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POPULATION PROCESSES SUBJECT TO A MASS MOVEMENT

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Abstract: This paper is concerned with three stochastic processes (Poisson, pure birth and pure death) subject to mass movement immigration or emigration. Such positive or negative mass movements may occur on a single occasion, or recur at times distributed as a Poisson process. For positive movements in the recurrent case, the resulting process is shown to be the sum of the Poisson, birth or death process and an independent immigration type process.

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1. Introduction

In this paper, we consider a population process $\{X(t), t \geq 0\}$ with probability generating function (pgf)

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$$F(u,t) = \sum_{j=0}^{N} p_j(t)u^j, \quad 0 < u \le 1, \; t \ge 0,$$

where N may be infinite. Gani and Swift [3] have shown that, subject to a catastrophe in time $(t, t+\delta t)$ with probability $\gamma\delta t + o(\delta t)$, the pgf will be

$$G(u,t) = e^{-\gamma t}F(u,t) + \int_0^t \gamma e^{-\gamma v} F(u,v)\, dv. \tag{1}$$

More often, a population $X(t)$ is subject to a mass movement of size $\pm M$ by immigration or emigration in the time $(t, t+\delta t)$, so that the population becomes $[X(t)+M]$ or $[X(t)-M]_+$, where

$$[X(t)-M]_+ = 0, \text{ if } X(t) \le M.$$

In these cases, the respective pgfs are

$$G_1(u,t) = e^{-\gamma t}F(u,t) + \int_0^t \gamma e^{-\gamma v} \sum_j p_j(v) u^{j+M} dv, \tag{2a}$$

and

$$G_2(u,t) = e^{-\gamma t}F(u,t) + \int_0^t \gamma e^{-\gamma v} \sum_j p_j(v) u^{[j-M]_+} dv. \tag{2b}$$

We proceed to examine some well known population processes subject to such mass movements.

2. Immigration with a Mass Movement

Let us first consider a Poisson immigration process, subject to a positive or negative mass movement of size M, where M is a discrete nonnegative random variable with pgf $H(u)$. The Poisson process has the pgf

$$F(u,t) = e^{-\lambda t(1-u)},$$

so that

$$\begin{aligned} G_1(u,t) &= e^{-(\gamma+\lambda-\lambda u)t} + \int_0^t \gamma e^{-(\gamma+\lambda-\lambda u)v} H(u)\, dv \\ &= e^{-(\gamma+\lambda-\lambda u)t} - \gamma H(u) \frac{e^{-(\gamma+\lambda-\lambda u)t} - 1}{\gamma + \lambda(1-u)}. \end{aligned} \tag{3}$$

One can write this more simply in terms of Laplace transforms, as follows:

$$G_1^*(u,s) = \int_0^\infty e^{-st} G_1(u,t)\, dt = \int_0^\infty e^{-(s+\gamma+\lambda-\lambda u)t}\, dt$$

$$+\int_0^\infty e^{-st}\,dt \int_0^t \gamma e^{-(\gamma+\lambda-\lambda u)v} H(u)\,dv.$$

Interchanging the integral signs, we find that

$$G_1^*(u,s) = \frac{1}{s+\gamma+\lambda-\lambda u} + \frac{\gamma H(u)}{\gamma+\lambda-\lambda u}\left[\frac{1}{s} - \frac{1}{\gamma+\lambda-\lambda u+s}\right]. \tag{4}$$

In much the same way, if the pgf for the r.v. M is

$$H(u) = \sum_{M=0}^{\infty} q_M u^M\,, \tag{5}$$

then we obtain

$$G_2(u,t) = e^{-(\gamma+\lambda-\lambda u)t} + \sum_{j=0}^{\infty}\left(\frac{\gamma}{\lambda+\gamma}\right)\left(\frac{\lambda u}{\lambda+\gamma}\right)^j h_j(u^{-1})P(j+1,(\gamma+\lambda)t), \tag{6}$$

where

$$\begin{aligned} h_j(u^{-1}) &= q_0 + q_1u^{-1} + \ldots + q_{j-1}u^{-(j-1)} + (q_j + \ldots)u^{-j} \\ &= \sum_{M=0}^{j-1} q_M u^{-M} + \sum_{M=j}^{\infty} q_M u^{-M} \end{aligned}$$

and

$$P(a,t) = \int_0^t e^{-v}\frac{v^{a-1}}{(a-1)!}\,dv$$

is the incomplete gamma function (cf. Abramowitz and Stegun [1], 6.5.1).

The associated Laplace transform is

$$G_2^*(u,s) = \frac{1}{\gamma+\lambda-\lambda u+s} + \frac{\gamma}{s}\sum_{j=0}^{\infty}\frac{\lambda^j}{(s+\lambda+\gamma)^{j+1}}\,u^j h_j(u^{-1}). \tag{7}$$

As an interesting example, we consider the pgf $G_2(u,t)$ from (6) for a Poisson immigration process subject to a (non-random) mass emigration of size M. The process has the following expressions for the transient probabilities

$$p_0(t) = G_2(0,t) = e^{-(\gamma+\lambda)t} + \gamma\int_0^t \sum_{j=0}^{M} e^{-(\gamma+\lambda)v}\frac{(\lambda v)^j}{j!}\,dv, \tag{8}$$

and for $k \geq 1$,

$$p_k(t) = \frac{\partial^k G_2(0,t)}{\partial u^k} = e^{-(\gamma+\lambda)t}\frac{(\lambda t)^k}{k!} + \int_0^t \gamma e^{-(\gamma+\lambda)v}\frac{(\lambda v)^{k+M}}{(k+M)!}\,dv. \tag{9}$$

A representative graph of the transient probabilities $p_0(t), p_1(t), p_2(t)$, and $p_3(t)$ is shown in Figure 1. Observe that these transient probabilities approach a non-zero steady-state.

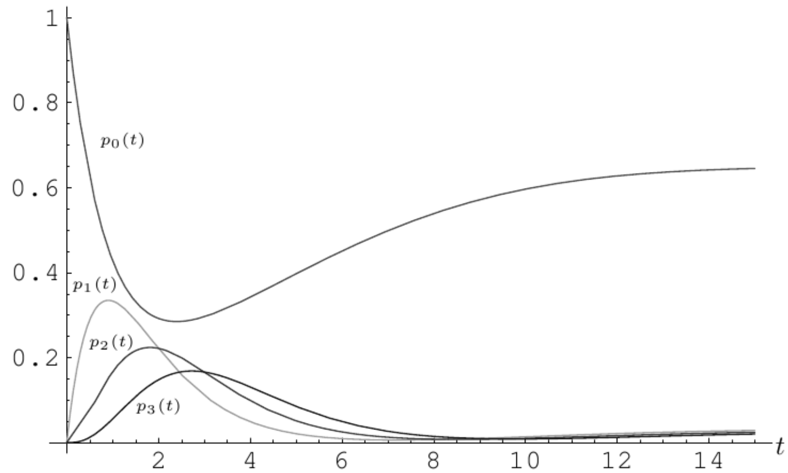

Figure 1: The first few transient probabilities $p_0(t), p_1(t), p_2(t)$, and $p_3(t)$ for a Poisson immigration process subject to a mass emigration of size M. Here $\lambda = 1$, $\gamma = 0.1$ and $M = 10$

If $\gamma = 0$ in equations (8) and (9), the equations reduce to the familiar Poisson process with rate λ and hence go to zero as $t \to \infty$. However, if $\gamma > 0$, a stationary distribution for (8) and (9) does exist. Letting $t \to \infty$ in both (8) and (9), one finds that

$$p_0 = \sum_{j=0}^{M} \frac{\gamma \lambda^j}{(\gamma + \lambda)^{j+1}} = 1 - \left(\frac{\lambda}{\gamma + \lambda} \right)^{M+1} \tag{10}$$

and

$$p_k = \left(\frac{\gamma}{\gamma + \lambda} \right) \left(\frac{\lambda}{\gamma + \lambda} \right)^{k+M}, \text{ for } k = 1, 2, \ldots . \tag{11}$$

Note that as M increases, the stationary distribution becomes $p_0 = 1, p_k = 0$, for $k \geq 1$, as one would expect.

It is also interesting to note the behavior of $p_0(t)$ for different values of γ. Figure 2 shows a graph of $p_0(t)$ for several values of γ. Note that the time at which $p_0(t)$ is a minimum decreases as γ increases.

By differentiating $p_0(t)$, the time $t_{\min}$ at which $p_0(t)$ is a minimum is found as a solution of the equation

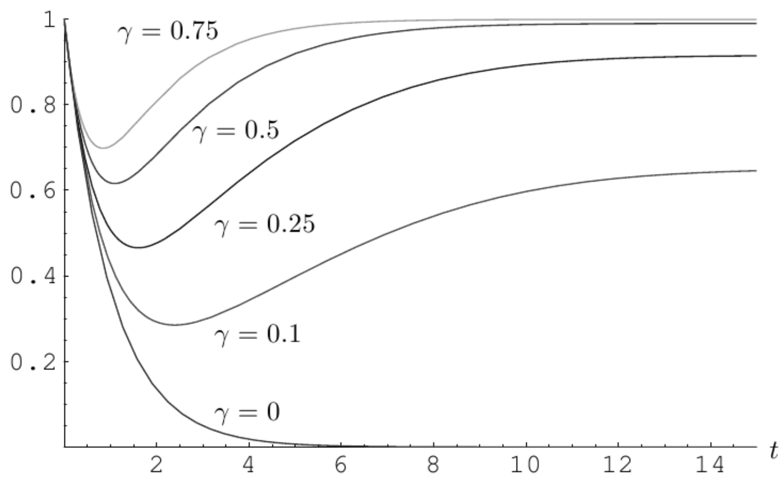

Figure 2: A graph of $p_0(t)$ for $\lambda = 1, M = 10$ with different values of γ

$$\frac{\gamma + \lambda}{\gamma} = \sum_{j=0}^{M} \frac{(\lambda t)^j}{j!}. \tag{12}$$

For large M, the right hand side of equation (12) is approximately $e^{\lambda t}$ so that the time at which $p_0(t)$ is a minimum is approximately

$$t_{\min} \approx \frac{1}{\lambda} \ln\left(\frac{\gamma + \lambda}{\gamma}\right). \tag{13}$$

Thus, by equation (13), as γ increases,

$$t_{\min} \approx \frac{1}{\lambda} \ln\left(1 + \frac{\lambda}{\gamma}\right) \to 0.$$

The expected population size at any time t can be readily found from equation (6) (or alternately, directly from equation (9)) as

$$E[X(t)] = \lambda t e^{-\gamma t} + \sum_{k=1}^{\infty} \frac{k\gamma\lambda^{k+M}}{(k+M)!} \int_0^t v^{k+M} e^{-(\gamma+\lambda)v}\, dv. \tag{14}$$

A graph of $E[X(t)]$ is shown in Figure 3. The steady-state expected value, using equations (10) and (11) is found as

$$E[X] = \left(\frac{\lambda}{\gamma}\right)\left(\frac{\lambda}{\gamma + \lambda}\right)^M.$$

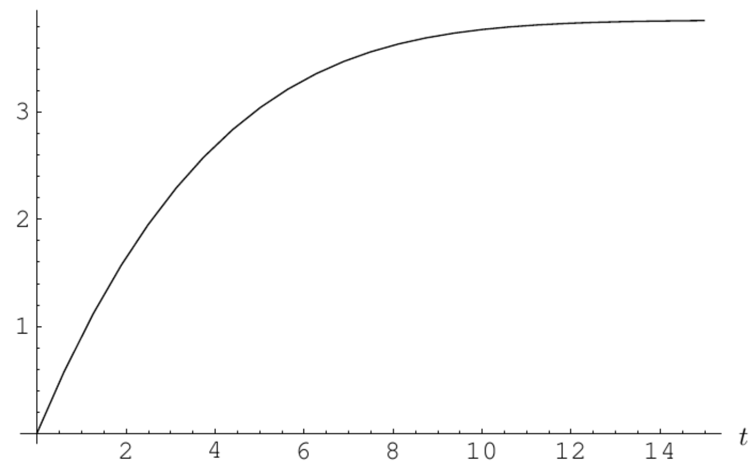

Figure 3: A graph of $E[X(t)]$ for $\lambda = 1, M = 10$ and $\gamma = 0.1$

A related problem is that in which the mass movement is recurrent at times distributed as a Poisson process with parameter γ. We consider the Kolmogorov forward equations for this process

$$p_n'(t) = -(\lambda + \gamma)p_n(t) + \lambda p_{n-1}(t) + \gamma \sum_{M=0}^{n} p_{n-M}(t) q_M \,, \tag{15}$$

where the pgf of the mass movement is given by equation (5). The pgf $\phi(u, t) = \sum_{n=0}^{\infty} p_n(t)u^n$ satisfies

$$\frac{\partial \phi(u, t)}{\partial t} = -(\lambda + \gamma)\phi(u, t) + \lambda u \phi(u, t) + \gamma \phi(u, t) H(u) \,,$$

so that

$$\frac{dt}{1} = \frac{d\phi}{(-\lambda - \gamma + \lambda u + \gamma H)\phi} = \frac{du}{0} \,, \tag{16}$$

which gives

$$u = C_1 \text{ (constant) and } \frac{d\phi}{\phi} = (\lambda u + \gamma H - (\lambda + \gamma))dt.$$

Integrating, we have

$$\ln \phi = \int (\lambda(u-1) + \gamma(H-1))dt = \lambda(u-1)t + \gamma(H-1)t + C_2 \,,$$

so that

$$\phi(u, t)e^{-(\lambda(u-1)+\gamma(H-1))t} = f(u). \tag{17}$$

When $t=0$, $\phi(u,0)=1$ and $1=f(u)$ so that

$$\phi(u,t)=e^{\lambda(u-1)t+\gamma(H(u)-1)t} \tag{18}$$

which as expected, is the pgf of the sum of a Poisson process and a compound Poisson process.

3. The Pure Birth Process with Mass Movement

A pure birth process $X(t)$ with $X(0)=1$ has pgf

$$\begin{aligned} F(u,t) &= \frac{ue^{-\lambda t}}{1-u(1-e^{-\lambda t})} \\ &= ue^{-\lambda t}\sum_{j=0}^{\infty} u^j(1-e^{-\lambda t})^j \\ &= ue^{-\lambda t}\sum_{j=0}^{\infty} u^j \sum_{k=0}^{j}(-1)^k\binom{j}{k}e^{-\lambda kt} \\ &= \sum_{j=0}^{\infty} u^{j+1} \sum_{k=0}^{j}(-1)^k\binom{j}{k}e^{-\lambda(k+1)t}, \end{aligned} \tag{19}$$

so that the pgf for the process $[X(t)-M]_+$ subject to mass movement of size $M\geq 0$ is

$$\begin{aligned} G(u,t) &= e^{-\gamma t-\lambda t}\frac{u}{1-u(1-e^{-\lambda t})} \\ &\quad +\int_0^t \gamma e^{-\gamma v}\sum_{j=0}^{\infty} u^{[j+1-M]_+}\sum_{k=0}^{j}\binom{j}{k}(-1)^k e^{-\lambda(k+1)v}\,dv \\ &= e^{-\gamma t-\lambda t}\frac{u}{1-u(1-e^{-\lambda t})} \\ &\quad +\gamma\sum_{j=0}^{\infty} u^{[j+1-M]_+}\sum_{k=0}^{j}(-1)^k\binom{j}{k}\left(\frac{1-e^{-(\lambda(k+1)+\gamma)t}}{\lambda(k+1)+\gamma}\right). \end{aligned} \tag{20}$$

More generally, if M is a nonnegative discrete random variable with pgf $H(u)$ as defined in equation (5) and $h_j(u^{-1})$ defined as after equation (6), we have

$$G(u,t) = e^{-(\gamma+\lambda)t}\frac{u}{1-u(1-e^{-\lambda t})}$$

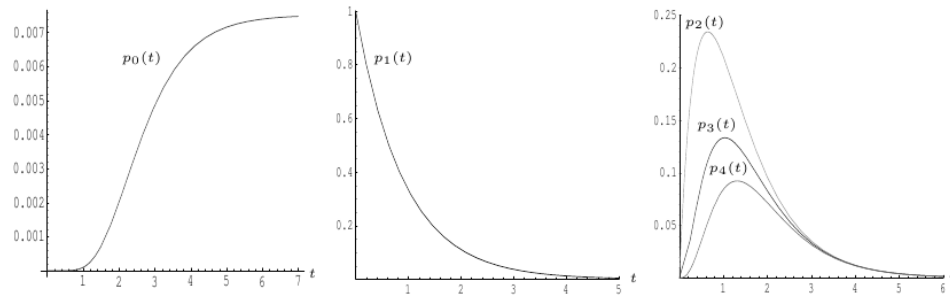

Figure 4: (a) $p_0(t)$ for $\gamma = 0.1, \lambda = 1$ and $M = 10$. (b) $p_1(t)$ for $\gamma = 0.1, \lambda = 1$ and $M = 10$. (c) $p_2(t), p_3(t)$ and $p_4(t)$ for $\gamma = 0.1, \lambda = 1$ and $M = 10$.

$$+\gamma \sum_{j=0}^{\infty} u^{j+1} h_{j+1}(u^{-1}) \sum_{k=0}^{j} (-1)^k \binom{j}{k} \frac{1 - e^{-[\lambda(k+1)+\gamma]t}}{\lambda(k+1)+\gamma}. \quad (21)$$

The Laplace transform of the pgf given in equation (21) is

$$G^*(u,s) = \sum_{j=0}^{\infty} u^{j+1} \sum_{k=0}^{j} (-1)^k \binom{j}{k} \frac{1}{\gamma + \lambda(k+1) + s}$$
$$+ \gamma \sum_{j=0}^{\infty} u^{j+1} h_{j+1}(u^{-1}) \sum_{k=0}^{j} (-1)^k \binom{j}{k} \left[\frac{1}{s(\gamma + \lambda(k+1) + s)} \right]. \quad (22)$$

From equation (20), the transient probabilities for a pure birth process subject to a mass movement of (nonrandom) size $M > 0$ are

$$p_0(t) = \gamma \sum_{k=0}^{M-1} (-1)^k \binom{M-1}{k} \left(\frac{1 - e^{-(\lambda(k+1)+\gamma)t}}{\lambda(k+1)+\gamma} \right) \quad (23)$$

and for $k \geq 1$,

$$p_k(t) = e^{-(\gamma+\lambda)t}(1 - e^{-\lambda t})^{k-1} + \sum_{r=0}^{M+k-1} (-1)^r \binom{M+k-1}{r} \left(\frac{1 - e^{-(\lambda(k+1)+\gamma)t}}{\lambda(k+1)+\gamma} \right). \quad (24)$$

Figure 4 shows a representative graph of the transient probabilities of a pure birth process with mass movement.

Letting $t \to \infty$ in equation (23) gives the stationary probability for p_0 as

$$p_0 = \gamma \sum_{k=0}^{M-1} (-1)^k \binom{M-1}{k} \left(\frac{1}{\lambda(k+1)+\gamma} \right)$$

$$= \left(\frac{\gamma}{\lambda}\right) \frac{(M-1)!\Gamma\left(\frac{\gamma+\lambda}{\lambda}\right)}{\Gamma(M+\frac{\gamma}{\lambda}+1)}, \tag{25}$$

where

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx$$

is the gamma function. Simplifying equation (25) with the identity $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$ gives

$$p_0 = \left(\frac{\gamma}{\lambda}\right)^2 \left(\frac{M!}{\prod_{k=1}^{M}(k+\frac{\gamma}{\lambda})}\right). \tag{26}$$

From equation (24), the stationary probabilities for p_k, $k \geq 1$, are

$$p_k = \sum_{r=0}^{M+k-1} (-1)^r \binom{M+k-1}{r} \left(\frac{1}{\lambda(k+1)+\gamma}\right) = 0. \tag{27}$$

These values agree with the behavior illustrated in Figure 4. Setting $\lambda = \gamma$ in equation (26) gives the simple expression

$$p_0 = \frac{1}{M+1}.$$

The expectation of the process can be readily obtained from equation (21) as

$$\begin{aligned} \frac{\partial G(1,t)}{\partial u} &= E[X(t)] = e^{(\lambda-\gamma)t} \\ &+\gamma \sum_{j=0}^{\infty} [j+1-M]_+ \sum_{k=0}^{j} (-1)^k \binom{j}{k} \frac{(1-e^{-[\lambda(k+1)+\gamma]t})}{\lambda(k+1)+\gamma} \\ = \quad & e^{(\lambda-\gamma)t} \\ &+\gamma \sum_{j=M}^{\infty} [j+1-M] \sum_{k=0}^{j} (-1)^k \binom{j}{k} \frac{(1-e^{-[\lambda(k+1)+\gamma]t})}{\lambda(k+1)+\gamma}. \end{aligned} \tag{28}$$

The pure birth process with positive mass movement recurrent at times distributed as a Poisson process with parameter γ can also be considered. The Kolmogorov forward equations for this process are

$$p_n'(t) = -(\lambda n+\gamma)p_n(t) + \lambda(n-1)p_{n-1}(t) + \gamma \sum_{M=0}^{n} p_{n-M}(t) q_M\,,$$

where the pgf $H(u)$ of the mass movement is given by equation (5).

The pgf $\phi(u,t) = \sum_{n=0}^{\infty} p_n(t) u^n$ satisfies

$$\frac{\partial \phi}{\partial t} = -\lambda u \frac{\partial \phi}{\partial u} - \gamma\phi + \lambda u^2 \frac{\partial \phi}{\partial u} + \gamma \phi H.$$

Hence

$$\frac{dt}{1} = \frac{-du}{\lambda u(u-1)} = \frac{d\phi}{\gamma\phi(H-1)}\,,$$

so that

$$\lambda t = \ln\left(\frac{u}{u-1}\right) + C_1 \text{ (constant)}$$

or

$$\frac{u}{u-1} e^{-\lambda t} = C_1' \text{ (constant)}.$$

Also

$$\begin{aligned}
\frac{d\phi}{\phi} &= \frac{\gamma}{\lambda}(H-1)\left(\frac{1}{u} - \frac{1}{u-1}\right) du \\
&= \frac{\gamma}{\lambda}\left[\frac{q_0 - 1}{u} + q_1 + q_2 u + q_3 u^2 + \dots \right. \\
&\quad \left. + H'(1) + \frac{(u-1)}{2!} H''(1) + \frac{(u-1)^2}{3!} H'''(1) + \dots \right] du
\end{aligned}$$

gives upon integrating

$$\begin{aligned}
\ln\phi &= \frac{\gamma}{\lambda}\left[(q_0 - 1)\ln u + q_1 u + q_2 \frac{u^2}{2} + q_3 \frac{u^3}{3} + \dots \right. \\
&\quad \left. + (u-1)H'(1) + \frac{(u-1)^2}{2 \cdot 2!} H''(1) + \frac{(u-1)^3}{3 \cdot 3!} H'''(1) + \dots \right] \\
&= \frac{\gamma}{\lambda}\left[J(u) + \psi(u-1)\right],
\end{aligned}$$

where

$$J(u) = (q_0 - 1)\ln u + q_1 u + q_2 \frac{u^2}{2} + q_3 \frac{u^3}{3} + \dots$$

and

$$\psi(u-1) = (u-1)H'(1) + \frac{(u-1)^2}{2 \cdot 2!} H''(1) + \frac{(u-1)^3}{3 \cdot 3!} H'''(1) + \dots.$$

We now have

$$\phi e^{-\frac{\gamma}{\lambda}[J(u) + \psi(u-1)]} = f\left(\frac{u}{u-1} e^{-\lambda t}\right)$$

and for $t = 0$, $\phi(u,0) = u$, so that

$$u e^{-\frac{\gamma}{\lambda}[J(u) + \psi(u-1)]} = f\left(\frac{u}{u-1}\right)$$

or

$$f(z) = \frac{z}{z-1} e^{-\frac{\gamma}{\lambda}[J(\frac{z}{z-1}) + \psi(\frac{1}{z-1})]}.$$

Hence

$$f\left(\frac{u}{u-1}e^{-\lambda t}\right) = \frac{ue^{-\lambda t}}{u(e^{-\lambda t}-1)+1} e^{-\frac{\gamma}{\lambda}[J\left(\frac{ue^{-\lambda t}}{1-u(1-e^{-\lambda t})}\right) + \psi\left(\frac{u-1}{1-u(1-e^{-\lambda t})}\right)]}$$

and so the pgf is

$$\phi(u,t) = \frac{ue^{-\lambda t}}{1-u(e^{-\lambda t}-1)} e^{-\frac{\gamma}{\lambda}[J\left(\frac{ue^{-\lambda t}}{1-u(1-e^{-\lambda t})}\right) - J(u) + \psi\left(\frac{u-1}{1-u(1-e^{-\lambda t})}\right) - \psi(u-1)]},$$

or the product of the pgf of the pure birth process and an independent immigration type process (see Gani and Stals [2] for further details).

4. Death Processes with Mass Movement

Let us now consider the death process $X(t)$ with mass movement. In this case of $X(0) = N$, then the pgf is

$$F(u,t) = (ue^{-\mu t} + 1 - e^{-\mu t})^N. \tag{29}$$

If there is a mass movement M of immigration in $(t, t+\delta t)$ with probability $\gamma\delta t$ then

$$\begin{aligned} G(u,t) &= e^{-\gamma t}F(u,t) + \int_0^t \gamma e^{-\gamma v}\,dv \sum_{j=0}^{N} \binom{N}{j} (e^{-\mu v})^{N-j}(1-e^{-\mu v})^j u^{N-j+M} \\ &= e^{-\gamma t}F(u,t) \\ &+ \gamma \sum_{j=0}^{N}\sum_{k=0}^{j} \frac{N!}{(N-j)!(j-k)!k!}(-1)^k \left(\frac{1-e^{-(\mu(N-j+k)+\gamma)t}}{\mu(N-j+k)+\gamma}\right) u^{N-j+M}. \end{aligned} \tag{30}$$

If M is a random variable with pgf $H(u)$ as defined in equation (5), then

$$\begin{aligned} G_1(u,t) = \sum_{j=0}^{N}\sum_{k=0}^{j} \binom{N}{j}\binom{j}{k}(-1)^k \\ \times \left(e^{-(\gamma+(N-j+k)\mu)t}u^{N-j} + \frac{1-e^{-(\gamma+(N-j+k)\mu)t}}{\gamma+(N-j+k)\mu}\gamma u^{N-j}H(u)\right), \end{aligned} \tag{31}$$

while for a mass movement M of emigration, where M is random with

$$h_{N-j}(u^{-1}) = \sum_{k=0}^{N-j-1} u^{-k}q_k + u^{-(N-j)} \sum_{k=N-j}^{\infty} q_k,$$

we have

$$G_2(u,t) = \sum_{j=0}^{N}\sum_{k=0}^{j}\binom{N}{j}\binom{j}{k}(-1)^k \times \left(e^{-(\gamma+(N-j+k)\mu)t}u^{N-j} + \frac{1-e^{-(\gamma+(N-j+k)\mu)t}}{\gamma+(N-j+k)\mu}\gamma u^{N-j}h_{N-j}(u^{-1})\right). \quad (32)$$

The Laplace transforms are respectively

$$G_1^*(u,s) = \sum_{j=0}^{N}\sum_{k=0}^{j}\binom{N}{j}\binom{j}{k}(-1)^k\left(\frac{u^{N-j}}{\gamma+(N-j+k)\mu+s} + \frac{\gamma u^{N-j}H(u)}{\gamma+(N-j+k)\mu}\left[\frac{1}{s}-\frac{1}{\gamma+(N-j+k)\mu+s}\right]\right), \quad (33)$$

and

$$G_2^*(u,s) = \sum_{j=0}^{N}\sum_{k=0}^{j}\binom{N}{j}\binom{j}{k}(-1)^k\left(\frac{u^{N-j}}{\gamma+(N-j+k)\mu+s} + \frac{\gamma u^{N-j}h_{N-j}(u^{-1})}{\gamma+(N-j+k)\mu}\left[\frac{1}{s}-\frac{1}{\gamma+(N-j+k)\mu+s}\right]\right). \quad (34)$$

The transient probabilities for the pure death process subject to a nonrandom mass movement of size $M>0$ can be found from equation (30) as

$$p_0(t) = (1-e^{-\mu t})^N e^{-\gamma t}, \quad (35)$$

and for $1 \le r \le M-1$

$$p_r(t) = \binom{N}{r}e^{-(\gamma+r\mu)t}(1-e^{-\mu t})^{N-r}. \quad (36)$$

When $r=M$, the transient probability is

$$p_M(t) = \binom{N}{M}e^{-(\gamma+M\mu)t}(1-e^{-\mu t})^{N-M} + \gamma\sum_{k=0}^{N}\binom{N}{k}(-1)^k\left(\frac{1-e^{-(k\mu+\gamma)t}}{k\mu+\gamma}\right), \quad (37)$$

and for $r=N+M$,

$$p_{N+M}(t) = \left(\frac{\gamma}{N\mu+\gamma}\right)(1-e^{-(N\mu+\gamma)t}). \quad (38)$$

Figures 5 and 6 show representative graphs of the transient probabilities of a pure death process with mass movement. We note the interesting behavior of $p_M(t)$, when $M=3$.

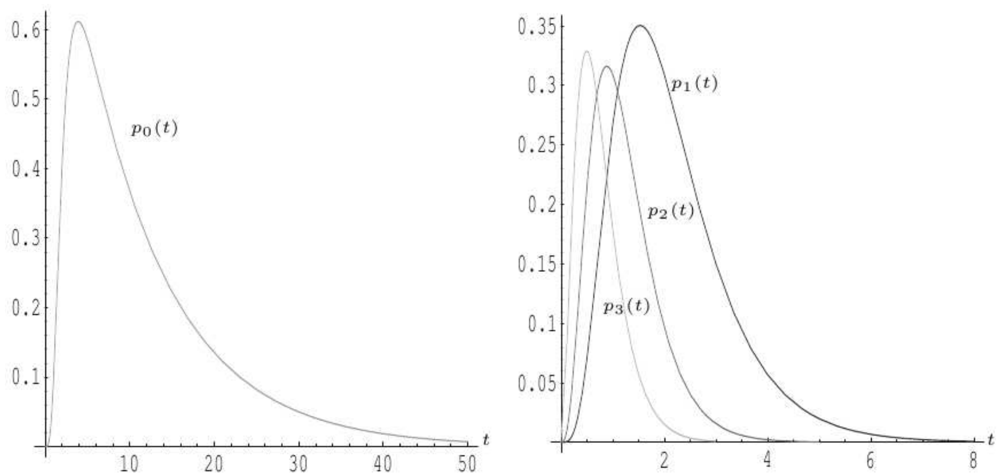

Figure 5: (a) $p_0(t)$ for $N = 5, \gamma = 0.1, \mu = 1$ and $M = 3$. (b) $p_1(t), p_2(t)$ and $p_3(t)$ for $N = 5, \gamma = 0.1, \mu = 1$ and $M = 3$

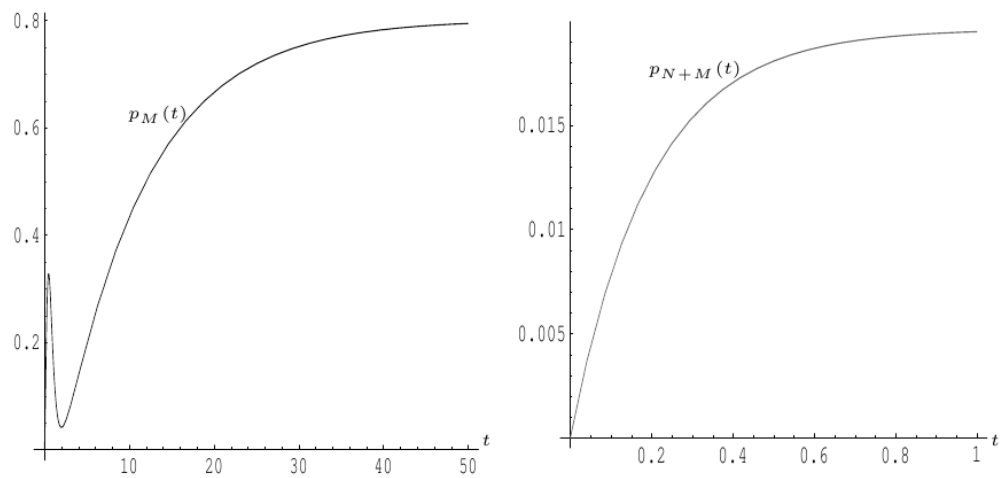

Figure 6: (a) $p_M(t)$ for $N = 5, \gamma = 0.1, \mu = 1$ and $M = 3$. (b) $p_{N+M}(t)$ for $N = 5, \gamma = 0.1, \mu = 1$ and $M = 3$

Letting $t \to \infty$ in equations (36)-(38) gives the stationary probabilities $p_r = 0$ for $0 \leq r \leq M-1$ and

$$p_M = \gamma \sum_{k=0}^{N} \frac{\binom{N}{k}(-1)^k}{k\mu + \gamma}, \tag{39}$$

while

$$p_{N+M} = \frac{\gamma}{N\mu + \gamma}. \tag{40}$$

To conclude this section, we consider the related problem of a pure death process with mass movement recurrent at times distributed as a Poisson process with parameter γ. The Kolmogorov forward equations for this process are

$$p_n'(t) = -(\gamma + n\mu)p_n(t) + (n+1)\mu p_{n+1}(t) + \gamma \sum_{M=0}^{n} p_{n-M}(t) q_M\,,$$

or, for the pgf $\phi(u,t) = \sum_{n=0}^{\infty} p_n(t) u^n$

$$\frac{\partial \phi(u,t)}{\partial t} = -\mu(u-1)\frac{\partial \phi(u,t)}{\partial u} + \gamma\phi(u,t)(H(u)-1)\,,$$

where the pgf $H(u)$ of the mass movement is as in equation (5).

Hence

$$\frac{dt}{1} = \frac{du}{\mu(u-1)} = \frac{d\phi}{\gamma\phi(H(u)-1)}$$

which lead to

$$(u-1)e^{-\mu t} = A \text{ (constant)},$$

and

$$\begin{aligned} \frac{d\phi}{\phi} = \frac{\gamma}{\mu}\frac{H(u)-1}{u-1}\,du &= \frac{\gamma}{\mu}\frac{H(1) - 1 + (u-1)H'(1) + \frac{(u-1)^2}{2!}H''(1) + \dots}{u-1}\,du \\ &= \frac{\gamma}{\mu}\left(H'(1) + \frac{(u-1)}{2!}H''(1) + \frac{(u-1)^2}{3!} + \dots\right)du. \end{aligned}$$

Integrating gives

$$\begin{aligned} \ln \phi(u,t) &= \frac{\gamma}{\mu}\left((u-1)H'(1) + \frac{(u-1)^2}{2\cdot 2!}H''(1) + \frac{(u-1)^3}{3\cdot 3!} + \dots\right) \\ &\quad + B \text{ (constant)} \\ &= \frac{\gamma}{\mu}\psi(u-1) + B, \end{aligned}$$

or

$$\phi(u,t)e^{-\frac{\gamma}{\mu}\psi(u-1)} = C \text{ (constant)}\ .$$

We can now write

$$\phi(u,t)e^{-\frac{\gamma}{\mu}\psi(u-1)} = f\left((u-1)e^{-\mu t}\right).$$

At $t = 0$, $\phi(u,0) = u^N$, so that

$$u^N e^{-\frac{\gamma}{\mu}\psi(u-1)} = f(u-1) \text{ or } f(z) = (z+1)^N e^{-\frac{\gamma}{\mu}\psi(z)}.$$

It follows that

$$\phi(u,t)e^{-\frac{\gamma}{\mu}\psi(u-1)} = \left((u-1)e^{-\mu t}+1\right)^N e^{-\frac{\gamma}{\mu}\psi((u-1)e^{-\mu t})},$$

or

$$\phi(u,t) = e^{-\frac{\gamma}{\mu}\{\phi((u-1)e^{-\mu t})-\psi(u-1)\}}\left[(u-1)e^{-\mu t}+1\right]^N.$$

Note that for $t=0$, $\phi(u,0)=u^N$, while for $u=1$, $\phi(1,t)=1$. We see that the combination of a death process with pgf $[(u-1)e^{-\mu t}+1]^N$ with a compound Poisson process with pgf $e^{\gamma(H(u)-1)t}$ results in the sum of two random variables, one of them the death process, and the other a modified immigration process with pgf $e^{-\frac{\gamma}{\mu}\{\phi((u-1)e^{-\mu t})-\psi(u-1)\}}$. The reader may refer to the paper by Gani and Stals [2] on stochastic processes with immigration for further details.

5. Concluding Remarks

In many of the standard stochastic processes, mass movements may occur in both immigration and emigration. This paper has attempted to derive equations for the pgfs of the Poisson, pure birth and pure death processes subject to positive and negative mass movements. Some knowledge of the probabilities of the population size in such processes has also been gained. Perhaps the most interesting results are those for recurrent positive mass movements occurring at times which follow the Poisson distribution. The resulting pgfs turn out to be products of the pgfs of the original processes and modified immigration processes, similar to those found in Gani and Stals [2]. It is hoped to extend the present research of mass movement to other stochastic processes.

References

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