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A CUBIC SPLINE METHOD FOR SOLVING A SYSTEM OF THIRD ORDER BOUNDARY VALUE PROBLEMS

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Abstract: We use uniform cubic polynomial splines to develop a new numerical method for computing approximations to the solution and its first and second derivatives for a system of third order boundary value problems. Such a system arise in physical oceanography and can be studied in the framework of variational inequality theory. It is shown that the present method is of order two and gives approximations which are better than those produced by other collocation, finite difference and spline methods when solving such a system. Numerical example is presented to illustrate applicability of the new method.

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Key Words: cubic spline, finite difference technique, third order boundary value problems, obstacle problems

1. Introduction

We consider using cubic spline function to develop a numerical method for obtaining approximations for the solution of a system of third order boundary value problem of the type

$$u''' = \begin{cases} f(x), & a \le x \le c, \\ q(x)u(x) + f(x) + r, & c \le x \le d, \\ f(x), & d \le x \le b, \end{cases} \tag{1.1a}$$

with the boundary conditions

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$$u(a) = \alpha, \quad u'(a) = \beta_1, \text{ and } \quad u'(b) = \beta_2, \tag{1.1b}$$

and the continuity conditions of u, u' and u'' at c and d. Here, f and q are continuous functions on $[a, b]$ and $[c, d]$, respectively. The parameters r, α, β_1 and β_2 are real finite constants. Such type of systems arises in the study of obstacle, unilateral, moving and free boundary value problems and has important applications in other branches of pure and applied sciences, see, for example [1]-[17] and the references therein. In general it is not possible to obtain the analytical solution of (1.1) for arbitrary choices of $f(x)$ and $q(x)$. We usually resort to some numerical methods for obtaining an approximate solution of (1.1).

Most of the available well known finite difference, collocation and spline methods are not suitable for solving system of boundary value problems of the form defined by (1.1). Such methods have a serious drawback in the accuracy regardless of the order of the convergent of the method being used, see [3], [5], [9], [15], [16], [17]. On the other hand, Al-Said [1], [2], Al-Said et al [4], [6] and Noor and Al-Said [14] have developed first and second order two-stage difference methods for solving (1.1) which gives numerical results that better than those produced by the first, second and third order methods considered in [3], [5], [9], [15], [16], [17].

In the present paper, we will use cubic spline functions to derive some new consistency relations which are used then to develop a numerical method for solving problem (1.1). The new method is of order two and capable of producing a continuous approximations for the solution of (1.1) as well as its first and second derivatives over the whole interval $[a, b]$. Our present method gives better numerical results than the other collocation, finite difference and spline methods when solving (1.1). In Section 2, we develop the cubic spline method for solving (1.1). Section 3 is devoted for the convergence analysis of the method. The numerical experiments are given in Section 4.

2. The Cubic Spline Method

For simplicity, we take $c = (3a + b)/4$ and $d = (a + 3b)/4$ in order to develop the cubic spline method for approximating the solution of the system of differential equations (1.1). For this purpose we divide the interval $[a, b]$ into n equal subintervals using the grid points $x_i = a + ih$, $i = 0, 1, 2, \ldots, n$, $x_0 = a$, $x_n = b$ and $h = (b - a)/n$, where n is a positive integer chosen such that both $n/4$ and $3n/4$ are also positive integers. Also, let $u(x)$ be the exact solution of (1.1) and s_i be an approximation to $u_i = u(x_i)$ obtained by the cubic $P_i(x)$ passing through the points (x_i, s_i) and (x_{i+1}, s_{i+1}). We write $P_i(x)$ in the form

$$P_i(x) = a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i \tag{2.1}$$

for $i = 0, 1, 2, \ldots, n-1$. Then the cubic spline defined by

$$s(x) = P_i(x), \quad i = 0, 1, 2, \ldots, n-1, \qquad s(x) \in C^2[a, b]. \tag{2.2}$$

We first develop explicit expressions for the four coefficients in (2.1). To do this we first designate

$$\begin{aligned} 2P_i(x_{i+\frac{1}{2}}) &= s_{i+\frac{1}{2}}, \quad P_i'(x_i) = D_i, \\ P_i''(x_{i+\frac{1}{2}}) &= Q_{i+\frac{1}{2}}, \quad P_i'''(x_{i+\frac{1}{2}}) = T_{i+\frac{1}{2}}, \end{aligned} \tag{2.3}$$

for $i = 0, 1, \ldots, n-1$, and

$$T_{i+\frac{1}{2}} = \begin{cases} f_{i+\frac{1}{2}}, & \text{for } 0 \le i \le \frac{n}{4} \text{ and } \frac{3n}{4} < i \le n-1, \\ q_{i+\frac{1}{2}} s_{i+\frac{1}{2}} + f_{i+\frac{1}{2}} + r, & \text{for } \frac{n}{4} < i \le \frac{3n}{4}, \end{cases} \tag{2.4}$$

where $f_i = f(x_i)$, and $q_i = q(x_i)$. Using (2.1) and (2.3) we obtain the following relations

$$a_i = \frac{1}{6} T_{i+\frac{1}{2}}, \tag{2.5a}$$

$$b_i = \frac{1}{2} Q_{i+\frac{1}{2}} - \frac{h}{4} T_{i+\frac{1}{2}}, \tag{2.5b}$$

$$c_i = D_i, \tag{2.5c}$$

$$d_i = s_{i+\frac{1}{2}} - \frac{h}{2} D_i - \frac{h^2}{8} Q_{i+\frac{1}{2}} + \frac{h^3}{24} T_{i+\frac{1}{2}}, \tag{2.5d}$$

for $i = 0, 1, 2, \ldots, n-1$.

Now from the continuity of the cubic spline $s(x)$ and its derivatives up to order two at the point (x_i, s_i) where the two cubics $P_{i-1}(x)$ and $P_i(x)$ join, we can have

$$P_{i-1}^{(m)}(x_i) = P_i^{(m)}(x_i), \quad m = 0, 1, 2. \tag{2.6}$$

Using (2.3), (2.5) and (2.6) we get the following consistency relations

$$h[D_i + D_{i-1}] = 2[s_{i+\frac{1}{2}} - s_{i-\frac{1}{2}}] - \frac{h^2}{4}[Q_{i+\frac{1}{2}} + 3Q_{i-\frac{1}{2}}] + \frac{h^3}{24}[T_{i+\frac{1}{2}} + T_{i-\frac{1}{2}}], \tag{2.7}$$

$$D_i - D_{i-1} = hQ_{i-\frac{1}{2}}, \tag{2.8}$$

$$h[D_{i+1} - 2D_i + D_{i-1}] = \frac{h^3}{2}[T_{i+\frac{1}{2}} + T_{i-\frac{1}{2}}]. \tag{2.9}$$

From equations (2.7)-(2.9) we obtain

$$hD_i = s_{i+\frac{1}{2}} - s_{i-\frac{1}{2}} - \frac{h^3}{48}[T_{i+\frac{1}{2}} + T_{i-\frac{1}{2}}]. \tag{2.10}$$

The elimination of D_i from (2.9) and (2.10) yields

$$-s_{i-\frac{5}{2}}+3s_{i-\frac{3}{2}}-3s_{i-\frac{1}{2}}+s_{i+\frac{1}{2}}=\frac{h^3}{48}h^3[T_{i-\frac{5}{2}}+23T_{i-\frac{3}{2}}+23T_{i-\frac{1}{2}}+T_{i+\frac{1}{2}}], \quad (2.11)$$

for $i = 3,4,\dots,n-1$. The recurrence relation (2.11) gives $(n-2)$ linear equations in the n unknowns s_i, $i = 1,2,\dots,n$. We need three more equations for the ends of the range of integration. These three equations are given by [2]

$$8s_0-9s_{\frac{1}{2}}+s_{\frac{3}{2}}=-3hD_0+\frac{18h^3}{48}T_{\frac{1}{2}}, \quad \text{for } i=1, \quad (2.12)$$

$$2s_{\frac{1}{2}}-3s_{\frac{3}{2}}+s_{\frac{5}{2}}=-hD_0+\frac{h^3}{48}[-2T_0+24T_{\frac{1}{2}}+24T_{\frac{3}{2}}], \quad \text{for } i=2, \quad (2.13)$$

and

$$-s_{n-\frac{5}{2}}+3s_{n-\frac{3}{2}}-2s_{n-\frac{1}{2}}=-hD_n+\frac{h^3}{48}[24T_{n-\frac{3}{2}}+24T_{n-\frac{1}{2}}-2T_n], \quad \text{for } i=n. \quad (2.14)$$

Now we can determine the values of $s_{i-\frac{1}{2}}$, for $i = 1,2,\dots,n$ by solving the system of linear equations defined by (2.11) - (2.14). Having the values of $s_{i-\frac{1}{2}}$, $i = 1,2,\dots,n$, we can compute $T_{i-\frac{1}{2}}$, for $i = 1,2,\dots,n$ using the differential equation (1.1a). Next we use equation (2.10) to compute D_i, for $i = 1,2,\dots,n-1$. Now, since $D_0 = u'(a)$ and $D_n = u'(b)$ are given then we can compute the values of $Q_{i-\frac{1}{2}}$, for $i = 1,2,\dots,n$ using equation (2.8). We remark that the knowledge of $s_{i-\frac{1}{2}}$, D_i, $Q_{i-\frac{1}{2}}$ and $T_{i-\frac{1}{2}}$, for $i = 1,2,\dots,n$ enables us to write down $P_i(x)$, $i = 0,1,\dots,n-1$ as given by (2.1). Thus, the cubic spline solution of (1.1) is determined.

3. Convergence Analysis

In this section, we investigate the convergence analysis of the cubic spline method developed in Section 2. For this purpose we first let $\mathbf{u} = (u_{i+\frac{1}{2}})$, $\mathbf{s} = (s_{i+\frac{1}{2}})$, $\mathbf{c} = (\bar{c}_i)$, $\mathbf{t} = (t_i)$ and $\mathbf{e} = (e_{i+\frac{1}{2}})$ be n-dimensional column vectors. Here $e_{i+\frac{1}{2}} = u_{i+\frac{1}{2}} - s_{i+\frac{1}{2}}$ is the discretization error and t_i is the local truncation error given by

$$t_i=\begin{cases} \frac{27}{1920}h^5u^{(5)}(\zeta_0)+O(h^6), \quad a<\zeta_0<x_{\frac{3}{2}} & \text{for } i=1\,, \\ -\frac{1}{1920}h^5u^{(5)}(\zeta_1)+O(h^6), \quad a<\zeta_1<x_{\frac{5}{2}} & \text{for } i=2\,, \\ -\frac{1}{24}h^5u^{(5)}(\zeta_i)+O(h^6), \quad x_{i-\frac{3}{2}}<\zeta_i<x_{i+\frac{3}{2}} & \text{for } 3\le i\le n-1\,, \\ -\frac{1}{1920}h^5u^{(5)}(\zeta_n)+O(h^6), \quad x_{n-\frac{3}{2}}<\zeta_n<b & \text{for } i=n, \end{cases} \quad (3.1)$$

Thus, we can write our method as follows

$$\mathbf{Au} = \mathbf{u}+\mathbf{t}, \tag{3.2a}$$
$$\mathbf{As} = \mathbf{c}, \tag{3.2b}$$
$$\mathbf{Ae} = \mathbf{t}, \tag{3.2c}$$

where

$$\mathbf{A} = \mathbf{A}_0 + \frac{1}{48}h^3\mathbf{BQ}, \tag{3.3}$$

$\mathbf{Q} = \mathrm{diag}(q_{i-\frac{1}{2}})$, $i = 1, 2, \ldots, n$, with $q_{i-\frac{1}{2}} \neq 0$ for $n/4 < i \leq 3n/4$, and the matrices $\mathbf{A}_0$ and $\mathbf{B}$ are defined by

$$\mathbf{A}_0 = \begin{bmatrix} 9 & -1 & 0 & \ldots & \ldots & \ldots & 0 \\ -2 & 3 & -1 & 0 & \ldots & \ldots & 0 \\ 1 & -3 & 3 & -1 & 0 & \ldots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & 1 & -3 & 3 & -1 \\ 0 & \ldots & \ldots & 0 & 1 & -3 & 2 \end{bmatrix}, \tag{3.4}$$

and

$$\mathbf{B} = \begin{bmatrix} 18 & 0 & 0 & \ldots & \ldots & \ldots & 0 \\ 24 & 24 & 0 & 0 & \ldots & \ldots & 0 \\ 1 & 23 & 23 & 1 & 0 & \ldots & 0 \\ 0 & \ddots & \ldots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & 1 & 23 & 23 & 1 \\ 0 & \ldots & \ldots & 0 & 0 & 24 & 24 \end{bmatrix}. \tag{3.5}$$

For the vector $\mathbf{c}$, we have

$$\bar{c}_i = \begin{cases} 8\alpha + 3h\beta_1 + \frac{18}{48}h^3\left[F_1\right], & i = 1, \\ h\beta_1 + \frac{18}{48}h^3\left[F_2\right], & i = 2, \\ \frac{1}{48}h^3\left[F_i\right], & 3 \leq i \leq \frac{n}{4} - 1 \text{ and } \frac{3n}{4} + 3 \leq i \leq n-1, \\ \frac{1}{48}h^3\left[F_i - r\right], & i = \frac{n}{4} \text{ and } i = \frac{3n}{4} + 2, \\ \frac{1}{48}h^3\left[F_i - 24r\right], & i = \frac{n}{4} + 1 \text{ and } i = \frac{3n}{4} + 1, \\ \frac{1}{48}h^3\left[F_i - 47r\right], & i = \frac{n}{4} + 2 \text{ and } i = \frac{3n}{4}, \\ \frac{1}{48}h^3\left[F_i - 48r\right], & \frac{n}{4} + 3 \leq i \leq \frac{3n}{4} - 1, \\ h\beta_2 + \frac{2}{48}h^3\left[F_n\right], & i = n, \end{cases} \tag{3.6}$$

where $F_i = \begin{cases} -f_{\frac{1}{2}}, & i = 1, \\ 2f_0 - 24[f_{\frac{1}{2}} + f_{\frac{3}{2}}], & i = 2, \\ -[f_{i-\frac{5}{2}} + 23f_{i-\frac{3}{2}} + 23f_{i-\frac{1}{2}} + f_{i+\frac{1}{2}}], & 3 \le i \le n-1, \\ -24[f_{n-\frac{3}{2}} + f_{n-\frac{1}{2}}] + 2f_n, & i = n. \end{cases}$

It has been shown in [1] that the matrix $\mathbf{A}_0$ is nonsingular and that its inverse satisfies

$$\|\mathbf{A}_0^{-1}\| = \frac{3h^3 - (b-a)h^2 + 4(b-a)^3}{48h^3}, \tag{3.7}$$

where $\|\cdot\|$ represent the ∞-norm in matrix vector. The following result gives the sufficient condition for which the system (3.2b) has a unique solution.

Theorem 3.1. (see [1]) *The discrete boundary value problem (3.2b) has a unique solution if* $\lambda|q(x)| < 1$, *where*

$$\lambda = \frac{1}{48}[3h^3 - (b-a)h^2 + 4(b-a)^3]\,. \tag{3.8}$$

Our main purpose now is to derive a bound on $\|\mathbf{e}\|$. From equation (3.2c) we have

$$\mathbf{e} = \mathbf{A}^{-1}\mathbf{t} = (\mathbf{A}_0 + \frac{1}{48}h^3\mathbf{BQ})^{-1}\mathbf{t} = (\mathbf{I} + \frac{1}{48}h^3\mathbf{A}_0^{-1}\mathbf{BQ})^{-1}\mathbf{A}_0^{-1}\mathbf{t} \tag{3.9}$$

and it follows that

$$\|\mathbf{e}\| \le \frac{\|\mathbf{A}_0^{-1}\|\|\mathbf{t}\|}{1 - \frac{1}{48}h^3\|\mathbf{A}_0^{-1}\|\|\mathbf{B}\|\|\mathbf{Q}\|} \tag{3.10}$$

provided that $\frac{1}{48}h^3\|\mathbf{A}_0^{-1}\|\|\mathbf{B}\|\|\mathbf{Q}\| < 1$. Now, from (3.1) we have

$$\|\mathbf{t}\| = \frac{1}{24}h^5 M_5, \quad M_5 = \max_x |u^{(5)}(x)|. \tag{3.11}$$

Thus, using (3.7) - (3.11) and the fact that $\|\mathbf{B}\| = 48$ and $\|\mathbf{Q}\| \le \max_x |q(x)|$, we get

$$\|\mathbf{e}\| \le \frac{\lambda M_5 h^2}{24[1 - \lambda \max_x |q(x)|]} \cong O(h^2). \tag{3.12}$$

This inequality shows that (3.2b) is a second order convergent method.

Now, we give a brief discussion regarding the order of accuracy of the relations (2.10) and (2.5b) which are used for computing the values of $D_i = P_i' \approx u_i'$ and $P_i'' = 6b_i \approx u_i''$, respectively. Using Taylor series one can easily show that the local truncation errors associated with (2.10) and (2.5b) are given by

$$-\frac{3}{640}h^5 u_i^{(5)}, \quad \text{and } -\frac{1}{8}h^4 u_i^{(4)}, \tag{3.13}$$

which indicate that the accuracy of the computed approximations of u_i' and u_i'' are of order four and two, respectively. However, since the order of the difference scheme (3.2b) is two and the computed values of $s_{i-\frac{1}{2}}$, for $i = 1, 2, \ldots, n$ are used to compute $T_{i-\frac{1}{2}}$, for $i = 1, 2, \ldots, n$, then the order of accuracy of the computed values of u_i' and u_i'' is reduced to two.

The above discussion suggest that the approximations of the solution and its first and second derivatives are second-order accurate approximations. This suggestion is supported by the numerical experiments given in the next section.

4. Application and Numerical Results

To illustrate the application of the spline method developed in the previous sections we consider the third-order obstacle boundary value problem of finding u such that

$$\begin{cases} -u''' \geq f & \text{on } \Omega = [0,1]\,, \\ u \geq \psi & \text{on } \Omega = [0,1]\,, \\ [-u''' - f][u-\psi] = 0 & \text{on } \Omega = [0,1]\,, \\ u(0) = 0, \quad u'(0) = 0, \quad u'(1) = 0\,, \end{cases} \tag{4.1}$$

where $f(x)$ is a continuous function and $\psi(x)$ is the obstacle function. Using the penalty function technique of Lewy and Stampacchia [10] and Noor [11-13], we can rewrite the obstacle boundary value problem (4.1) in the following equivalent form:

$$\begin{cases} -u''' + \nu(u-\psi)(u-\psi) = f, \quad 0 < x < 1\,, \\ u(0) = u'(0) = u'(1) = 0\,, \end{cases}, \tag{4.2}$$

where

$$\nu(t) = \begin{cases} 1\,, & \text{for } t \geq 0\,, \\ 0\,, & \text{for } t < 0, \end{cases} \tag{4.3}$$

is a discontinuous function and is known as the penalty function, and ψ is the given obstacle function defined by

$$\psi(x) = \begin{cases} -1, & \text{for } 0 \leq x \leq \frac{1}{4} \text{ and } \frac{3}{4} \leq x \leq 1\;, \\ 1, & \text{for } \frac{1}{4} \leq x \leq \frac{3}{4}. \end{cases} \tag{4.4}$$

From equations (4.2) - (4.4), we obtain the following system of differential equations

$$u''' = \begin{cases} f\,, & \text{for } 0 \leq x \leq \frac{1}{4} \text{ and } \frac{3}{4} \leq x \leq 1\,, \\ u + f - 1\,, & \text{for } \frac{1}{4} \leq x \leq \frac{3}{4}, \end{cases} \tag{4.5}$$

with the boundary conditions

$$u(0) = u'(0) = u'(1) = 0 \tag{4.6}$$

h	$\max_i \lvert u_i - s_i \rvert$	$\max_i \lvert u'_i - s'_i \rvert$	$\max_i \lvert u''_i - s''_i \rvert$
$\frac{1}{16}$	2.81×10^{-6}	7.86×10^{-5}	1.48×10^{-5}
$\frac{1}{32}$	7.01×10^{-7}	1.95×10^{-5}	3.87×10^{-6}
$\frac{1}{64}$	1.73×10^{-7}	4.88×10^{-6}	9.70×10^{-7}
$\frac{1}{128}$	4.31×10^{-8}	1.21×10^{-6}	2.31×10^{-7}

Table 1: Observed maximum errors

and the condition of continuity of u, u' and u'' at $x = \frac{1}{4}$ and $\frac{3}{4}$. Note that the system of differential equations (4.5) is a special form of the system (1.1a) with $q(x) = 1$ and $r = -1$.

Example. When $f = 0$, the system of differential equations (4.5) reduces to

$$u''' = \begin{cases} 0, & \text{for } 0 \le x \le 1/4 \text{ and } 3/4 \le x \le 1\,, \\ u - 1, & \text{for } 1/4 \le x \le 3/4\,, \end{cases} \tag{4.7}$$

with the boundary conditions (4.6). The analytical solution for this problem is

$$u(x) = \begin{cases} \frac{1}{2} a_1 x^2, & 0 \le x \le \frac{1}{4}\,, \\ 1 + a_2 e^x + e^{-x/2}[a_3 \cos \frac{\sqrt{3}}{2} x + a_4 \sin \frac{\sqrt{3}}{2} x], & \frac{1}{4} \le x \le \frac{3}{4}\,, \\ \frac{1}{2} a_5 x(x-2) + a_6, & \frac{3}{4} \le x \le 1\,. \end{cases} \tag{4.8}$$

We can find the constants a_i, for $i = 1, 2, \ldots, 6$ by solving a system of linear equations constructed by applying the continuity conditions of u, u' and u'' at $x = \frac{1}{4}$ and $x = \frac{3}{4}$, see [1] for more details.

The system of differential equations (4.7) along with the boundary conditions (4.6) was solved using the spline method described in Sections 2 and 3 with a variety of h values. The observed maximum errors (in absolute value) associated with $u_i^{(m)}$, for $m = 0, 1, 2$ are given in Table 1. It may be noted from this table that halving the stepsize h reduces the value of the maximum errors associated with u_i, u'_i and u''_i by a factor of approximately $\frac{1}{4}$, which confirms that our cubic spline method is a second order convergent process as predicted in Section 3.

The system of differential equations (4.7) along with the boundary conditions (4.6) was also solved in [16] using an $O(h^3)$ collocation method with quintic B-spline as basis functions and in [1]-[6], [9], [14], [15], [17] using first and second order finite difference and spline methods. The numerical results for some of these methods are given in Table 2.

It may be noted from Tables 1 and 2 that our present method gives better numerical results than the others. Further more, the present method is capable of producing second order approximations for the solutions as well as its first and second derivatives over the whole interval. We mention here in passing

h	[1]	[2]	[4]	[6]	[14]	[17]
$\frac{1}{32}$	4.89×10^{-5}	2.97×10^{-5}	2.98×10^{-5}	4.68×10^{-4}	3.76×10^{-4}	4.05×10^{-4}
$\frac{1}{64}$	1.22×10^{-6}	7.43×10^{-6}	7.44×10^{-6}	2.31×10^{-4}	9.40×10^{-5}	2.24×10^{-4}
$\frac{1}{128}$	3.06×10^{-6}	1.84×10^{-6}	1.86×10^{-6}	1.15×10^{-4}	2.35×10^{-5}	1.15×10^{-4}

Table 2: Observed maximum errors

that the numerical results for the first order finite difference method developed in [5], and the second and third order methods discussed in [3], [9], [15], [16] are worse than those produced in [6] and they are not included in Table 2.

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