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FUNCTIONAL INEQUALITIES FOR SUPERQUADRATIC FUNCTIONS

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Abstract: Using some characterizations of superquadratic function we obtain a sequence of inequalities for linear isotonic functionals and superquadratic functions which is analogous to the classical inequalities such as Jensen's, its converse, Slater's, Hölder's, Popoviciu's and Minkowski's inequalities. If the considered function is convex at the same time, we get refinements of the well-known classical inequalities.

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1. Introduction

Jensen's inequality for superquadratic functions, like many other inequalities for that class of functions, can be generalized for the isotonic linear functionals. First we define isotonic linear functionals.

Let E be a non-empty set and L be a linear class of real-valued functions $f : E \to \mathbf{R}$ having the properties:

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L1: $f, g \in L \Rightarrow (\alpha f + \beta g) \in L$ for all $\alpha, \beta \in \mathbf{R}$;

L2: $\mathbf{1} \in L$, i.e., if $f(t) = 1$ for $t \in E$, then $f \in L$.

Let $A : L \to \mathbf{R}$ be a functional with properties:

A1: $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for $f, g \in L$, $\alpha, \beta \in \mathbf{R}$ (A is linear);

A2: $f \in L$, $f(t) \geq 0$ on $E \Rightarrow A(f) \geq 0$ (A is isotonic).

Futhermore, if the functional A has a property

A3: $A(\mathbf{1}) = 1$, where $\mathbf{1}(t) = 1$ for all $t \in E$, then we will say that A is normalized.

The Jensen inequality for a non-normalized isotonic linear functional A and convex function φ is given in [8, p. 113] and it is a "weighted" version of Jessen's result from [7].

Theorem 1. *Let L satisfy conditions* L1, L2 *and A satisfy conditions* A1, A2 *on a non-empty set E. Suppose that $k \in L$ with $k \geq 0$ and $A(k) > 0$ and that $\varphi : I \to \mathbf{R}$ is a continuous convex function. Then for an arbitrary function $f : E \to \mathbf{R}$ such that kf, $k\varphi(f) \in L$ we have*

$$\varphi\left(\frac{A(kf)}{A(k)}\right) \leq \frac{A(k\varphi(f))}{A(k)}. \tag{1}$$

Consequences of this result are Hölder's and Minkowski's inequalities for isotonic functionals, given in the following theorems (see [8, pp. 113-114]).

Theorem 2. *Let L and A be as in the previous theorem. Let $p > 1$ and define q by $\frac{1}{p} + \frac{1}{q} = 1$. Then for all non-negative functions g, h on E such that $gh, g^p, h^q \in L$ and $A(h^q) > 0$ the following inequality*

$$A(gh) \leq A^{1/p}(g^p)A^{1/q}(h^q) \tag{2}$$

holds. In the case $0 < p < 1$ and $A(h^q) > 0$ (or $p < 0$ and $A(g^p) > 0$) the inequality in (2) *is reversed.*

Theorem 3. *Let L and A be as in the previous theorem. If $p > 1$, then for all non-negative functions g, h on E such that $(g+h)^p, g^p, h^q \in L$ the following inequality*

$$A^{1/p}((g+h)^p) \leq A^{1/p}(g^p) + A^{1/p}(h^p) \tag{3}$$

holds. If $0 < p < 1$ or $p < 0$ and $A(g^p) > 0$, $A(h^p) > 0$ the inequality in (3) *is reversed.*

Functional reverse of Jensen's inequality for convex functions is given in the following theorem (see [8, p. 124]).

Theorem 4. *Let L satisfy conditions* L1, L2 *and A satisfy conditions* A1, A2 *on a non-empty set* E *and let* $\varphi : I \to \mathbf{R}$ *be a convex function on* $I \subseteq \mathbf{R}$. *Assume that* $k \in L$ *is non-negative function on* E, $f : E \to I$ *and* $a, u \in \mathbf{R}$ *such that* $kf, k\varphi(f) \in L$, $0 < A(k) < u$ *and* $\frac{ua - A(kf)}{u - A(k)} \in I$, *then*

$$\varphi\left(\frac{ua - A(kf)}{u - A(k)}\right) \geq \frac{u\varphi(a) - A(k\varphi(f))}{u - A(k)}. \tag{4}$$

We also quote here a consequence of the above result, namely Popoviciu's inequality for isotonic functionals:

Theorem 5. *Let* L *and* A *be as in the previous theorem. If* $p > 1$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f, g : E \to [0, \infty)$ *are such that* $f^p, g^q, fg \in L$, *and* f_0, g_0 *are positive real numbers such that* $g_0^q - A(g^q) > 0$, $f_0^p - A(f^p) > 0$ *and* $A(g^q) > 0$, *then the inequality*

$$f_0 g_0 - A(fg) \geq \left(f_0^p - A(f^p)\right)^{1/p} \left(g_0^q - A(g^q)\right)^{1/q} \tag{5}$$

holds. In the case $0 < p < 1$ *and* $A(g^q) > 0$ *(or* $p < 0$ *and* $A(f^p) > 0$*) the inequality in* (5) *is reversed.*

The following converse of Jensen's inequality for isotonic linear functional and convex function is proved in [6] by Beesack and Pečarić.

Theorem 6. *Let* $\varphi : [m, M] \to \mathbf{R}$ *be a convex function, let* L *satisfy properties* L1, L2 *and* A *be any isotonic linear functional on* L *with* $A(\mathbf{1}) = 1$. *Then for every* $f \in L$ *such that* $\varphi(f) \in L$ *(so that* $m \leq f(t) \leq M$, *for all* $t \in E$*) we have*

$$A(\varphi(f)) \leq \frac{M - A(f)}{M - m}\varphi(m) + \frac{A(f) - m}{M - m}\varphi(M). \tag{6}$$

Now we quote a definition and state some basic properties of superquadratic functions established in [2] and [3]. For other interesting properties of superquadratic functions we refer reader to recent papers [1]-[5].

Definition 7. (see [2]) A function $\varphi : [0, \infty) \to \mathbf{R}$ is *superquadratic* provided that for all $x \geq 0$ there exists a constant $C(x) \in \mathbf{R}$ such that

$$\varphi(y) \geq \varphi(x) + C(x)(y - x) + \varphi(|y - x|) \tag{7}$$

for all $y \geq 0$. We say that f is *subquadratic* if $-\varphi$ is a superquadratic function.

It is easy to verify that for a subquadratic function inequality (7) is reversed.

Lemma 8. (see [2]) *Let* φ *be a superquadratic function with* $C(x)$ *as in Definition 7. Then:*

(i) $\varphi(0) \leq 0$.

(ii) If $\varphi(0) = \varphi'(0) = 0$, then $C(x) = \varphi'(x)$ whenever φ is differentiable at $x > 0$.

(iii) If $\varphi \geq 0$, then φ is convex and $\varphi(0) = \varphi'(0) = 0$.

At the end of this section we give an important result, a theorem with characterizations of the superquadratic functions, which are analogous to the well known characterizations of the convex functions.

Theorem 9. (see [4]) *For the function $\varphi : [0, \infty) \to \mathbf{R}$ the following conditions are equivalent:*

(i) The function φ is a superquadratic function, i.e., there exists a function $C : [0, \infty) \to \mathbf{R}$ such that

$$\varphi(y) \geq \varphi(x) + C(x)(y - x) + \varphi(|y - x|), \quad \forall x, y \geq 0. \tag{8}$$

(ii) For any two non-negative n-tuples $(x_1, ..., x_n)$ and $(p_1, ..., p_n)$ such that $P_n = \sum_{i=1}^n p_i > 0$ the following inequality

$$\varphi(\overline{x}) \leq \frac{1}{P_n} \sum_{i=1}^n p_i \varphi(x_i) - \frac{1}{P_n} \sum_{i=1}^n p_i \varphi(|x_i - \overline{x}|), \tag{9}$$

holds, where $\overline{x} = \frac{1}{P_n} \sum_{i=1}^n p_i x_i$.

(iii) The following inequality

$$\begin{aligned} \varphi(\lambda y_1 + (1 - \lambda) y_2) \quad &\leq \quad \lambda \varphi(y_1) + (1 - \lambda) \varphi(y_2) \\ &-\lambda \varphi((1 - \lambda)|y_1 - y_2|) - (1 - \lambda) \varphi(\lambda |y_1 - y_2|) \end{aligned} \tag{10}$$

holds for all $y_1, y_2 \geq 0$ and $\lambda \in [0, 1]$.

(iv) For all $x, y_1, y_2 \geq 0$, such that $y_1 < x < y_2$ we have

$$\begin{aligned} \varphi(x) \quad &\leq \quad \frac{y_2 - x}{y_2 - y_1} (\varphi(y_1) - \varphi(x - y_1)) \\ &+ \quad \frac{x - y_1}{y_2 - y_1} (\varphi(y_2) - \varphi(y_2 - x)) \end{aligned} \tag{11}$$

or equivalently

$$\frac{\varphi(y_1) - \varphi(x) - \varphi(x - y_1)}{y_1 - x} \leq \frac{\varphi(y_2) - \varphi(x) - \varphi(y_2 - x)}{y_2 - x}. \tag{12}$$

Proof. (i)⇒(ii) Let φ be a superquadratic function, then (8) holds for all $x, y \geq 0$. Choosing $x = \overline{x}$ and $y = x_i$ $(i = 1, ...n))$ in (8) we get that

$$\varphi(x_i) \geq \varphi(\overline{x}) + C(\overline{x})(x_i - \overline{x}) + \varphi(|x_i - \overline{x}|)$$

holds for all $i \in \{1, ..., n\}$. Multiplying this inequality by $p_i \geq 0$ and summing it over $i = 1, \ldots, n$ we get

$$\sum_{i=1}^{n} p_i \varphi(x_i) \geq P_n \varphi(\overline{x}) + C(\overline{x}) \left(\sum_{i=1}^{n} p_i x_i - P_n \overline{x} \right) + \sum_{i=1}^{n} p_i \varphi(|x_i - \overline{x}|),$$

and hence we have

$$\sum_{i=1}^{n} p_i \varphi(x_i) \geq P_n \varphi(\overline{x}) + \sum_{i=1}^{n} p_i \varphi(|x_i - \overline{x}|).$$

Finally we divide this by P_n to obtain (9).

(ii)$\Rightarrow$(iii) Let (9) holds for all non-negative n-tuples $(x_1, ..., x_n)$ and $(p_1, ..., p_n)$ such that $P_n = \sum_{i=1}^{n} p_i > 0$. In the special case for $n = 2$, $p_1 = \lambda \in [0,1]$, $p_2 = 1 - \lambda$ and for $x_1 = y_1 \geq 0$, $x_2 = y_2 \geq 0$ from (9) we obtain (10).

(iii)$\Rightarrow$(iv) Suppose that (10) holds for all $y_1, y_2 \geq 0$ and $\lambda \in [0,1]$. Let $x, y_1, y_2 \geq 0$ be such that $y_1 < x < y_2$. Then there exists $\lambda \in [0,1]$ such that $x = \lambda y_1 + (1 - \lambda) y_2$. Hence we have $\lambda = \frac{y_2 - x}{y_2 - y_1}$ and $1 - \lambda = \frac{x - y_1}{y_2 - y_1}$. By substitution of these expressions in (10) we get (11), and multiplying this inequality with $y_2 - y_1 > 0$ we have

$$\begin{aligned} \varphi(x)(y_2 - x + x - y_1) &\leq (y_2 - x)(\varphi(y_1) - \varphi(x - y_1)) \\ &\quad + (x - y_1)(\varphi(y_2) - \varphi(y_2 - x)), \end{aligned}$$

and

$$\begin{aligned} &[\varphi(x) - \varphi(y_1) + \varphi(x - y_1)](y_2 - x) \\ &\leq (x - y_1)[\varphi(y_2) - \varphi(x) - \varphi(y_2 - x)]. \end{aligned}$$

Dividing the last inequality by $(x - y_1)(y_2 - x) > 0$ we obtain (12).

$(iv) \Rightarrow (i)$ Suppose that (12) holds for all $x,\ y_1,\ y_2 \geq 0$, such that $y_1 < x < y_2$. By fixing $y_2 > x$, we get an upper bound of the left fraction in (12), so a supremum (over the all $y_1 \in (0, x)$) of that fraction exists:

$$C(x) \equiv \sup_{0 < y_1 < x} \frac{\varphi(y_1) - \varphi(x) - \varphi(x - y_1)}{y_1 - x}.$$

Hence, for the arbitrary $x,\ y_1,\ y_2 \geq 0$, such that $y_1 < x < y_2$ inequality

$$\frac{\varphi(y_1) - \varphi(x) - \varphi(x - y_1)}{y_1 - x} \leq C(x) \leq \frac{\varphi(y_2) - \varphi(x) - \varphi(y_2 - x)}{y_2 - x}$$

holds. From the above inequality for all $y \geq 0$ such that $y < x$ we get

$$\varphi(y) \geq \varphi(x) + C(x)(y - x) + \varphi(x - y),$$

at the same time for all $y > x$ we obtain

$$\varphi(y) \geq \varphi(x) + C(x)(y - x) + \varphi(y - x).$$

The last two inequalities imply (8), so the function φ is superquadratic. $\square$

Using these inequalities in the sequel we obtain a series of functional inequalities for superquadratic functions. In the second part of this paper we give functional versions of the Jensen inequality for superquadratic functions, the Hölder-type and Minkowski-type inequalities. The third and the fourth sections are devoted to the converse and reversal of the Jensen inequality and to the Slater inequality for superquadratic functions and refinements of those type inequalities which hold for convex functions.

2. Jensen's and Related Inequalities

The first theorem that we prove is the Jensen inequality for isotonic linear functional and for superquadratic functions.

Theorem 10. *Let L satisfy conditions* L1, L2 *and A satisfy conditions* A1, A2 *on a non-empty set E. Suppose that $k \in L$ with $k \geq 0$ and $A(k) > 0$ and that $\varphi : [0, \infty) \to \mathbf{R}$ is a continuous superquadratic function. Then for all non-negative $f \in L$ such that kf, $k\varphi(f)$, $k\varphi\left(\left|f - \frac{A(kf)}{A(k)} \cdot \mathbf{1}\right|\right) \in L$ we have*

$$\varphi\left(\frac{A(kf)}{A(k)}\right) \leq \frac{A(k\varphi(f)) - A\left(k\varphi\left(\left|f - \frac{A(kf)}{A(k)} \cdot \mathbf{1}\right|\right)\right)}{A(k)}. \tag{13}$$

If φ is a subquadratic function, then a reversed inequality in (13) holds.

Proof. Since φ is a superquadratic function, inequality (7) holds for all $x, y \geq 0$. The functions k and f are non-negative and A is isotonic, so we have $\frac{A(kf)}{A(k)} \geq 0$. Setting $x = \frac{A(kf)}{A(k)}$ and $y = f(t)$ for any $t \in E$ we get

$$\begin{aligned} \varphi(f(t)) \geq \varphi\left(\frac{A(kf)}{A(k)}\right) &+ C\left(\frac{A(kf)}{A(k)}\right)\left(f(t) - \frac{A(kf)}{A(k)}\right) \\ &+\varphi\left(\left|f(t) - \frac{A(kf)}{A(k)}\right|\right), \end{aligned}$$

for all $t \in E$. Multiplying the above inequality with $k(t)$, $t \in E$, we obtain that

$$\begin{aligned} k(t)\varphi(f(t)) \quad \geq \quad & \varphi\left(\frac{A(kf)}{A(k)}\right)k(t) \\ &+C\left(\frac{A(kf)}{A(k)}\right)\left(k(t)f(t) - \frac{A(kf)}{A(k)}k(t)\right) \\ &+k(t)\varphi\left(\left|f(t) - \frac{A(kf)}{A(k)}\right|\right) \end{aligned}$$

holds for all $t \in E$. Hence for the functions we have the following inequality

$$k\varphi(f) \geq \varphi\left(\frac{A(kf)}{A(k)}\right)k + C\left(\frac{A(kf)}{A(k)}\right)\left(kf - \frac{A(kf)}{A(k)}k\right) + k\varphi\left(\left|f - \frac{A(kf)}{A(k)} \cdot \mathbf{1}\right|\right).$$

Now applying linear functional A on the above inequality and using the fact that A is isotonic we obtain

$$\begin{aligned} A(k\varphi(f)) &\geq \varphi\left(\frac{A(kf)}{A(k)}\right) \cdot A(k) \\ &\quad + C\left(\frac{A(kf)}{A(k)}\right)\left(A(kf) - \frac{A(kf)}{A(k)} \cdot A(k)\right) \\ &\quad + A\left(k\varphi\left(\left|f - \frac{A(kf)}{A(k)} \cdot \mathbf{1}\right|\right)\right) \\ &= \varphi\left(\frac{A(kf)}{A(k)}\right) \cdot A(k) + A\left(k\varphi\left(\left|f - \frac{A(kf)}{A(k)} \cdot \mathbf{1}\right|\right)\right). \end{aligned}$$

Dividing this inequality by $A(k)$ we get (13). □

Remark 11. If A is a normalized linear isotonic functional and if $k \equiv \mathbf{1}$ we get a normalized Jensen's inequality for superquadratic function:

$$\varphi(A(f)) \leq A(\varphi(f)) - A(\varphi(|f - A(f) \cdot \mathbf{1}|)). \tag{14}$$

If $A(f) = \int_\Omega f(s)d\mu(s)$, where μ is a probability measure, then inequality (14) is obtain in paper [2].

Remark 12. In the case of non-negative superquadratic function (and therefore a convex function), since we have

$$A\left(k\varphi\left(\left|f - \frac{A(kf)}{A(k)} \cdot \mathbf{1}\right|\right)\right) \geq 0,$$

Theorem 10 represents a refinement of Theorem 1.

By suitable choice of the functions f and k in the above theorem we get the following result, a refinement of the functional Hölder's inequality.

Theorem 13. *Let L satisfy conditions* L1, L2 *and A satisfy conditions* A1, A2 *on a non-empty set E. Let $p \geq 2$ and define q by $\frac{1}{p} + \frac{1}{q} = 1$. Then for all non-negative functions $g, h \in L$ such that $gh, g^p, h^q, |g - h^{q-1}\frac{A(gh)}{A(h^q)}|^p \in L$ and $A(h^q) > 0$ the following inequality*

$$A(gh) \leq \left[A(g^p) - A\left(\left|g - h^{q-1}\frac{A(gh)}{A(h^q)}\right|^p\right)\right]^{\frac{1}{p}} A^{\frac{1}{q}}(h^q) \tag{15}$$

holds. In the case $0 < p < 2$ *the inequality in* (15) *is reversed.*

Proof. First we assume that $p \geq 2$. By applying the following substitution in (13):

$$\varphi(x) = x^p, \quad f = gh^{-\frac{q}{p}}, \quad k = h^q,$$

we obtain

$$\left(\frac{A(gh)}{A(h^q)}\right)^p \leq \frac{A(g^p) - A\left(\left|g - h^{q-1}\frac{A(gh)}{A(h^q)}\right|^p\right)}{A(h^q)}. \tag{16}$$

Multiplying this inequality by $A(h^q)^p > 0$ we obtain

$$A(gh)^p \leq \left[A(g^p) - A\left(\left|g - h^{q-1}\frac{A(gh)}{A(h^q)}\right|^p\right)\right] \cdot A^{\frac{p}{q}}(h^q).$$

The last inequality implies (15).

In the case $0 < p < 2$ the function $\varphi(x) = x^p$ is subquadratic and inequality (16) is reversed. Hence, in that case we get reversal of inequality (15). □

Since A is isotonic functional we have

$$A\left(\left|g - h^{q-1}\frac{A(gh)}{A(h^q)}\right|^p\right) \geq 0,$$

so the inequality (15) represents a refinement of the classical Hölder's inequality (2). When functional A is an integral, corresponding result is given in [9].

Similarly, we obtain a functional Minkowski's inequality for superquadratic functions.

Theorem 14. *Let* L *and* A *be as in the previous theorem. If* $p \geq 2$*, then for all non-negative functions* g, h *on* E *such that* $(g+h)^p, g^p, h^q \in L$ *the following inequality*

$$A^{1/p}((g+h)^p) \leq \left(A(g^p) - A\left(|g - (g+h)\frac{A(g(g+h)^{p-1}}{A(g+h)^p}|^p\right)\right)^{1/p}$$
$$+\left(A(h^p) - A\left(|h - (g+h)\frac{A(h(g+h)^{p-1}}{A(g+h)^p}|^p\right)\right)^{1/p} \tag{17}$$

holds.

Proof. For $p \geq 2$ we have $(g+h)^p = g(g+h)^{p-1} + h(g+h)^{p-1}$, i.e.

$$A((g+h)^p) = A(g(g+h)^{p-1}) + A(h(g+h)^{p-1}).$$

Using (15) with conjugate exponents p and $q = \frac{p}{p-1}$ we have:

$$A((g+h)^p)$$

$$\leq \left(A(g^p) - A\left(|g-(g+h)\frac{A(g(g+h)^{p-1}}{A(g+h)^p}|^p\right)\right)^{1/p} A^{\frac{p-1}{p}}((g+h)^p)$$
$$+ \left(A(h^p) - A\left(|h-(g+h)\frac{A(h(g+h)^{p-1}}{A(g+h)^p}|^p\right)\right)^{1/p} A^{\frac{p-1}{p}}((g+h)^p)$$

from which the (17) follows. □

3. Converse of Jensen's Inequality and Slater's Inequality

In the following theorem we prove a functional version of converse of Jensen's inequality for superquadratic functions.

Theorem 15. *Let L satisfy conditions* L1, L2 *and A satisfy conditions* A1, *and* A2 *on a non-empty set E. Let $k \in L$ be a non-negative function. Suppose that $\varphi : [0,\infty) \to \mathbf{R}$ is a superquadratic function. Then for every $f \in L$, $f : E \to [m,M] \subseteq [0,\infty)$ such that $kf, k(\varphi \circ f) \in L$, we have*

$$A(k\varphi(f)) + \Delta_c \leq \frac{MA(k) - A(kf)}{M-m}\varphi(m) + \frac{A(kf) - mA(k)}{M-m}\varphi(M), \quad (18)$$

where

$$\Delta_c = \frac{1}{M-m}A((Mk - kf)\varphi(f - m\cdot\mathbf{1}) + (kf - mk)\varphi(M\cdot\mathbf{1} - f)).$$

Proof. Since the function φ is superquadratic, inequality (11) holds for all $y_1, y_2 \geq 0$ and $x \in \langle y_1, y_2\rangle$. By setting $x = f(t)$, $t \in E$, $y_1 = m$, $y_2 = M$ we get

$$\begin{aligned}\varphi(f(t)) &+ \frac{M - f(t)}{M-m}\varphi(f(t) - m) + \frac{f(t) - m}{M-m}\varphi(M - f(t)) \\ &\leq \frac{M - f(t)}{M-m}\varphi(m) + \frac{f(t) - m}{M-m}\varphi(M),\end{aligned}$$

for all $t \in E$. Multiplying the above inequality by $k(t)$ and rewriting inequality in the form of inequality of functions we have the following inequality

$$\begin{aligned}k\varphi(f) &+ \frac{1}{M-m}[(Mk - kf)\varphi(f - m\cdot\mathbf{1}) \\ &\qquad + (kf - mk)\varphi(M\cdot\mathbf{1} - f)] \\ &\leq \frac{Mk - kf}{M-m}\varphi(m) + \frac{kf - mk}{M-m}\varphi(M).\end{aligned}$$

Since the functional A satisfies conditions A1-A2, applying A on the above inequality we get (18). □

Remark 16. Under the assumptions of the Theorem 15, and if A satisfies A3 with $k \equiv \mathbf{1}$ and if φ is a non-negative and superquadratic function (and therefore a convex) we have both of the results, (6) and (18). Since the term Δ_c is non-negative in this case, then the inequality (18) refines the inequality (6).

Another estimate of the expression $A(\varphi(f))$, i.e. an upper bound for it in the case of superquadratic function φ is given in the following theorem. We obtain a functional Slater's type inequality for superquadratic functions.

Theorem 17. *Let L satisfy conditions* L1, L2 *and A satisfy conditions* A1 *and* A2 *on a non-empty set E. Suppose that $\varphi : [0, \infty) \to \mathbf{R}$ is a superquadratic function, C is the function as in* (8), *and $k, f \in L$ are non-negative functions such that $k\varphi(f)$, $kC(f)$, $kfC(f)$, $k\varphi(|f - M \cdot \mathbf{1}|) \in L$. If*

$$M = \frac{A(kfC(f))}{A(kC(f))} \geq 0,$$

then the following inequality

$$A(k\varphi(f)) \leq \varphi(M) A(k) - A(k\varphi(|f - M \cdot \mathbf{1}|)) \tag{19}$$

holds.

Proof. By setting $y = M$ and $x = f(t)$, $t \in E$ in (7) and multiplying by $k(t) \geq 0$ we obtain

$$\varphi(M) k(t) \geq k(t)\varphi(f(t)) + k(t)C(f(t))(M - f(t))$$
$$+k(t)\varphi(|M - f(t)|),$$

for all $t \in E$, and hence we have the following inequality for the functions

$$\varphi(M) k \geq k\varphi(f) + MkC(f) - kfC(f) + k\varphi(|M \cdot \mathbf{1} - f|).$$

Applying isotonic linear functional A on the above inequality we get

$$\varphi(M) A(k) \geq A(k\varphi(f)) + MA(kC(f)) - A(kfC(f))$$
$$+A(k\varphi(|M \cdot \mathbf{1} - f|)),$$

and this imply inequality (19). □

4. Reversal of Jensen's Inequality and Popoviciu's Type Inequality

In the paper [5] authors proved a discrete reversal of Jensen's inequality for superquadratic functions. This result is given in the following theorem.

Theorem 18. *Let $(p_1, \ldots, p_n)$ be a real n-tuple such that*

$$p_1 > 0, \quad p_i \le 0 \quad (i = 2, \ldots, n), \quad P_n = \sum_{i=1}^{n} p_i > 0.$$

If $x_i \ge 0$ $(i = 1, \ldots, n)$ and $\overline{x} = \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i \ge 0$, then for a superquadratic function $\varphi : [0, \infty) \to \mathbf{R}$ the following inequality

$$\varphi\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \ge \frac{1}{P_n}\sum_{i=1}^{n} p_i \varphi(x_i) + \varphi(|\overline{x} - x_1|) - \frac{1}{P_n}\sum_{i=2}^{n} p_i \varphi(|x_i - x_1|). \tag{20}$$

holds.

Using the Theorem 18 we are able to prove a functional version of the reversal of Jensen's inequality for superquadratic functions.

Theorem 19. *Let L satisfy conditions* L1, L2 *and A satisfy conditions* A1, A2 *on a non-empty set E and let $\varphi : [0, \infty) \to \mathbf{R}$ be a superquadratic function. If $k, f \in L$ are non-negative functions and $a, u \in [0, \infty)$ are real numbers such that $kf, k\varphi(f) \in L$, $k\varphi\left(\left|\frac{A(kf)}{A(k)} - f\right|\right)|) \in L$, $0 < A(k) < u$ and $ua - A(kf) \ge 0$, then*

$$\varphi\left(\frac{ua - A(kf)}{u - A(k)}\right) \ge \frac{u\varphi(a) - A(k\varphi(f))}{u - A(k)} + \Delta_{RJ}, \tag{21}$$

where

$$\Delta_{RJ} = \frac{1}{u - A(k)}\left[A\left(k\varphi\left(\left|\frac{A(kf)}{A(k)} \cdot \mathbf{1} - f\right|\right)\right)\right.$$
$$\left. + A(k)\varphi\left(\left|\frac{A(kf)}{A(k)} - a\right|\right) + (u - A(k))\varphi\left(\frac{A(k)}{u - A(k)}\left|\frac{A(kf)}{A(k)} - a\right|\right)\right].$$

Proof. Putting in Theorem 18 $n = 2$, $p_1 = u$, $p_2 = -A(k)$, $x_1 = a$, $x_2 = \frac{A(kf)}{A(k)}$ we obtain the following

$$\varphi\left(\frac{ua - A(kf)}{u - A(k)}\right) \ge \frac{u\varphi(a) - A(k)\varphi\left(\frac{A(kf)}{A(k)}\right)}{u - A(k)} + \varphi\left(\frac{A(k)}{u - A(k)}\left|\frac{A(kf)}{A(k)} - a\right|\right) + \frac{A(k)}{u - A(k)}\varphi\left(\left|\frac{A(kf)}{A(k)} - a\right|\right).$$

Using Jensen's inequality for superquadratic function (Theorem 10) we get

$$A(k)\varphi\left(\frac{A(kf)}{A(k)}\right) \leq A(k\varphi(f)) - A\left(k\varphi\left(\left|f - \frac{A(kf)}{A(k)}\cdot \mathbf{1}\right|\right)\right).$$

Combining these two inequalities we get inequality (21). □

Remark 20. In the case when φ is a non-negative superquadratic function, and therefore (by Lemma 8 (iii)) a convex too, the results of Theorem 4 and Theorem 19 are both valid for such function. Under the assumptions of these theorems: A is isotonic functional, k is non-negative function and $u - A(k) > 0$, so in this case we have $\Delta_{RJ} \geq 0$ and the result of Theorem 19 refines a result of Theorem 4.

Considering the superquadratic function $\varphi(x) = x^p$, $p \geq 2$, and applying it in the previous result (21) we obtain the Popoviciu type inequality for isotonic functionals.

Theorem 21. *Let A and L be as in Theorem 19. Let $p \geq 2$ and q be real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. If $g, h : E \to [0, \infty)$ are functions and g_0, h_0 are positive real numbers such that $g^p, h^q, gh, |h^{q/p}A(gh) - gA(h^q)|^p \in L$, $h_0^q - A(h^q) > 0$, $g_0^p - A(g^p) > 0$ and $A(h^q) > 0$ then*

$$\begin{aligned} g_0h_0 - A(gh) &\geq \left[(g_0^p - A(g^p))(h_0^q - A(h^q))^{p/q} + \Delta_P\right]^{1/p} \\ &\geq (g_0^p - A(g^p))^{1/p}(h_0^q - A(h^q))^{1/q}, \end{aligned} \tag{22}$$

where

$$\Delta_P = \frac{(h_0^q - A(h^q))^{p/q}}{A^p(h^q)} A\left(|h^{q/p}A(gh) - gA(h^q)|^p\right)$$
$$+ \left|g_0h_0^{-q/p}A(h^q) - A(gh)\right|^p \cdot \left(1 + \frac{(h_0^q - A(h^q))^{p/q}}{A^{p/q}(h^q)}\right).$$

Proof. By applying the substitutions: $\varphi(x) = x^p$, $p \geq 2$,

$$u \to h_0^q, \quad k \to h^q, \quad a \to h_0^{-q/p}g_0, \quad f \to h^{-q/p}g$$

in (21) we obtain inequality (22), while the second inequality follows from the positivity of Δ_P. □

As we can see, the result of the previous theorem is, in fact, the refinement of the classical Popoviciu's inequality (5). Results of this section in the case when $A(f) = \int_\Omega f(s)d\mu(s)$, where $(\Omega, \mathcal{A}, \mu)$ is a measure space, are considered in [5].

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