$.

3. Generators for $k(l)_*SO(2n+1)/v_l$-torsion

This section is devoted to proving the following result.

Theorem 3.1. *Suppose that* $2^l \leq n < \infty$. *Let* $m = [n/2]$.

1. All torsion in $k(l)_*SO(2n+1)$ *is annihilated by* v_l.

2. Suppose that $s = 2^{j-1}(2i+1) \geq 2^l$ *and* $m < s \leq n$. *Then there exists a unique element* $\zeta_s \in k(l)_{4s-2^{l+1}+1}SO(2n+1)$ *that projects to* $\gamma_{2^j}(\overline{\beta}_{2i+1})Q_l(\gamma_{2^j}(\overline{\beta}_{2i+1})) \in H_*SO(2n+1)$ *and satisfies* $v_l\zeta_s = \overline{\alpha}_{2s-1}$.

3. A simple system of generators for $k(l)_*SO(2n+1)/v_l$*-torsion over* $k(l)_*$

is given by

$$\{\zeta_s \mid m < s \leq n\} \cup \{\overline{\beta}_i \mid n - 2^l + 1 \leq i < n\} \cup \{\overline{\beta}_{2i} \mid 2^{l-1} \leq i \leq m - 2^{l-1}\} \cup \{\widehat{\gamma}_{ij} \mid 0 \leq i < 2^{l-1}, 1 \leq j < p(i)\},$$

if $2^{l+1} \leq n$, *and by*

$$\{\zeta_s \mid 2^l \leq s \leq n\} \cup \{\overline{\beta}_i \mid n - 2^l + 1 \leq i < n\} \cup \{\widehat{\gamma}_{ij} \mid 0 \leq i < 2^{l-1}, 1 \leq j < p(i)\},$$

if $2^l \leq n < 2^{l+1}$.

We will separate out some of the steps as lemmas to improve the clarity of the proof.

Lemma 3.2. *Let* $A = \mathbb{Z}/2[x_1, \ldots, x_n, y_1, \ldots, y_n]/(x_i^2, y_i^2)$. *Let* d *be the derivation defined by* $d(x_i) = 0$, $d(y_1) = x_1$, $d(y_i) = y_{i-1}d(y_{i-1}) + x_i$ *for* $2 \leq i \leq n$. *Then*

$$H_*(A, d) = \mathbb{Z}/2[x_2, \ldots, x_n, y_n d(y_n)]/(x_i^2, (y_n d(y_n))^2).$$

Proof. It is easy to see that $A = \bigotimes_{i=1}^n \mathbb{Z}/2[y_i, d(y_i)]/(y_i^2, (d(y_i))^2)$. It follows that $H_*(A, d) = \bigotimes_{i=1}^n \mathbb{Z}/2[y_i d(y_i)]/((y_i d(y_i))^2)$. The conclusion follows from the fact that $y_i d(y_i)$ is homologous to $-x_{i+1}$ for $1 \leq i < n$. □

For $0 \leq i < n$, define $r(i)$ by $n < 2^{r(i)}(2i+1) \leq 2n$. Dualizing the classical description of $H^*SO(2n+1)$ and the action of the Steenrod algebra on it we see that

$$H_*SO(2n+1) = \bigotimes_{i=0}^{n-1} \Gamma_{r(i)+1}(\overline{\beta}_i), \tag{1}$$

$$Q_j \gamma_{2^{s+1}}(\overline{\beta}_i) = \gamma_{2^s}(\overline{\beta}_i) Q_j \gamma_{2^s}(\overline{\beta}_i) + \overline{\beta}_{2^{s+1}(2i+1)-2^{j+1}+1} \tag{2}$$

(Alternatively, we can use [8], Theorem 1.1 and the proof of [11], Lemma 5.2.)

To simplify notation, if $q = 2^t(2i+1) \leq 2n$, we denote $\gamma_{2^t}(\overline{\beta}_i)$ in $H_q SO(2n+1)$ by g_q. □

Lemma 3.3. *Let* Δ *be the diagonal of* $SO(2n+1)$. *Suppose that* $q = 2^t(2i+1) \leq 2n$. *Let* $x = g_q Q_l(g_q)$. *Then*

$$\Delta_*(x) - (1 \otimes x + x \otimes 1) = Q_l \left(\sum_{a=1}^{2^{t+1}-1} \gamma_a(\overline{\beta}_i) \otimes \gamma_{2^{t+1}-a}(\overline{\beta}_i) \right).$$

Proof. As $H_*SO(2n+1)$ injects into H_*SO, it is enough to prove this for

SO. Let $j = 2^{t+1}(2i+1) - 2^{l+1} + 1$. In H_*SO, $x + \overline{\beta}_j = Q_l(\gamma_{2^{t+1}}(\overline{\beta}_i))$. So,

$$\begin{aligned}\Delta_*(x) &= Q_l(\Delta_*(\gamma_{2^{t+1}}(\overline{\beta}_i))) - \Delta_*(\overline{\beta}_j) \\ &= 1 \otimes (x + \overline{\beta}_j) + (x + \overline{\beta}_j) \otimes 1 \\ &\quad + Q_l\left(\sum_{a=1}^{2^{t+1}-1} \gamma_a(\overline{\beta}_i) \otimes \gamma_{2^{t+1}-a}(\overline{\beta}_i)\right) \\ &\quad - (1 \otimes \overline{\beta}_j + (\overline{\beta}_j \otimes 1)\end{aligned}$$

This proves the lemma. $\square$

Lemma 3.4. *Suppose that $n \geq 2^l$. Then as a Hopf algebra, $H_*(H_*SO(2n+1), Q_l)$ is isomorphic to*

$$\begin{aligned}&\bigotimes_{i=0}^{2^{l-1}-1} \Gamma_{p(i)}(\gamma_2(\overline{\beta}_i)) \otimes \bigotimes_{i=2^{l-1}}^{m-2^{l-1}} E(\overline{\beta}_{2i}) \otimes \bigotimes_{i=n-2^l+1}^{n-1} E(\overline{\beta}_i) \\ &\quad \otimes \bigotimes_{i=0}^{[(n-1)/2]} E(\gamma_{2^{r(i)}}(\overline{\beta}_i) Q_l(\gamma_{2^{r(i)}}(\overline{\beta}_i))) \qquad \text{if } 2^{l+1} \leq n \\ &\bigotimes_{i=0}^{2^{l-1}-1} \Gamma_{p(i)}(\gamma_2(\overline{\beta}_i)) \otimes \bigotimes_{i=2^{l-1}}^{n-1} E(\overline{\beta}_i) \otimes \bigotimes_{i=2^l}^{n} E(g_{2i} Q_l(g_{2i})) \\ &\qquad \text{if } 2^l + 2^{l-1} \leq n < 2^{l+1} \\ &\bigotimes_{i=0}^{n-2^l} \Gamma_{p(i)}(\gamma_2(\overline{\beta}_i)) \otimes \bigotimes_{i=n-2^l+1}^{2^{l-1}-1} \Gamma_{p(i)+1}(\overline{\beta}_i) \otimes \bigotimes_{i=2^{l-1}}^{n-1} E(\overline{\beta}_i) \\ &\quad \otimes \bigotimes_{i=2^l}^{n} E(g_{2i} Q_l(g_{2i})) \qquad \text{if } 2^l \leq n < 2^l + 2^{l-1} - 1\end{aligned}$$

as a Hopf algebra.

Proof. First consider the case $n \geq 2^{l+1}$. Using (1), we see that as an algebra, we can write $H_*SO(2n+1)$ as

$$\begin{aligned}&\bigotimes_{i=n-2^l+1}^{n-1} E(\overline{\beta}_i) \otimes \bigotimes_{i=0}^{2^{l-1}-1} \Gamma_{p(i)}(\gamma_2(\overline{\beta}_i)) \\ &\quad \otimes E(\gamma_{2^j}(\overline{\beta}_i), \overline{\beta}_{2^{j-1}(2i+1)-2^l} \mid 0 \leq i < 2^{l-1}, p(i) \leq j < r(i)) \\ &\quad \otimes E(\gamma_{2^j}(\overline{\beta}_i), \overline{\beta}_{2^{j-1}(2i+1)-2^l} \mid 2^{l-1} \leq i < n, 1 \leq j < r(i))\end{aligned}$$

Now (2) implies that Lemma 3.2 applies to the last two factors. It follows that the algebra structure is as claimed. Lemma 3.3 implies that $[\gamma_{2^j}(\overline{\beta}_i)Q_l(\gamma_{2^j}(\overline{\beta}_i))]$ is primitive in $H_*(H_*SO(2n{+}1), Q_l)$. The rest of the coalgebra structure follows from that of $H_*SO(2n+1)$.

The proofs of the other two cases are similar, except for the decomposition of $H_*SO(2n+1)$ as an algebra. These are

$$\bigotimes_{i=0}^{2^{l-1}-1} \Gamma_{p(i)}(\gamma_2(\overline{\beta}_i)) \otimes \bigotimes_{i=2^{l-1}}^{n-1} E(\overline{\beta}_i) \otimes \bigotimes_{i=2^l}^{n} E(g_{2i}, \overline{\beta}_{2i-2^{l+1}+1})$$

if $2^l + 2^{l-1} \leq n < 2^{l+1}$ and

$$\bigotimes_{i=0}^{n-2^l} \Gamma_{p(i)}(\gamma_2(\overline{\beta}_i)) \otimes \bigotimes_{i=n-2^l+1}^{2^{l-1}-1} \Gamma_{p(i)+1}(\overline{\beta}_i) \otimes \bigotimes_{i=2^{l-1}}^{n-1} E(\overline{\beta}_i) \otimes \bigotimes_{i=2^l}^{n} E(g_{2i}, \overline{\beta}_{2i-2^{l+1}+1})$$

if $2^l \leq n < 2^l + 2^{l-1} - 1$. The details are left to the reader. □

Proof of Theorem 3.1. We proceed by induction on n. The base case of $n = 2^l$ is similar to the induction step. The difference is that the only ζ_s that is relevant is for $s = 2^l$, and the claim at the end of the next paragraph is trivially true. We will leave the details to the reader.

Note that for $m < s < n$, the image of $\zeta_s \in k(l)_*SO(2n-1)$ under the homomorphism induced by inclusion has the properties listed in 2 of the theorem. Let X be

$$\begin{aligned}\{\zeta_s \mid m < s < n\} &\cup \{\overline{\beta}_i \mid n - 2^l + 1 \leq i \leq n-1\}\\ &\cup \{\overline{\beta}_{2i} \mid 2^{l-1} \leq i \leq m - 2^{l-1}\}\\ &\cup \{\widehat{\gamma}_{ij} \mid 0 \leq i \leq 2^{l-1}-1, 1 \leq j < p(i) - 1\}\end{aligned}$$

if $2^{l+1} \leq n$, and

$$\begin{aligned}\{\zeta_s \mid 2^l \leq s < n\} &\cup \{\overline{\beta}_i \mid n - 2^l + 1 \leq i \leq n-1\}\\ &\cup \{\widehat{\gamma}_{ij} \mid 0 \leq i \leq 2^{l-1}-1, 1 \leq j < p(i) - 1\}\end{aligned}$$

if $2^l \leq n < 2^{l+1}$. Using the induction assumption and Lemma 3.4, it is easy to show that the image of X in $H_*(H_*SO(2n+1), Q_l)$ together with $g_{2n}Q_l(g_{2n})$ gives a simple system of generators for $H_*(H_*SO(2n+1), Q_l)$.

There is an $x \in k(l)_*SO(2n{+}1)$ such that $v_l x = \overline{\alpha}_{2n-1}$ because $\overline{\alpha}_{2n-1}$ maps to 0 in $H_*SO(2n+1)$. The induction assumption and [9], Theorem 1.1 imply that $\{x\} \cup X$ is simple system of generators for $K(l)_*SO(2n+1)$. So

$$\dim_{\mathbb{Z}/2} H_*(H_*SO(2n+1), Q_l) = \dim_{K(l)_*} K(l)_*SO(2n+1)$$

Hence the Bockstein spectral sequence collapses at E^2. It follows that v_l-torsion in $k(l)_*SO(2n+1)$ is annihilated by v_l and that there exists $y \in k(l)_*SO(2n+1)$ that maps to $g_{2n}Q_l(g_{2n})$. Consequently, $\{y\} \cup X$ projects to a simple system of generators of $H_*(H_*SO(2n+1), Q_l)$. Hence $\{y\} \cup X$ is a simple system of generators for $k(l)_*SO(2n+1)/v_l$-torsion

Now degree considerations imply that $x = ey + x_1 + x_2$, where $e = 0$ or 1, x_1 is a sum of simple monomials in X and x_2 is v_l-torsion. If $e = 0$, then $v_l x = \overline{\alpha}_{2n-1}$ would be in the subalgebra of $K(l)_*SO(2n+1)$ generated by X, contradicting [9], Theorem 1.1. So $e = 1$.

Let Δ be the diagonal map of $SO(2n+1)$. As $\overline{\alpha}_{2n-1}$ originates in $BP_*\Sigma\Omega$ $SO(2n+1)$, $\Delta_*(\overline{\alpha}_{2n-1}) = 1\times\overline{\alpha}_{2n-1}+\overline{\alpha}_{2n-1}\times 1$. It follows that $v_l(\Delta_*(x)-1\times x-x\times 1) = 0$ and that the image of x in $H_*(H_*SO(2n+1), Q_l)$ is primitive. Same is true of y by Lemma 3.3. As $v_l x_2 = 0$, x_2 maps to 0 in $H_*(H_*SO(2n+1), Q_l)$. Thus the image of $x_1 = x - y$ in $H_*(H_*SO(2n+1), Q_l)$ is primitive. This image is a linear combination of simple monomials in Y. Using Lemma 3.4, we see that this must be trivial. It follows that x_1 is the sum of a v_l-divisible element and a v_l-torsion element. Changing x_2 if necessary, we can assume that x_1 is divisble by v_l. Then $v_l(x - x_2) = v_l x = \overline{\alpha}_{2n-1}$ and $x - x_2$ and y have the same image in $H_*SO(2n+1)$. □

4. Remarks on the Algebra Structure

Proposition 4.1. *Suppose that* $n \geq 2^l$.

1. *Torsion elements are central in* $k(l)_*SO(2n+1)$.

2. *The subalgebra of* $k(l)_*SO(2n+1)$ *generated by* $\overline{\beta}_i$, $0 \leq i < n$ *and* ζ_s, $\min(2^l-1, n/2) < s \leq n$ *is exterior.*

3. *If* $n \geq 2^{l+1}$, *then the subalgebra of* $k(l)_*SO(2n+1)$ *generated by* $\widehat{\gamma}_{ij}$, $0 \leq i < 2^{l-1}$, $1 \leq j < p(i)$ *is commutative.*

Proof. Commutators in $k(l)_*SO(2n+1)$ are divisible by v_l because $H_*SO(2n+1)$ is commutative. Combining this with Theorem 3.1.1 shows that any commutator that is v_l-torsion must be trivial.

Suppose that x, y in $\widetilde{k(l)}_*SO(2n+1)$ have filtration 1 in the bar spectral sequence. Then $[x, y]$ trivial because the bar spectral sequence is a spectral sequence of commutative algebras, $E^*_{*,\mathrm{odd}} = 0$, and $E^*_{0*} = k(l)_*(\mathrm{pt})$. So the $\overline{\beta}_i$'s and $\overline{\alpha}_j$'s generate a commutative subalgebra. It follows that $[\overline{\beta}_i, \zeta_s]$ and $[\zeta_s, \zeta_t]$ are v_l-torsion. So they must be trivial. A similar argument shows that $\overline{\beta}_i^2 = 0$

and $\zeta_s^2 = 0$.

To prove the last statement, observe that by the calculations in [11], section 5, the commutators of the form $[\widehat{\gamma}_{ij}, \widehat{\gamma}_{st}]$ lie in the ideal generated by $\overline{\beta}_i$, $0 \leq i < 2^l$. But the latter elements are trivial in $K(l)_*SO(2^{l+2}+1)$ by [9], Lemma 3.3. □

Lemma 4.2. *If $n \geq 2^l - 3$, then $K(l)_*SO(2n+1)$ is a cocommutative Hopf algebra and $K(l)^*SO(2n+1)$ is the dual Hopf algebra.*

Proof. Q_{l-1} acts trivially on the image of the homology suspension $BP_*\Omega SO(2n+1) \to K(l)_{*+1}SO(2n+1)$. This includes $\overline{\beta}_i$'s and $\overline{\alpha}_{2j+1}$'s. It follows from [9], Lemma 3.3 and [11], Lemma 5.2 that Q_{l-1} acts trivially on $\widehat{\gamma}_{ij}$'s. These generate $K(l)_*SO(2n+1)$ as an algebra. As Q_{l-1} is an algebra derivation, it follows that Q_{l-1} acts trivially on all of $K(l)_*SO(2n+1)$. Now the first claim follows from the $K(l)$-theory analog of [10], Corollary 1.6. The second follows by a similar diagram chase. □

Remark. The next result follows from [2]. A direct proof from our earlier calculations is simple enough that we give it here.

Lemma 4.3. *$K(l)_*SO$ is a bicommutative Hopf algebra, and as a coalgebra, is given by*

$$\bigotimes_{i=2^{l-1}}^{\infty} E(\overline{\beta}_{2i}) \otimes \bigotimes_{i=0}^{2^{l-1}-1} \Gamma_{p(i)}(\gamma_{i1}) \,.$$

Proof. The same argument as above shows that Q_{l-1} acts trivially on $K(l)_*SO$. Hence the left hand square in the diagram below, where τ denotes the appropriate switch map, commutes.

$$\begin{array}{ccccc}
K(l)_*SO \otimes K(l)_*SO & \longrightarrow & K(l)_*(SO \times SO) & \xrightarrow{\text{mult}_*} & K(l)_*SO \\
\tau\big\downarrow & & \tau_*\big\downarrow & & \big\| \\
K(l)_*SO \otimes K(l)_*SO & \longrightarrow & K(l)_*(SO \times SO) & \xrightarrow{\text{mult}_*} & K(l)_*SO
\end{array}$$

The right hand square commutes because SO is a commutative H-space. Thus $K(l)_*SO$ is a bicommutative Hopf algebra. It follows that there are no coalgebra extension problems in the Bar spectral sequence. □

Remark. It follows from Proposition 4.1 and [11], Proposition 5.8 that the only non-trivial algebra extensions are given by the relations

$$\widehat{\gamma}_{i,p(i)}^2 = v_l \widehat{\gamma}_{2^{p(i)-1}(2i+1)-2^{l-1},1}.$$

In view of the above, it is worthwhile to determine the kernal of

$K(l)_*SO(2n+1) \to K(l)_*SO$. We will do so by slightly modifying the generators obtained in the last section.

Let $[-1]t$ be the $[-1]$-series for $k(l)_*$. Then we can write

$$([-1]t)^q = t^q + \sum_{i=1}^{\infty} a_{qi} v_l^i t^{q+i(2^l-1)} . \tag{3}$$

Define

$$\overline{\eta}_{2i+1} = \sum_{1 \le s < (2i+2^l)/(2^l-1)} a_{2i+s-(s-1)(2^l-1),s} v_l^{s-1} \overline{\beta}_{2i+s-(s-1)(2^l-1)} .$$

Lemma 4.4. *$v_l \overline{\eta}_{2i+1} = 0$ in $k(l)_*SO(2n+1)$ for $i < n/2 - 2^{l-1}$.*

Proof. It follows from [7], Theorem 1.1(6) that

$$\left(1 + \sum_{i=1}^{n-1} \beta_j t^j\right)\left(1 + \sum_{i=1}^{n-1} \beta_j([-1]t)^j\right) = 1 \bmod t^n$$

in $k(l)_*\Omega SO(2n+1)[[t]]$. Substituting from (3), expanding and equating the coefficients of t^{2i+2^l} gives

$$\beta^2_{i+2^{l-1}} + \sum_{\substack{1 \le j,s \\ j+s(2^l-1)=2i+2^l}} a_{js} v_s^l \beta_j + \sum_{\substack{1 \le j,s,u \\ j+u+s(2^l-1)=2i+2^l}} a_{js} v_s^l \beta_j \beta_u = 0 \tag{4}$$

in $\widetilde{k(l)}_*\Omega SO(2n+1)$. Using the fact that the homology suspension $\widetilde{k(l)}_*\Omega SO(2n+1) \to \widetilde{k(l)}_{*+1} SO(2n+1)$ annihilates decomposables, we see that the second term in (4) has trivial homology suspension, that is $v_l \overline{\eta}_{2i+1} = 0$. □

Lemma 4.5. *If $n/2 < s < 2^{l-1} + n/2$ and $q \ge 2s$, then $\zeta_s + \overline{\beta}_{2s-2^l} \in k(l)_*SO(2n+1)$ maps to a v_l-torsion element in $k(l)_*SO(2q+1)$.*

Proof. $\overline{\alpha}_{2s-1} = v_l \overline{\beta}_{2s-2^l}$ in $k(l)_*SO(2q+1)$ because $[2]t = v_l t^{2^l}$ in $k(l)_*$. Hence $v_l(\zeta_s + \overline{\beta}_{2s-2^l}) = \overline{\alpha}_{2s-1} + v_l \overline{\beta}_{2s-2^l}$ maps trivially into $k(l)_*SO(2q+1)$. □

Proposition 4.6. *For $n \ge 2^{l+1}$, the kernal of $K(l) * SO(2n+1) \to K(l)_*SO$ is the right ideal generated by $\overline{\eta}_{2i+1}$, $n/2 - 2^{l-1} \le i < (n-1)/2$ and $\zeta_s + \overline{\beta}_{2s-2^l}$, $n/2 < s < 2^{l-1} + n/2$.*

Proof. Let

$$\begin{aligned} A &= \{\zeta_s + \overline{\beta}_{2s-2^l} \mid m < s < 2^{l-1} + n/2\} \\ &\quad \cup \{\overline{\eta}_{2i+1} \mid n - 2^l \le 2i < n-1\} , \\ B &= \{\zeta_s \mid 2^{l-1} + n/2 \le s \le n\} \cup \{\overline{\beta}_{2i} \mid 2^l \le 2i < n\} \\ &\quad \cup \{\widehat{\gamma}_{ij} \mid 0 \le i < 2^{l-1}, 1 \le j < p(i)\} . \end{aligned}$$

Note that $a_{2f+1,1} = 1$: By [4], , $[-1]t = t + v_l t^{2^l} \bmod v^{2^l+1}$ in $k(l)_*$. So, $([-1]t)^{2f+1} = t^{2f+1} + v_l t^{2f+2^l} \bmod v^{2^l+1}$. Combining this with Proposition 4.1, we see that the simple monomials in $A \cup B$ are related to the simple monomials in the generators given in Theorem 3.1.3 by an invertible linear transformation over $k(l)_*$.

By Lemma 4.5 and Lemma 4.4, the elements of A map trivially into $K(l)_*SO$. Lemma 4.5 also implies that ζ_s maps to $\overline{\beta}_{2s-2^l}$ if $2^{l-1} + n/2 \leq s \leq n$. It follows from [8], Theorem 1.1 that simple monomials in B map to linearly independent elements in the E^∞-term of the bar spectral sequence converging to $K(l)_*SO$. Thus the kernal of $K(l)_*SO(2n+1) \to K(l)_*SO$ is the $K(l)_*$-submodule generated by those simple monomials in $A \cup B$ that contain one or more elements of A. □

Proposition 4.7. *Let $n \geq 2^{l+1}$. If $x \in k(l)_*SO(2n+1)$ is in the $k(l)_*$-submodule generated by $\{\overline{\beta}_i \mid i < n\} \cup \{\zeta_s \mid m < s \leq n\}$, then $[\widehat{\gamma}_{ij}, x] - [\widehat{\gamma}_{i,j-1}, x]\widehat{\gamma}_{i,j-1}$ is in the $k(l)_*$-submodule generated by*

$$\{\zeta_s + \overline{\beta}_{2s-2^l} \mid m < s < 2^{l-1} + n/2\} \cup \{\overline{\eta}_{2i+1} \mid n - 2^l \leq 2i < n-1\}.$$

Proof. As $H_*SO(2n+1)$ is commutative, $[\widehat{\gamma}_{ij}, x] - [\widehat{\gamma}_{i,j-1}, x]\widehat{\gamma}_{i,j-1}$ reduces to 0 in $\mathbb{Z}/2$-homology. Thus we can write it as $v_l y$.

Let A and B be as in the previous proof. Then we can write $y = z + w$ where z is v_l-torsion and $w = \sum w_i$ where each w_i is a simple monomial in $A \cup B$. As $K(l)_*SO(2n+1)$ is a Hopf algebra and x maps to a primitve element of $K(l)_*SO(2n+1)$, [-, x] is an Hopf algebra derivation. Hence, $[\widehat{\gamma}_{ij}, x] - [\widehat{\gamma}_{i,j-1}, x]\widehat{\gamma}_{i,j-1} = v_l y = v_l w$ maps to a primitive element of $K(l)_*SO(2n+1)$. The Hopf algebra structure of the E^∞-term of the bar spectral sequence implies that each w_i consists of a single factor. By Proposition 4.6, this factor must be from A. □

References

[1] Raoul Bott, The space of loops on a Lie group, *Michigan Math. J.*, **5**, (1958), 35-61.

[2] Nitu Kitchloo, Laures, Gerd, Wilson, W. Stephen, The Morava K-theory of spaces related to BO, *Adv. Math.*, **189**, No. 1 (2004), 192-236.

[3] Mamoru Mimura, Nishimoto, Tetsu, Hopf algebra structure of Morava K-theory of the exceptional Lie groups, In: *Recent Progress in Homotopy*

Theory, Baltimore, MD (2000); *Contemp. Math.*, **293**, Amer. Math. Soc., Providence, RI, (2002), 195-231.

[4] M.R.F. Moreira, Formal groups and the BP-spectrum, *J. Pure. App. Alg*, **18**, No. 1 (1980), 79-89.

[5] Tetsu Nishimoto, Higher torsion in the Morava K-theory of $\mathrm{SO}(m)$ and $\mathrm{Spin}(m)$, *J. Math. Soc. Japan*, **53**, No. 2 (2001).

[6] Tetsu Nishimoto, Hopf algebra structure of Morava K-theory of the exceptional Lie groups. II, *Math. J. Okayama Univ.*, **44**, (2002), 57-121.

[7] Vidhyānāth K. Rao, The Hopf algebra structure of the complex bordism of the loop spaces of the special orthogonal groups, *Indiana Univ. Math. J.*, **38**, No. 2 (1989), 277-291.

[8] Vidhyānāth K. Rao, The bar spectral sequence converging to $h_*(\mathrm{SO}(2n+1))$, *Manuscripta Math.*, **65**, No. 1 (1989), 47-61.

[9] Vidhyānāth K. Rao, On the Morava K-theories of $\mathrm{SO}(2n+1)$, *Proc. Amer. Math. Soc.*, **108**, No. 4 (1990), 1031-1038.

[10] Vidhyānāth K. Rao, $\mathrm{Spin}(n)$ is not homotopy nilpotent for $n \geq 7$, *Topology*, **32**, No. 2 (1993), 239-249.

[11] Vidhyānāth K. Rao, Towards the algebra structure of the Morava K-theory of the orthogonal groups, *Manuscripta Math.*, **94**, No. 3, (1997), 287-301.