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SOME PROPERTIES OF HYPERGEOMETRIC FUNCTIONS FOR A CERTAIN SUBCLASS OF UNIFORMLY CONVEX FUNCTIONS

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Abstract: The purpose of this paper is to give a sufficient condition for a (Gaussian) hypergeometric function to be in a subclass of uniformly convex functions of order α, which is also necessary condition under additional restrictions. Furthermore, an integral operator related to the hypergeometric function is also considered.

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1. Introduction

Let $\mathcal{A}$ be the class consisting of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

that are analytic in the open unit disk $\Delta = \{z : |z| < 1\}$. Let $\mathcal{S}$, $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ denote the subclass of $\mathcal{A}$ consisting of univalent, starlike and convex functions of order α, respectively. For convenience, we write $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{K}(0) = \mathcal{K}$ (see, e.g., Srivastava and Owa [15]).

Motivated by geometric considerations, Goodman [4, 5] introduced the classes $\mathcal{UCV}$ and $\mathcal{UST}$ of uniformly convex and starlike functions. Ronning [9] (also, see [6]) gave a more applicable one variable analytic characterization for $\mathcal{UCV}$. That is, a function $f(z)$ of the form (1) is in $\mathcal{UCV}$ if and only if

$$\mathcal{R}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq \left|\frac{zf''(z)}{f'(z)}\right| \quad (z \in \Delta).$$

We note [4] that the classical Alexander's result, $f(z) \in \mathcal{K} \Leftrightarrow zf'(z) \in \mathcal{S}^*$ does not hold between the classes $\mathcal{UCV}$ and $\mathcal{UST}$. Later on, Ronning [10] introduced the class $\mathcal{S}_p$ consisting of functions such that $f(z) \in \mathcal{UCV} \Leftrightarrow zf'(z) \in \mathcal{S}_p$. And also in [9], Ronning generalized the classes $\mathcal{UCV}$ and $\mathcal{S}_p$ by introducing a parameter α in the following way. That is, a function $f(z)$ of the form (1) is in $\mathcal{S}_p(\alpha)$ if it satisfies the analytic characterization

$$\mathcal{R}\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} \geq \left|\frac{zf'(z)}{f(z)} - 1\right| \quad (\alpha \in \mathbb{R}, z \in \Delta),$$

and $f(z) \in \mathcal{UCV}(\alpha)$, the class of uniformly convex functions of order α, if and only if $zf'(z) \in \mathcal{S}_p(\alpha)$. More over, Srivastava and Mishra [14] presented a systematic and unified study of the classes $\mathcal{UCV}(\alpha)$, $\mathcal{S}_p(\alpha)$.

Definition 1.1. A function $f(z)$ of the form (1) is in $\mathcal{U}(\lambda, \alpha)$, if it satisfies the analytic characterization

$$\mathcal{R}\left[\frac{\lambda z^3 f'''(z) + (1+2\lambda)z^2 f''(z) + zf'(z)}{\lambda z^2 f''(z) + zf'(z)} - \alpha\right] \geq \left|\frac{\lambda z^3 f'''(z) + (1+2\lambda)z^2 f''(z) + zf'(z)}{\lambda z^2 f''(z) + zf'(z)} - 1\right| \quad (\alpha \in \mathbb{R}, 0 \leq \lambda \leq 1, z \in \Delta). \tag{2}$$

We also note that $\mathcal{U}(0, \alpha) = \mathcal{UCV}(\alpha)$.

We denote by T, the subclass of $\mathcal{S}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0) \tag{3}$$

and let $\mathcal{U}_T(\lambda, \alpha) = \mathcal{U}(\lambda, \alpha) \cap T$.

Let $F(a, b; c; z)$ be the (Gaussian) hypergeometric function defined by

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n,$$

where $c \neq 0, -1, -2, \dots$ and $(\beta)_n$ is the Pochhammer symbol defined by

$$(\beta)_n = \begin{cases} 1 & \text{if } n = 0\,, \\ \beta(\beta+1)\dots(\beta+n-1) & \text{if } n \in \mathbb{N} = \{1, 2, \dots\}\,. \end{cases}$$

We note that $F(a, b; c; 1)$ converges for $\mathcal{R}(c - a - b) > 0$ and is related to the Gamma functions by

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}. \tag{4}$$

Merkes and Scott [7] and Ruscheweyh and Singh [11] used continued fractions to find sufficient conditions for $zF(a, b; c; 1)$ to be in $\mathcal{S}^*(\alpha)$ $(0 \leq \alpha < 1)$ for various choices of the parameters a, b and c. Carlson and Shaffer [2] showed how some convolution results about $\mathcal{S}^*(\alpha)$ may be expressed in terms of a linear operator acting on hypergeometric functions. Recently, Silverman [12] gave necessary and sufficient condition for $zF(a, b; c; z)$ to be in $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ (see also [8, 13]).

In the present paper, we determine sufficient conditions for $zF(a, b; c; z)$ to be in $\mathcal{U}(\lambda, \alpha)$ and also give necessary and sufficient conditions for $zF(a, b; c; z)$ to be in $\mathcal{U}_T(\lambda, \alpha)$ with appropriate restrictions on a, b, c. Furthermore, we consider an integral operator related to the hypergeometric function.

2. Some Sufficent Conditions

To establish our main results, we need the following lemmas.

Lemma 2.1. *A sufficient condition for $f(z)$ of the form (1) to be in $\mathcal{U}(\lambda, \alpha)$ $(0 \leq \lambda \leq 1, -1 \leq \alpha < 1)$ is that*

$$\sum_{n=2}^{\infty} [2\lambda n(n-1)(n-2) + [2 + (3-\alpha)\lambda] n(n-1) + (1-\alpha)n]\, |a_n|$$

$$\leq 1-\alpha. \quad (5)$$

Proof. In view of the definition of $\mathcal{U}(\lambda,\alpha)$, it is sufficient if we verify the condition

$$\left|\frac{\lambda z^3 f'''(z)+(1+2\lambda)z^2 f''(z)+zf'(z)}{\lambda z^2 f''(z)+zf'(z)}-1\right| \leq \mathcal{R}\left[\frac{\lambda z^3 f'''(z)+(1+2\lambda)z^2 f''(z)+zf'(z)}{\lambda z^2 f''(z)+zf'(z)}-1\right]+1-\alpha.$$

We have

$$\begin{aligned}
&\left|\frac{\lambda z^3 f'''(z)+(1+2\lambda)z^2 f''(z)+zf'(z)}{\lambda z^2 f''(z)+zf'(z)}-1\right] \\
&\quad -\Re\left[\frac{\lambda z^3 f'''(z)+(1+2\lambda)z^2 f''(z)+zf'(z)}{\lambda z^2 f''(z)+zf'(z)}-1\right] \\
&\leq 2\left|\frac{\lambda z^3 f'''(z)+(1+2\lambda)z^2 f''(z)+zf'(z)}{\lambda z^2 f''(z)+zf'(z)}-1\right| \\
&\leq \frac{\sum_{n=2}^{\infty}\left[2\lambda n(n-1)(n-2)+2(1+\lambda)n(n-1)\right]|a_n||z|^n}{|z|-\sum_{n=2}^{\infty}[\lambda n(n-1)+n]|a_n||z|^n} \\
&\leq \frac{\sum_{n=2}^{\infty}\left[2\lambda n(n-1)(n-2)+2(1+\lambda)n(n-1)\right]|a_n|}{1-\sum_{n=2}^{\infty}[\lambda n(n-1)+n]|a_n|}.
\end{aligned}$$

The above expression is bounded by $1-\alpha$ if and only if equation (5) is satisfied and the proof is complete. □

Now, we show that the sufficient condition of Lemma 2.1 for $\mathcal{U}(\lambda,\alpha)$ is also a necessary condition for $\mathcal{U}_T(\lambda,\alpha)$.

Lemma 2.2. *A necessary and sufficient condition for $f(z)$ of the form (3) to be in $\mathcal{U}_T(\lambda,\alpha)$ $(0\leq\lambda\leq 1, -1\leq\alpha<1)$ is that*

$$\sum_{n=2}^{\infty}\left[2\lambda n(n-1)(n-2)+(2+(3-\alpha)\lambda)n(n-1)+(1-\alpha)n\right]a_n \leq 1-\alpha. \quad (6)$$

Proof. In view of Lemma 2.1, we need to only show that $f(z)\in\mathcal{U}_T(\lambda,\alpha)$ satisfies the inequality (6).

If $f(z)\in\mathcal{U}_T(\lambda,\alpha)$, then the definition of $\mathcal{U}_T(\lambda,\alpha)$ yields

$$(1-\alpha)+\mathcal{R}\left[\frac{\lambda z^3 f'''(z)+(1+2\lambda)z^2 f''(z)+zf'(z)}{\lambda z^2 f''(z)+zf'(z)}-1\right] \geq \left|\frac{\lambda z^3 f'''(z)+(1+2\lambda)z^2 f''(z)+zf'(z)}{\lambda z^2 f''(z)+zf'(z)}-1\right|.$$

Or equivalently,

$$(1-\alpha)-\mathcal{R}\left[\frac{\sum_{n=2}^{\infty}\left[\lambda n(n-1)(n-2)+(1+\lambda)n(n-1)\right]a_nz^n}{z-\sum_{n=2}^{\infty}[\lambda n(n-1)+n]a_nz^n}\right] \geq \left|\frac{\sum_{n=2}^{\infty}\left[\lambda n(n-1)(n-2)+(1+\lambda)n(n-1)\right]a_nz^n}{z-\sum_{n=2}^{\infty}[\lambda n(n-1)+n]a_nz^n}\right|.$$

Choosing values of z on the real axis, so that the left side of this inequality is real, and letting $z \to 1$, we obtain

$$\left\{(1-\alpha)\left[1-\sum_{n=2}^{\infty}(\lambda n(n-1)+n)a_n\right] - \sum_{n=2}^{\infty}[\lambda n(n-1)(n-2)+(1+\lambda)n(n-1)]a_n\right\} \geq \sum_{n=2}^{\infty}[\lambda n(n-1)(n-2)+(1+\lambda)n(n-1)]a_n,$$

which gives the required result (6). □

By using Lemma 2.1, we now derive the following theorem.

Theorem 2.1. *If $a,b>0$ and $c>a+b+3$, then a sufficient condition for $zF(a,b;c;z)$ to be in $\mathcal{U}(\lambda,\alpha)$ $(0\leq\lambda\leq 1, -1\leq\alpha<1)$ is that*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\times\left[\frac{2\lambda(a)_3(b)_3}{(1-\alpha)(c-a-b-3)_3}+\frac{[2+(9-\alpha)\lambda](a)_2(b)_2}{(1-\alpha)(c-a-b-2)_2} + \frac{[2+(3-\alpha)(1+2\lambda)]ab}{(1-\alpha)(c-a-b-1)}+1\right]\leq 2. \quad (7)$$

Condition (7) is necessary and sufficient for F_1 defined by $F_1(a,b;c;z)=z(2-F(a,b;c;z))$ to be in $\mathcal{U}_T(\lambda,\alpha)$.

Proof. Since

$$zF(a,b;c;z)=z+\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}z^n,$$

according to Lemma 2.1, we need only to show that

$$\sum_{n=2}^{\infty}[2\lambda n(n-1)(n-2)+(2+(3-\alpha)\lambda)n(n-1)+(1-\alpha)n]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \leq (1-\alpha).$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty}\Big[2\lambda n(n-1)(n-2)+(2+(3-\alpha)\lambda)n(n-1)\\
&\quad+(1-\alpha)n\Big]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\\
&=\sum_{n=0}^{\infty}\Big[2\lambda(n+2)(n+1)n+(2+(3-\alpha)\lambda)(n+2)(n+1)\\
&\quad+(1-\alpha)(n+2)\Big]\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}. \qquad (8)
\end{aligned}$$

Noting that $(\beta)_n=\beta(\beta+1)_{n-1}$ and then applying (4), we may express equation (8) as

$$\begin{aligned}
&\frac{2\lambda(a)_3(b)_3}{(c)_3}\sum_{n=2}^{\infty}\frac{(a+3)_{n-2}(b+3)_{n-2}}{(c+3)_{n-2}(1)_{n-2}}\\
&\quad+\frac{(2+(9-\alpha)\lambda)(a)_2(b)_2}{(c)_2}\sum_{n=1}^{\infty}\frac{(a+2)_{n-1}(b+2)_{n-1}}{(c+2)_{n-1}(1)_{n-1}}\\
&\quad+\frac{(2+(3-\alpha)(1+2\lambda))ab}{c}\sum_{n=0}^{\infty}\frac{(a+1)_{n}(b+1)_{n}}{(c+1)_{n}(1)_{n}}\\
&\quad+(1-\alpha)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}\\
&=\frac{2\lambda(a)_3(b)_3}{(c)_3}\frac{\Gamma(c+3)\Gamma(c-a-b-3)}{\Gamma(c-a)\Gamma(c-b)}\\
&\quad+\frac{(2+(9-\alpha)\lambda)(a)_2(b)_2}{(c)_2}\frac{\Gamma(c+2)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)}\\
&\quad+\frac{(2+(3-\alpha)(1+2\lambda))ab}{c}\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\\
&\quad+(1-\alpha)\left[\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}-1\right]\\
&=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\\
&\times\quad\left[\frac{2\lambda(a)_3(b)_3}{(c-a-b-3)_3}+\frac{(2+(9-\alpha)\lambda)(a)_2(b)_2}{(c-a-b-2)_2}\right.\\
&\quad\left.+\frac{(2+(3-\alpha)(1+2\lambda))ab}{c-a-b-1}+(1-\alpha)\right]-(1-\alpha).
\end{aligned}$$

But this last expression is bounded by $1-\alpha$ if and only if (7) holds, thereby completing the proof. □

Since

$$F_1(a,b;c;z) = z - \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} z^n,$$

the necessity of (7) for F_1 to be in $\mathcal{U}_T(\lambda,\alpha)$ follows from Lemma 2.2.

Theorem 2.2. *If $a,b>-1$, $ab<0$ and $c>a+b+3$, then a necessary and sufficient condition for $zF(a,b;c;z)$ to be in $\mathcal{U}_T(\lambda,\alpha)$ $(0\le\lambda\le 1, -1\le\alpha<1)$ is that*

$$\begin{aligned} &2\lambda(a)_3(b)_3 + (2+(9-\alpha)\lambda)(a)_2(b)_2(c-a-b-3) \\ &+ (2+(3-\alpha)(1+2\lambda))ab(c-a-b-3)_2 \\ &+ (1-\alpha)(c-a-b-3)_3 \ge 0. \end{aligned} \tag{9}$$

Proof. Since

$$\begin{aligned} zF(a,b;c;z) &= z + \frac{ab}{c}\sum_{n=2}^{\infty} \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} z^n \\ &= z - \left|\frac{ab}{c}\right| \sum_{n=2}^{\infty} \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} z^n, \end{aligned} \tag{10}$$

according to Lemma 2.2, we must show that,

$$\begin{aligned} &\sum_{n=2}^{\infty}\Big[2\lambda n(n-1)(n-2) + (2+(3-\alpha)\lambda)n(n-1) \\ &\quad + (1-\alpha)n\Big] \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} \\ &\le \left|\frac{c}{ab}\right|(1-\alpha). \end{aligned} \tag{11}$$

Now

$$\begin{aligned} &\sum_{n=0}^{\infty}[2\lambda(n+2)(n+1)n + (2+(3-\alpha)\lambda)(n+2)(n+1) \\ &\quad + (1-\alpha)(n+2)] \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} \\ &= 2\lambda\sum_{n=2}^{\infty} \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n-2}} + (2+(9-\alpha)\lambda)\sum_{n=1}^{\infty} \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n-1}} \\ &\quad + (2+(3-\alpha)(1+2\lambda))\sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} \end{aligned}$$

$$
\begin{aligned}
&+(1-\alpha)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}\\
&=\frac{2\lambda(a+1)_2(b+1)_2}{(c+1)_2}\frac{\Gamma(c+3)\Gamma(c-a-b-3)}{\Gamma(c-a)\Gamma(c-b)}\\
&\quad+\frac{(2+(9-\alpha)\lambda)(a+1)(b+1)}{(c+1)}\frac{\Gamma(c+2)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)}\\
&\quad+(2+(3-\alpha)(1+2\lambda))\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\\
&\quad+(1-\alpha)\frac{c}{ab}\left[\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}-1\right]\\
&=\frac{\Gamma(c+1)\Gamma(c-a-b-3)}{\Gamma(c-a)\Gamma(c-b)}\\
\times\quad&\Big[2\lambda(a+1)_2(b+1)_2+(2+(9-\alpha)\lambda)(a+1)(b+1)(c-a-b-3)\\
&\quad+(2+(3-\alpha)(1+2\lambda))(c-a-b-3)_2\\
&\quad+\frac{(1-\alpha)}{ab}(c-a-b-3)_3\Big]-(1-\alpha)\frac{c}{ab}.
\end{aligned}
$$

Hence (11) is equivalent to

$$
\begin{aligned}
&\frac{\Gamma(c+1)\Gamma(c-a-b-3)}{\Gamma(c-a)\Gamma(c-b)}\\
&\times\Big[2\lambda(a+1)_2(b+1)_2+(2+(9-\alpha)\lambda)(a+1)(b+1)(c-a-b-3)\\
&+(2+(3-\alpha)(1+2\lambda))(c-a-b-3)_2+\frac{(1-\alpha)}{ab}(c-a-b-3)_3\Big]\\
&\quad\leq(1-\alpha)\left[\frac{c}{|ab|}+\frac{c}{ab}\right]=0. \qquad (12)
\end{aligned}
$$

Thus (12) is valid if and only if

$$
\begin{aligned}
&2\lambda(a+1)_2(b+1)_2+(2+(9-\alpha)\lambda)(a+1)(b+1)(c-a-b-3)\\
&+(2+(3-\alpha)(1+2\lambda))(c-a-b-3)_2\\
&+\frac{(1-\alpha)}{ab}(c-a-b-3)_3\leq 0,
\end{aligned}
$$

or equivalently,

$$
\begin{aligned}
&2\lambda(a)_3(b)_3+(2+(9-\alpha)\lambda)(a)_2(b)_2(c-a-b-3)\\
&+(2+(3-\alpha)(1+2\lambda))ab(c-a-b-3)_2\\
&+(1-\alpha)(c-a-b-3)_3\geq 0.
\end{aligned}
$$

Remark 2.1. Letting $\lambda = 0$ in Theorem 2.1 and Theorem 2.2 above, we get the results of Cho et al [3, pp. 308-309].

3. An Integral Operator

In this section, we obtain similar type results in connection with a particular integral operator $G(a,b;c;z)$ acting on $F(a,b;c;z)$ as follows:

$$G(a,b;c;z) = \int_0^z F(a,b;c;t)dt. \tag{13}$$

Theorem 3.1. *(i) If $a, b > 1$ and $c > a+b+2$, then a sufficient condition for $G(a,b;c;z)$ defined by (13) to be in $\mathcal{U}(\lambda,\alpha)$ $(0 \leq \lambda \leq 1, -1 \leq \alpha < 1)$ is that*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[\frac{2\lambda(a)_2(b)_2}{(c-a-b-2)_2} + \frac{(2+(3-\alpha)\lambda)ab}{c-a-b-1} + (1-\alpha)\right] \leq 2(1-\alpha). \tag{14}$$

(ii) If $a, b > -1$, $ab < 0$ and $c > \max\{0, a+b+2\}$, then $G(a,b;c;z)$ defined by (13) is in $\mathcal{U}_T(\lambda,\alpha)$ $(0 \leq \lambda \leq 1, -1 \leq \alpha < 1)$ if and only if

$$\frac{\Gamma(c)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)}\Big[2\lambda(a+1)(b+1) + (2+(3-\alpha)\lambda)(c-a-b-2) + \frac{(1-\alpha)(c-a-b-2)_2}{ab}\Big] \leq 0. \tag{15}$$

Proof. Since

$$G(a,b;c;z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_n} z^n,$$

we note that

$$\sum_{n=2}^{\infty}\Big[2\lambda n(n-1)(n-2) + (2+(3-\alpha)\lambda)n(n-1) + (1-\alpha)n\Big]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_n}$$
$$= \sum_{n=1}^{\infty}\Big[2\lambda(n+1)n(n-1) + (2+(3-\alpha)\lambda)(n+1)n + (1-\alpha)(n+1)\Big]\frac{(a)_n(b)_n}{(c)_n(1)_{n+1}}$$

$$
\begin{aligned}
&= 2\lambda \sum_{n=2}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_{n-2}} + (2+(3-\alpha)\lambda) \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_{n-1}} \\
&\quad + (1-\alpha) \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} \\
&= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \\
\times &\left[\frac{2\lambda(a)_2(b)_2}{(c-a-b-2)_2} + \frac{(2+(3-\alpha)\lambda)ab}{c-a-b-1} + (1-\alpha)\right] - (1-\alpha),
\end{aligned}
$$

which is bounded above by $(1-\alpha)$ if and only if (14) holds, which completes the proof of (i).

To prove (ii) we apply Lemma 2.2 to

$$
G(a,b;c;z) = z - \frac{|ab|}{c} \sum_{n=2}^{\infty} \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_n} z^n.
$$

It sufficies to show that

$$
\begin{aligned}
&\sum_{n=2}^{\infty} \Big[2\lambda n(n-1)(n-2) + (2+(3-\alpha)\lambda)n(n-1) \\
&\quad + (1-\alpha)n\Big] \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_n} \leq (1-\alpha)\frac{c}{|ab|}.
\end{aligned} \tag{16}
$$

Now

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \Big[2\lambda(n+2)(n+1)n + (2+(3-\alpha)\lambda)(n+2)(n+1) \\
&\quad + (1-\alpha)(n+2)\Big] \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+2}} \\
&= 2\lambda \sum_{n=1}^{\infty} \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n-1}} + (2+(3-\alpha)\lambda) \sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} \\
&\quad + (1-\alpha) \sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} \\
&= \frac{\Gamma(c+1)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)} \\
\times &\Big[2\lambda(a+1)(b+1) + (2+(3-\alpha)\lambda)(c-a-b-2) \\
&\quad + \frac{(1-\alpha)(c-a-b-2)_2}{ab}\Big] - (1-\alpha)\frac{c}{ab}.
\end{aligned}
$$

Hence (16) is equivalent to

$$\frac{\Gamma(c+1)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)} \times\Bigg[2\lambda(a+1)(b+1)+(2+(3-\alpha)\lambda)(c-a-b-2) + \frac{(1-\alpha)(c-a-b-2)_2}{ab}\Bigg] \leq (1-\alpha)\left[\frac{c}{|ab|}+\frac{c}{ab}\right]=0. \tag{17}$$

Thus (17) is valid if and only if

$$\frac{\Gamma(c+1)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)} \times\Bigg[2\lambda(a+1)(b+1)+(2+(3-\alpha)\lambda)(c-a-b-2) + \frac{(1-\alpha)(c-a-b-2)_2}{ab}\Bigg] \leq 0,$$

which completes the proof. □

Remark 3.1. Letting $\lambda = 0$ in Theorem 3.1, we obtain the earlier results of Cho et al [3, p. 312].

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