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ON GENERALIZATIONS OF LAMBERT'S SERIES

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Abstract: The classical Lambert's series makes it possible to generate many remarkable transformations of series. These Lambert's series are all constructed from the function $z/(1-z)$. In this paper we show how to generalize these series by using an arbitrary function in place of $z/(1-z)$. Series transformations exhibiting beautiful symmetry are obtained. In addition, a double contour integral is found which represents these series. Our method is compared to a general procedure introduced by MacMahon.

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1. Introduction

Let

$$B(z) = \sum_{n=1}^{\infty} b_n z^n,$$

then the series

$$L(z) = \sum_{n=1}^{\infty} \frac{b_n}{1 - z^n} \tag{1.1}$$

is called a Lambert's series. These series are mentioned briefly in the classical texts by Abramowitz and Stegun [1], Bromwich [13], Crystal [16], Hardy and

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Wright [28], Knopp [36], MacMahon [39], Polya and Szegö [42], and Titchmarsch [48]. With all the $b_n = 1$, the series is an example of an analytic function defined inside the unit circle, which cannot be continued to a larger domain (see Titchmarsch [48]).

Starting with the classical paper by Knopp [35] in 1913 and continuing to the present day, these series have entered into the theory of numbers, the theory of Weierstrass's elliptic functions, and the theory of basic hypergeometric series. Much of this research was motivated by the ideas of Ramanujan. Lambert series also occur in the expansion of Eisenstein series, a particular kind of modular form. Agarwal [2] gives an excellent survey of these results. The series (1.1) can be transformed into another series

$$L(z) = \sum_{n=1}^{\infty} \frac{b_n}{1 - z^n} = \sum_{n=1}^{\infty} B\left(z^n\right) \tag{1.2}$$

and can be expanded in a Taylor's series

$$L(z) = \sum_{n=1}^{\infty} \left(\sum_{d|n} b_d \right) z^n \,, \tag{1.3}$$

where the inner sum is over all divisors d of n. The series transformation (1.2) and the Taylor's series (1.3) are discussed by Knopp in [35] and [36].

In [36], Knopp lists several examples of (1.2) which he calls "remarkable". Two of these identities are

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{1 - z^n} = \sum_{n=1}^{\infty} \frac{z^n}{1 + z^n}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n} \frac{z^n}{1 - z^n} = \sum_{n=1}^{\infty} \log\left(\frac{z^n}{1 + z^n} \right).$$

In this paper we will discuss several generalizations of the Lambert's series. One generalization is the following: Suppose

$$A(z) = \sum_{n=1}^{\infty} a_n z^n \qquad \text{and} \qquad B(z) = \sum_{n=1}^{\infty} b_n z^n.$$

Then the series transformation (1.2) generalizes to

$$L(z) = \sum_{n=1}^{\infty} b_n A\left(z^n\right) = \sum_{n=1}^{\infty} a_n B\left(z^n\right) \tag{1.4}$$

and the Taylor's series (1.3) now becomes

$$L(z) = \sum_{n=1}^{\infty} \left(\sum_{d|n} a_d b_{n/d} \right) z^n. \tag{1.5}$$

The series (1.4) and (1.5) first appeared in a paper by I.I. Zogin [52] in 1958. In 1981, Audinarayana Moorthy [11] studied the same series, and may not have been aware of Zogin's earlier paper. The paper by Spira [47] also contains related information. We see that these generalizations involve replacing the specific function $z/(1-z)$ in (1.2) and (1.3) by the arbitrary function $A(z)$ in (1.4) and (1.5). We should also note that Julia [31], in 1913, studied the possibility of expanding a general function in a series of the form shown in (1.4).

In this paper we give a double contour integral representation of all these series

$$L(z) = \frac{1}{(2\pi i)^2} \iint_C G(s,t;z) A(s) B(t) dt ds, \tag{1.6}$$

where the kernel is given by

$$G(s,t;z) = \sum_{m,n=0}^{\infty} \frac{z^{mn}}{s^{m+1} t^{n+1}}. \tag{1.7}$$

The contour integral (1.6) and the importance of the function $G(s,t;z)$ in (1.7) may be presented here for the first time. All the technical details concerning the convergence of the above series, the nature of the contours of integration and the analyticity of the functions are discussed in detail in Section 3.

Two examples of our series transformations include

$$\sum_{n=0}^{\infty} \frac{a^n}{n!} \exp\left(bz^n\right) = \sum_{n=0}^{\infty} \frac{b^n}{n!} \exp\left(az^n\right) \tag{1.8}$$

and

$$\sum_{n=0}^{\infty} \frac{a^n \left(cz^n + d\right)^n}{n!} \exp\left(bcz^n\right) = \sum_{n=0}^{\infty} \frac{c^n \left(az^n + b\right)^n}{n!} \exp\left(adz^n\right). \tag{1.9}$$

The symmetry of these examples is surprising. More examples along with the necessary conditions for convergence are discussed in Section 4.

Finally, a more general series transformation and associated Taylor's series is presented. Let $F(s,t)$ be a given function, then

$$\sum_{n=0}^{\infty} \frac{D_t^n F\left(z^n, 0\right)}{n!}$$
$$= \sum_{n=0}^{\infty} \frac{D_s^n F\left(0, z^n\right)}{n!} = F(1,0) + F(0,1) - F(0,0) + \sum_{n=1}^{\infty} c_n z^n, \quad (1.10)$$

where

$$c_n = \sum_{d|n} \frac{D_t^d D_s^{n/d} F\left(0,0\right)}{d!(n/d)!}.$$

The previous transformation (1.4) is the special case of (1.10) in which $F(s,t) = A(s)B(t)$. To the best of our knowledge, (1.10) is new.

The final two sections compare our generalizations with the work of MacMahon.

2. Intuitive Insight

Before giving rigorous proofs of our results in the next section, we present a simple formal argument using power series which will give insight into why our series transformations are valid. Let $F(s,t)$ have a power series expansion without the constant term ($F(0,0) = 0$). Then

$$\begin{aligned} F(s,t) &= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} f_{j,k} s^j t^k \qquad (2.1) \\ &= f_{1,1}s^1t + f_{1,2}s^1t^2 + f_{1,3}s^1t^3 + \cdots \\ &\quad + f_{2,1}s^2t + f_{2,2}s^2t^2 + f_{2,3}s^2t^3 + \cdots \\ &\quad + f_{3,1}s^3t + f_{3,2}s^3t^2 + f_{3,3}s^3t^3 + \cdots \\ &\qquad \vdots \\ &\quad + f_{n,1}s^nt + f_{n,2}s^nt^2 + f_{n,3}s^nt^3 + \cdots \\ &\qquad \vdots \end{aligned}$$

Next we differentiate n times partially with respect to s, and see that the first $n-1$ rows drop out to obtain

$$\begin{aligned} &D_s^n F(s,t) = n!f_{n,1}t + n!f_{n,2}t^2 + n!f_{n,3}t^3 + \cdots \\ &+(n+1)!f_{n+1,1}s^1t + (n+1)!f_{n+1,2}s^1t^2 + (n+1)!f_{n+1,3}s^1t^3 + \cdots \\ &+(n+2)!f_{n+2,1}s^2t + (n+2)!f_{n+2,2}s^2t^2 + (n+2)!f_{n+2,3}s^2t^3 + \cdots \end{aligned}$$

$$\vdots$$

If we set $s = 0$ all but the first row in the above expression drops out and we get

$$D_s^n F(0,t) = n! f_{n,1} t + n! f_{n,2} t^2 + n! f_{n,3} t^3 + \cdots$$

Dividing by $n!$ and letting $t = z^n$ we get

$$\frac{D_s^n F(0,z^n)}{n!} = f_{n,1} z^n + f_{n,2} z^{2n} + f_{n,3} z^{3n} + \cdots \quad (2.2)$$

Summing this last expression on n we get

$$\sum_{n=1}^{\infty} \frac{D_s^n F(0,z^n)}{n!} = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{n,k} z^{kn}. \quad (2.3)$$

Definition 2.1. We will call the sum $\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{n,k} z^{kn}$ the *Lambert power series derived from the function* $F(s,t)$.

Notice that (2.2) is the n-th row of the sum $\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{n,k} z^{kn}$.

Definition 2.2. We call (2.3) the *sum by rows*, or the R sum of the Lambert power series.

The power series on the right side of (2.3) can be rewritten as

$$\sum_{n=1}^{\infty} \frac{D_s^n F(0,z^n)}{n!} = \sum_{n=1}^{\infty} \left(\sum_{d|n} f_{d,n/d} \right) z^n.$$

It is not difficult to see that interchanging the roles of s and t in the above argument would have given us

$$\sum_{n=1}^{\infty} \frac{D_t^n F(z^n,0)}{n!} = \sum_{n=1}^{\infty} \left(\sum_{d|n} f_{d,n/d} \right) z^n. \quad (2.4)$$

Since the right-hand sides of the last two expressions are identical, we have finally our transformation

$$\sum_{n=1}^{\infty} \frac{D_s^n F(0,z^n)}{n!} = \sum_{n=1}^{\infty} \frac{D_t^n F(z^n,0)}{n!} = \sum_{n=1}^{\infty} \left(\sum_{d|n} f_{d,n/d} \right) z^n.$$

Notice that $\frac{D_s^n F(0,z^n)}{n!}$ is the n-th column of the Lambert power series $\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{n,k} z^{kn}$.

Definition 2.3. We call (2.4) the *column sum or the C sum of the Lambert power series.*

Notice also that

$$f_{d,n/d} \quad = \quad \frac{D_s^d D_t^{n/d} F(0,0)}{d!(n/d)!}$$

which is the familiar Taylor's series coefficient in two variables.

This completes our intuitive analysis of our main new result. We left out the minor modification needed to remove the restriction that $F(0,0) = 0$ to keep the analysis simple and clear. The above argument could be made rigorous, but we choose a contour integral approach for that purpose in the next section.

3. Proofs of the Main Results

Having given a quick intuitive look at why our transformation of series should be valid, we now present rigorous statements and proofs. We could begin by expanding all functions in Taylor's series, but we choose to go a different route. We will define a function of three complex variables by a geometric like series

$$G(s,t;z) \quad = \quad \sum_{m,n=0}^{\infty} \frac{z^{mn}}{s^{m+1}t^{n+1}}.$$

This function will then serve as the Kernel in an integral transform

$$L(z) = \frac{1}{(2\pi i)^2} \iint_C G(s,t;z)F(s,t)dtds$$

of the function $F(s,t)$. By expanding the kernel $G(s,t;z)$ in various ways, we will obtain our desired results. This method resembles the way in which the Taylor's series is derived from the geometric series using Cauchy's integral formula.

We begin by defining an important region.

Definition of the region *R*. The region R is the space of the three complex variables s, t and z restricted to

$$R = \{|z| \leq 1,\ |s| \geq \rho > 1,\ |t| \geq \rho > 1\}.$$

Here ρ can be as close to 1 as we please.

Three lemmas regarding the kernel $G(s,t;z)$ will be needed before we can prove our main result.

Lemma 3.1. *The series* $G(s,t;z) = \sum_{m,n=0}^{\infty} \frac{z^{mn}}{s^{m+1}t^{n+1}}$ *is absolutely and uniformly convergent in the three complex variables* s, t *and* z *when confined to any compact subset of the region* R. *Thus* $G(s,t;z)$ *is analytic in* s, t *and* z *on* R.

Proof. For s, t and z on R, we have

$$\left|\frac{z^{mn}}{s^m t^n}\right| \le \frac{1}{\rho^{m+n}}.$$

Since

$$\sum_{m,n=0}^{\infty} \frac{1}{\rho^{m+n}} = \frac{\rho^2}{(\rho-1)^2},$$

the Weierstrass M test tells us that the series for $G(s,t;z)$ converges absolutely and uniformly in every compact subset of the region R. Thus $G(s,t;z)$ is an analytic function of the three variables on the region R. □

Lemma 3.2. *For s, t and z on R, we have*

$$G(s,t;z) = \sum_{m=0}^{\infty} \frac{1}{t^{m+1}(s-z^m)} \tag{3.1}$$

and

$$G(s,t;z) = \sum_{m=0}^{\infty} \frac{1}{s^{m+1}(t-z^m)}. \tag{3.2}$$

Both series converge absolutely and uniformly on the region R.

Proof. Since the series defining G is absolutely convergent, we can evaluate the sum using any arrangement of the terms. Using only the sum of a geometric series we have

$$\begin{aligned} G(s,t;z) = \sum_{m,n=0}^{\infty} \frac{z^{mn}}{s^{m+1}t^{n+1}} \\ = \sum_{m=0}^{\infty} \frac{1}{ts^{m+1}} \sum_{n=0}^{\infty} \left(\frac{z^m}{t}\right)^n = \sum_{m=0}^{\infty} \frac{1}{s^{m+1}(t-z^m)}. \end{aligned}$$

This proves (3.2). Since $G(s,t;z) = G(t,s;z)$ we have (3.1) also. Since

$$\left|\frac{1}{t^{m+1}(s-z^m)}\right| < \frac{1}{\rho^{m+1}(\rho-1)},$$

we see by the Weierstrass M test that the series (3.1) is absolutely and uniformly convergent. The same is true for (3.2). □

Lemma 3.3. *For s, t and z on R, we have*

$$G(s,t;z) = \frac{1}{st} + \frac{1}{st(s-1)} + \frac{1}{st(t-1)} + \frac{1}{st}\sum_{k=1}^{\infty} c_k(s,t)z^k, \tag{3.3}$$

where

$$c_k(s,t) = \sum_{d|k} \frac{1}{s^d t^{k/d}}. \tag{3.4}$$

Here the index of summation d is over all divisors of k.

Proof. We will partition this double sum into four parts:

Part 1: $m = n = 0$.

Part 2: $n = 0$; $m = 1, 2, 3, \cdots$.

Part 3: $m = 0$; $n = 1, 2, 3, \cdots$.

Part 4: $m = 1, 2, 3, ...$; $n = 1, 2, 3, \cdots$.

The right-hand side of (3.3) consists of four terms, and each term is the result of summing over one of the respective four parts just listed. The first term is obvious. The second and third terms use only the sum of a geometric series. The fourth sum is a double sum in which we collect terms in which the product $mn = k$. □

With the above three lemmas available, we now derive our main result.

Theorem 3.1. *Let $F(s,t)$ be analytic in the two complex variables s and t for $|s| < r$ and $|t| < r$, where $r > \rho$ (ρ is defined in the definition of the region R). Then for $|z| \le 1$,*

$$\sum_{n=0}^{\infty} \frac{D_t^n F(z^n, 0)}{n!} = \sum_{n=0}^{\infty} \frac{D_s^n F(0, z^n)}{n!} \tag{3.5}$$

$$= F(1,0) + F(0,1) - F(0,0) + \sum_{n=1}^{\infty} c_n z^n, \tag{3.6}$$

where

$$c_n = \sum_{d|n} \frac{D_t^d D_s^{n/d} F(0,0)}{d!(n/d)!}$$

and the symbol $D_s^n F(a,b)$ means the n-th partial derivative of $F(s,t)$ with respect to s evaluated at $s = a$ and $t = b$.

Proof. Consider the double contour integral

$$L(z) = \frac{1}{(2\pi i)^2} \iint_C G(s,t;z) F(s,t) dt ds\,, \tag{3.7}$$

where the contour C is a circle $|s| = r_0$ in the region of analyticity with $1 < \rho < r_0 < r$ in the complex s-plane, and the same circle in the complex t-plane. From Lemma 3.1 we know that the integrand $F(s,t)G(s,t;z)$ is analytic in this double-annulus. Expanding G with (3.1) we get

$$L(z) = \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^2} \iint_C \frac{F(s,t)}{t^{m+1}(s - z^m)} dtds. \tag{3.8}$$

Here the interchange of summation and integration is valid because the series converges uniformly in s and t over C. Using Cauchy's integral formula we have

$$\frac{1}{2\pi i} \int_C \frac{F(s,t)}{t^{m+1}} dt = \frac{D_t^m F(s,0)}{m!}$$

so our double integral in (3.8) reduces to

$$L(z) = \sum_{m=0}^{\infty} \frac{1}{2\pi i} \frac{1}{m!} \int_C \frac{D_t^m F(s,0)}{s - z^m} ds.$$

Again using Cauchy's integral formula we have

$$L(z) = \sum_{m=0}^{\infty} \frac{D_t^m F(z^m,0)}{m!}.$$

If we evaluate the original double integral using (3.2) in the same way we get our first result (3.5).

To get (3.6) we start again with the double integral (3.7) for $L(z)$ and now expand $G(s,t;z)$ using (3.3). We get

$$L(z) = \frac{1}{(2\pi i)^2} \iint_C G(s,t;z)F(s,t)dtds = \frac{1}{(2\pi i)^2} \iint_C \frac{F(s,t)}{st} dtds$$
$$+ \frac{1}{(2\pi i)^2} \iint_C \frac{F(s,t)}{st(t-1)} dtds + \frac{1}{(2\pi i)^2} \iint_C \frac{F(s,t)}{st(s-1)} dtds + \sum_{n=1}^{\infty} c_n z^n,$$

where

$$c_n = \sum_{d|n} \frac{1}{(2\pi i)^2} \iint_C \frac{F(s,t)}{s^{d+1} t^{n/d+1}} dtds.$$

The first term is $F(0,0)$ using Cauchy's integral formula. The second term is $-F(0,0) + F(0,1)$. The third term is $-F(0,0) + F(1,0)$. These three terms sum to $F(1,0) + F(0,1) - F(0,0)$ and thus the first three terms of (3.6) are verified. Using Cauchy's integral formula, the fourth term above is easily seen to be the fourth term in (3.6). This completes the proof of the theorem. □

Having proved our main result, we next give two corollaries which describe the result for special forms of $F(s,t)$. These two forms are a product $F(s,t) = A(s)B(t)$ and a composite function $F(s,t) = f(as + bt + c)$.

Corollary 3.1. *Let*

$$A(z)=\sum_{n=1}^{\infty}a_nz^n \qquad \text{and} \qquad B(z)=\sum_{n=1}^{\infty}b_nz^n,$$

where both series converge inside the circle $|z|<r$, $(1<r)$. *Then for* $|z|\le 1$, *we have*

$$\sum_{n=0}^{\infty}b_nA\left(z^n\right)=\sum_{n=0}^{\infty}a_nB\left(z^n\right) \tag{3.9}$$

$$= \quad a_0B(1)+b_0A(1)-a_0b_0+\sum_{n=1}^{\infty}\left(\sum_{d|n}a_db_{n/d}\right)z^n. \tag{3.10}$$

Proof. Let $F(s,t)=A(s)B(t)$ in Theorem 3.1. This corollary follows immediately from the three relations:

$$\frac{D_t^nF(z^n,0)}{n!}=A(z^n)\frac{D_t^nB(0)}{n!}=A(z^n)b_n,$$

$$\frac{D_s^nF(0,z^n)}{n!}=B(z^n)\frac{D_s^nA(0)}{n!}=B(z^n)a_n,$$

and

$$\frac{D_s^mD_t^nF(0,0)}{m!n!}=\frac{D_s^mA(0)D_t^nB(0)}{m!n!}=a_mb_n.$$

□

Corollary 3.2. *Let* a, b *and* c *be fixed complex numbers and let the real number* $r>1$. *Let the function* $f(z)$ *be analytic inside the circle* $|z-c|<(|a|+|b|)r$. *Then for* $|z|\le 1$,

$$\sum_{n=0}^{\infty}\frac{a^nf^{(n)}\left(bz^n+c\right)}{n!}=\sum_{n=0}^{\infty}\frac{b^nf^{(n)}\left(az^n+c\right)}{n!} \tag{3.11}$$

$$=f(a+c)+f(b+c)-f(c)+\sum_{n=1}^{\infty}\left(\sum_{d|n}\frac{a^db^{n/d}f^{(d+n/d)}(c)}{d!(n/d)!}\right)z^n. \tag{3.12}$$

Proof. Let $F(s,t)=f(as+bt+c)$. The circle in which the function $f(z)$ is analytic is large enough for $F(s,t)$ to be analytic inside the required region R of Theorem 3.1. Since

$$D_s^nF(s,t)=a^nf^{(n)}(as+bt+c) \qquad \text{and} \qquad D_t^nF(s,t)=b^nf^{(n)}(as+bt+c),$$

the corollary follows from the result of Theorem 1. □

4. Examples

The following examples of series transformations illustrate specific cases of relation (3.5) in Theorem 3.1 of the previous section. All of the examples were checked on a computer for various numerical values of the parameters using the software program *Mathcad*.

Example 4.1. Let $F(s,t) = e^{as+bt}$. Then for all a and b and $|z| \leq 1$, we have

$$\sum_{n=0}^{\infty} \frac{a^n}{n!} \exp\left(bz^n\right) = \sum_{n=0}^{\infty} \frac{b^n}{n!} \exp\left(az^n\right). \tag{4.1}$$

Example 4.2. Let $F(s,t) = e^{(as+b)(ct+d)}$. Then for all a, b, c, and d and $|z| \leq 1$, we have

$$\sum_{n=0}^{\infty} \frac{a^n\left(cz^n+d\right)^n}{n!} \exp\left(bcz^n\right) = \sum_{n=0}^{\infty} \frac{c^n\left(az^n+b\right)^n}{n!} \exp\left(adz^n\right). \tag{4.2}$$

Example 4.3. Let $F(s,t) = (1-as)^{-1}(1-bt)^{-1}$. Then for all $|a| < 1$ and $|b| < 1$ and $|z| \leq 1$, we have

$$\sum_{n=0}^{\infty} \frac{b^n}{1-az^n} = \sum_{n=0}^{\infty} \frac{a^n}{1-bz^n}. \tag{4.3}$$

Example 4.4. Let $F(s,t) = (as+bt+c)^p$. Then for all $|a+b| < |c|$ and $|z| \leq 1$, we have

$$\sum_{n=0}^{\infty} \binom{p}{n} a^n \left(bz^n+c\right)^{p-n} = \sum_{n=0}^{\infty} \binom{p}{n} b^n \left(az^n+c\right)^{p-n}. \tag{4.4}$$

Example 4.5. Let $F(s,t) = (1+as)^p(1+bt)^q$. Then for all $|a| < 1$ and $|b| < 1$ and $|z| \leq 1$, we have

$$\sum_{n=0}^{\infty} \binom{q}{n} b^n \left(1+az^n\right)^p = \sum_{n=0}^{\infty} \binom{p}{n} a^n \left(1+bz^n\right)^q. \tag{4.5}$$

Example 4.6. Let $F(s,t) = (1+as)^p e^{bs}$. Then for all b and p, and for $|a| < 1$ and $|z| \leq 1$, we have

$$\sum_{n=0}^{\infty} \frac{b^n}{n!} \left(1+az^n\right)^p = \sum_{n=0}^{\infty} \binom{p}{n} a^n \exp\left(bz^n\right). \tag{4.6}$$

5. The R, C, RC and CR Sums of MacMahon

In [39], MacMahon gave an interesting and very general way of deriving Lambert series. MacMahon starts with the Lambert power series,

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} f_{m,n} z^{mn}.$$

He does not consider the associated function $F(s,t)$ which we use as a starting point from which to derive the Lambert power series. He introduces four different ways to sum the series called the R, C, RC and CR sums. He can always sum by the R and C methods, but the RC and CR sums can only be evaluated conveniently in a few special cases. In this section we will explain these four sums, and relate our method to MacMahon's. Consider once again the power series expansion

$$F(s,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} f_{m,n} s^m t^n.$$

We now define the series of partial rows

$$R_n(s,t) = \sum_{k=n}^{\infty}\sum_{m=0}^{\infty} f_{m,k} s^m t^k. \tag{5.1}$$

Notice that $R_n(s,t)$ is the same as the sum for $F(s,t)$ with columns $0,1,2,\cdots,n-1$ removed. In the same way we define the series of partial columns

$$C_m(s,t) = \sum_{k=m}^{\infty}\sum_{n=0}^{\infty} f_{k,n} s^k t^n, \tag{5.2}$$

where $C_m(s,t)$ is the same as the sum for $F(s,t)$ with rows $0,1,2,\cdots,m-1$ removed. We observe that

$$D_s^m R_n(s,t) = m! \sum_{k=n}^{\infty}\sum_{h=0}^{\infty} f_{m+h,k} s^h t^k$$

and hence

$$\frac{D_s^m R_n(0,z^m)}{m!} = \sum_{k=n}^{\infty} f_{m,k} z^{km} \tag{5.3}$$

is the partial sum of the Lambert power series

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} f_{m,n} z^{mn}$$

consisting of the m-th row starting at the n-th column.

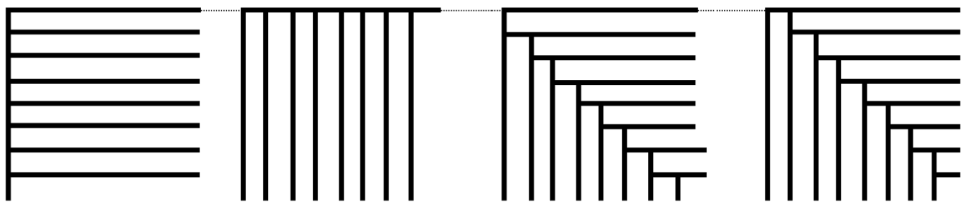

Figure 1: The four methods of sums

In a similar way

$$\frac{D_t^n C_m(z^n, 0)}{n!} = \sum_{k=m}^{\infty} f_{k,n} z^{kn} \tag{5.4}$$

is the partial sum of the Lambert power series consisting of the n-th column starting at the m-th row.

In Section 2 we showed why $\sum_{m=0}^{\infty} \frac{D_s^m R_n(0,z^m)}{m!}$ is the sum of the Lambert power series $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} f_{m,n} z^{mn}$ by rows (R sum), and why $\sum_{n=0}^{\infty} \frac{D_s^n R_m(z^n,0)}{n!}$ is the sum of the Lambert power series by columns (C sum). With the definitions (5.1) and (5.2) and the series (5.3) and (5.4) we can now sum the Lambert power series in many other ways besides simply rows, or simply columns. In particular, following MacMahon, we can define the RC and CR sums explained by the diagram in Figure 1. The diagram shows how the sums are taken over the matrix of indices of the Lambert power series.

Using (5.3) and (5.4) we can write the RC sum as

$$\frac{D_s^0 R_0\left(0, z^0\right)}{0!} + \frac{D_t^0 C_1\left(z^0, 0\right)}{0!} + \frac{D_s^1 R_1\left(0, z^1\right)}{1!} + \frac{D_t^1 C_2\left(z^1, 0\right)}{1!} + \cdots,$$

which is

$$\sum_{n=0}^{\infty} \left(\frac{D_s^n R_n(0, z^n)}{n!} + \frac{D_t^n C_{n+1}(z^n, 0)}{n!} \right). \tag{5.5}$$

The CR sum is

$$\frac{D_t^0 C_1\left(z^0, 0\right)}{0!} + \frac{D_s^0 R_1\left(0, z^0\right)}{0!} + \frac{D_t^1 C_1\left(z^1, 0\right)}{0!} + \frac{D_s^1 R_2\left(0, z^1\right)}{1!} \cdots,$$

which is

$$\sum_{n=0}^{\infty} \left(\frac{D_t^n C_n\left(z^n, 0\right)}{n!} + \frac{D_s^n R_{n+1}\left(0, z^n\right)}{n!} \right). \tag{5.6}$$

Thus (5.5) and (5.6) give us two additional ways to write the Lambert power series, and obtain new generalizations of Lambert series. Unfortunately, there are very few functions for which the partial rows and partial columns, (5.1) and (5.2), can be conveniently evaluated. This makes the RC and CR sums impossible to calculate in most examples. However, MacMahon has shown in [39] that in the examples in which these sums are possible, some very interesting results are obtained. We explore some of these in the next section.

6. Examples Using *RC*, *CR*, *RRC* and *CCR* Sums

Consider the function

$$F(s,t)=\frac{s^a t^d}{\left(1-\alpha s^a\right)\left(1-\beta t^c\right)}, \tag{6.1}$$

where a, b, c, and d are positive integers and $|\alpha|\leq 1$ and $|\beta|\leq 1$. It is clear that the series expansion

$$F(s,t)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\alpha^m\beta^n s^{am+b}t^{cn+d} \tag{6.2}$$

is absolutely convergent for $|s|<1$ and $|t|<1$. In this case we can easily calculate the partial rows (5.1) and partial columns (5.2). We have

$$\begin{aligned}R_{cn+d}(s,t) &= \sum_{m=0}^{\infty}\sum_{k=n}^{\infty}\alpha^m\beta^k s^{am+b}t^{ck+d}\\ &= \frac{\beta^n s^b t^{cn+d}}{1-\alpha s^a}\sum_{k=0}^{\infty}\beta^k t^{ck}\\ &= \frac{\beta^n s^b t^{cn+d}}{\left(1-\alpha s^a\right)\left(1-\beta t^c\right)}.\end{aligned}$$

In a similar way we have

$$C_{am+b}(s,t)=\frac{\alpha^m s^{am+b}t^d}{\left(1-\alpha s^a\right)\left(1-\beta t^c\right)}.$$

From (5.3) and (5.4) we can calculate the partial row and column sums

$$\frac{D_s^{am+b}R_{cn+d}\left(0,z^{am+b}\right)}{(am+b)!}=\frac{\alpha^m\beta^n z^{(am+b)(cn+d)}}{1-\beta z^{c(am+b)}} \tag{6.3}$$

and

$$\frac{D_t^{cn+d}C_{am+b}\left(z^{am+b},0\right)}{(cn+d)!}=\frac{\alpha^m\beta^n z^{(am+b)(cn+d)}}{1-\alpha z^{a(cn+d)}}. \tag{6.4}$$

The R, C, RC, and CR sums of the Lambert power series derived from the function $F(s,t)$ are given respectively by the four series

$$\sum_{n=0}^{\infty} \frac{D_s^{an+b} F\left(0, z^{an+b}\right)}{(an+b)!} = \sum_{n=0}^{\infty} \frac{D_t^{cn+d} F\left(z^{cn+d}, 0\right)}{(cn+d)!}$$
$$= \sum_{n=0}^{\infty} \left(\frac{D_s^{an+b} R_{cn+d}\left(0, z^{an+b}\right)}{(an+b)!} + \frac{D_t^{a(n+1)+b} C_{a(n+1)+b}\left(z^{cn+d}, 0\right)}{(cn+d)!} \right)$$
$$= \sum_{n=0}^{\infty} \left(\frac{D_s^{an+b} R_{cn+d}\left(0, z^{an+b}\right)}{(an+b)!} + \frac{D_t^{a(n+1)+b} C_{a(n+1)+b}\left(z^{cn+d}, 0\right)}{(cn+d)!} \right).$$

Using (6.3) and (6.4) these simplify to the following generalized Lambert series respectively:

$$\sum_{n=0}^{\infty} \frac{\alpha^n z^{d(an+b)}}{1-\beta z^{c(an+b)}} \tag{6.5}$$
$$= \sum_{n=0}^{\infty} \frac{\beta^n z^{b(cn+d)}}{1-\alpha z^{a(cn+d)}} \tag{6.6}$$
$$= \sum_{n=0}^{\infty} \left(\frac{\alpha^n \beta^n z^{(an+b)(cn+d)}}{1-\beta z^{c(an+b)}} + \frac{\alpha^{n+1} \beta^n z^{(a(n+1)+b)(cn+d)}}{1-\alpha z^{a(cn+d)}} \right) \tag{6.7}$$
$$= \sum_{n=0}^{\infty} \left(\frac{\alpha^n \beta^{n+1} z^{(an+b)(c(n+1)+d)}}{1-\beta z^{c(an+b)}} + \frac{\alpha^n \beta^n z^{(an+b)(cn+d)}}{1-\alpha z^{a(cn+d)}} \right). \tag{6.8}$$

In a similar way we find the RRC and CRR sums associated with the function (6.1) and get respectively

$$\sum_{n=0}^{\infty} \left(\frac{\alpha^{2n} \beta^n z^{(2an+b)(cn+d)}}{1-\beta z^{c(2an+b)}} + \frac{\alpha^{2n+1} \beta^n z^{(a(2n+1)+b)(cn+d)}}{1-\beta z^{c(a(2n+1)+b)}} + \frac{\alpha^{2n+2} \beta^n z^{(a(2n+2)+b)(cn+d)}}{1-\alpha z^{a(cn+d)}} \right) \tag{6.9}$$

and

$$\sum_{n=0}^{\infty} \left(\frac{\alpha^n \beta^{2n} z^{(an+b)(2cn+d)}}{1-\alpha z^{a(2cn+d)}} + \frac{\alpha^n \beta^{2n+1} z^{(an+b)(c(2n+1)+d)}}{1-\alpha z^{a(c(2n+1)+d)}} + \frac{\alpha^n \beta^{2n+2} z^{(an+b)(c(2n+2)+d)}}{1-\beta z^{c(an+b)}} \right). \tag{6.10}$$

Of course, the six series 6.5 to 6.10 are all equal. MacMahon in [39] finds

(6.5) to (6.8) for the two special cases $\alpha = \beta = 1$ and $\alpha = \beta = -1$.

We checked these series ((6.5) to (6.10)) on a computer for various values of the parameters. We found that our restriction of a, b, c and d to positive integers seems to be unnecessary. The series appear to be valid when these parameters are any positive numbers. This suggests using appropriate fractional derivatives to prove our theorems.

If we let $\alpha = \beta = -1$, $a = 2$, and $b = c = d = 1$, the above six series become respectively

$$\begin{aligned}
&\frac{z}{1+z} - \frac{z^3}{1+z^3} + \frac{z^5}{1+z^5} - \frac{z^7}{1+z^7} + \frac{z^9}{1+z^9} - \frac{z^{11}}{1+z^{11}} + \cdots \\
= &\frac{z}{1+z} - \frac{z^2}{1+z^4} + \frac{z^3}{1+z^6} - \frac{z^4}{1+z^8} + \frac{z^5}{1+z^{10}} - \frac{z^6}{1+z^{12}} + \cdots \\
= &\frac{z}{1+z^2} - \frac{z^3}{1+z^2} + \frac{z^6}{1+z^3} - \frac{z^{10}}{1+z^4} + \frac{z^{15}}{1+z^5} - \frac{z^{21}}{1+z^6} + \cdots \\
= &\frac{z}{1+z^2} - \frac{z^2}{1+z} + \frac{z^6}{1+z^4} - \frac{z^9}{1+z^3} + \frac{z^{15}}{1+z^6} - \frac{z^{20}}{1+z^5} + \cdots \\
= &\frac{z}{1+z} - \frac{z^3}{1+z^3} + \frac{z^5}{1+z^2} - \frac{z^{10}}{1+z^5} + \frac{z^{14}}{1+z^7} - \frac{z^{18}}{1+z^4} + \cdots \\
= &\frac{z}{1+z^2} - \frac{z^2}{1+z^4} + \frac{z^3}{1+z} - \frac{z^9}{1+z^6} + \frac{z^{12}}{1+z^8} - \frac{z^{15}}{1+z^3} + \cdots .
\end{aligned}$$

MacMahon points out that the identity between the first and third of the above series is a celebrated relation of Jacobi. We could continue and find RCR, CRC, RCC and CRR sums and more, but we choose to end the paper here.

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