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ONE-STEP 7-STAGE HERMITE-BIRKHOFF-TAYLOR ODE SOLVER OF ORDER 13

Truong Nguyen-Ba[1], Vladan Bozic[2], Rémi Vaillancourt[3 §]
[1,2,3]Department of Mathematics and Statistics
Faculty of Science
University of Ottawa
140 Louis-Pasteur, Ottawa, Ontario, K1N 6N5, CANADA
[1]e-mail: tnguyen@mathstat.uottawa.ca
[2]e-mail: vladan.bozic@gmail.com
[3]e-mail: remi@uottawa.ca

Abstract: A one-step Hermite-Birkhoff-Taylor method of order 13 with seven stages, denoted by HBT(13)7, is constructed for solving nonstiff systems of first-order differential equations of the form $y' = f(x, y)$, $y(x_0) = y_0$. The method uses derivatives y' to $y^{(8)}$, as in Taylor methods, combined with a 7-stage Runge-Kutta method of order 6. Forcing Taylor's expansions of the numerical and true solutions to agree leads to Taylor- and Runge-Kutta-type order conditions which are reorganized into Vandermonde-type linear systems whose solutions are the coefficients of the method. The new method has larger scaled interval of absolute stability than Dormand-Prince DP(8,7)13M. The stepsize is controlled by means of two high order derivatives. HBT(13)7 is superior to DP(8,7)13M and Taylor method of order 13 in solving several test problems for higher-order ODE solvers on the basis the number of steps, CPU time, and maximum global error, thus showing the benefits of adding high order derivatives to Runge-Kutta methods.

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[§]Correspondence author

1. Introduction

A Taylor method of order 8, denoted here by T8, and a 7-stage Runge-Kutta method of order 6 are cast into a one-step 7-stage Hermite-Birkhoff-Taylor method of order 13, named HBT(13)7 because it uses Hermite-Birkhoff interpolation polynomials and the derivatives y' to $y^{(8)}$ for solving $y' = f(x, y)$ at step points, x_n, like in Taylor methods. The link between the two types of methods is that values at off-step points are obtained by means of predictors which use values of derivatives of different orders at the current step point. By construction, HBT(13)7 uses lower order derivatives than the traditional Taylor method of order 13, denoted by T13.

Taylor methods have been an excellent choice in astronomical calculations Barrrio et al [2], sensitivity analysis of ODEs/DAEs Barrio [1], solving general problems Corliss et al [4] and validating solutions of ODEs/DAEs by means of interval analysis Hoefkens et al [7], Nedialkov [12]. Deprit and Zahar [5] proved that recurrent power series in Taylor methods are very effective in achieving high accuracy, even with a small value of computing time and large step-sizes. On the other hand, multiderivative, multistep methods, which are a new form of the classical Adams-Cowell methods, were introduced by Huang and Innanen Huang et al [9]. Some of these methods have larger stability interval and smaller truncation error than classical multistep methods. The above results prompted the effective addition of high order derivatives to an ODE solver.

HBT(13)7 is designed for solving nonstiff systems of first-order initial value problems of the form

$$y' = f(x, y), \qquad y(x_0) = y_0, \qquad \text{where} \quad {}' = \frac{d}{dx}. \tag{1}$$

The high order derivatives y'' to $y^{(8)}$ can be obtained by differentiating $f(x, y(x))$ in the right-hand side of equation (1). But this approach is useful only in theoretical studies because of the computational complexity of high order derivatives.

Following the pioneering work of Steffensen [21] and Rabe [19], another approach is to use fast automatic differentiation (AD) techniques to compute sums, differences, products and powers of power series, to name but a few (see Barrio et al [2], Lara et al [11], and references therein). Formulae for generating these high order derivatives may be found in textbooks (see for instance Hairer et al [6, pp. 46-49]).

Forcing a Taylor expansion of the numerical solution to agree with an expansion of the true solution leads to a combination of Taylor- and Runge-Kutta-type order conditions which are reorganized into linear Vandermonde-type systems. The coefficients of this one-step method are obtained once for all as solutions of these systems by means, say, of Gaussian elimination. Moreover, with HBT(13)7, there are no rejected steps because the stepsize is chosen in order to obtain the required precision level once the series are generated.

The *C++* performances of HBT(13)7, DP(8,7)13M Prince et al [18] and T13, were compared on several problems frequently used to test higher order ODE solvers. It is seen that HBT(13)7 requires fewer steps, uses less CPU time, and has higher accuracy than DP(8,7)13M and T13.

Section 2 introduces HBT(13)7. Order conditions are listed in Section 3. In Section 4, HBT(13)7 is represented in terms of Vandermonde-type systems. Section 5 considers the region of absolute stability of the constant step method. Section 6 deals with the step control. In Section 7, two criteria are used to compare the performance of the methods considered in this paper. Appendix lists 39 Runge-Kutta-type order conditions, the formulae of the method and the recurrent computation of high order derivatives.

2. One-Step HBT(13)7

The defining formulae of HBT(13)7 depend on the Runge-Kutta parameters listed in the following Butcher tableau

$$\begin{array}{c|ccccccc}
c_1 & & & & & & & \\
c_2 & a_{21} & & & & & & \\
c_3 & a_{31} & a_{32} & & & & & \\
c_4 & a_{41} & a_{42} & a_{43} & & & & \\
c_5 & a_{51} & a_{52} & a_{53} & a_{54} & & & \\
c_6 & a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & & \\
c_7 & a_{71} & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & \\
\hline
 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7
\end{array}$$

and on the Taylor expansion parameters $\gamma_{\ell j}$. Seven predictors, P_2, $\mathrm{P}_3,\ldots,\mathrm{P}_6$ and P_7, and an integration formula, IF, are needed to perform the integration step from x_n to x_{n+1}.

For $\ell = 2$, an Hermite-Birkhoff polynomial of degree $\ell+6$ is used as predictor P_ℓ to obtain y_{n+c_ℓ} to order 8 and for $\ell = 3, \ldots, 7$, Hermite-Birkhoff polynomials of degree $\ell + 6$ are used as predictors P_ℓ to obtain y_{n+c_ℓ} to order 9,

$$y_{n+c_\ell} = y_n + h_{n+1}\sum_{j=1}^{\ell-1} a_{\ell j} f_{n+c_\ell} + \sum_{j=2}^{8} h_{n+1}^{j}\gamma_{\ell j} f_n^{(j-1)}, \quad \ell = 2,3,\dots,7. \tag{2}$$

A Hermite-Birkhoff polynomial of degree 13 is used as integration formula IF to obtain y_{n+1} to order 13,

$$y_{n+1} = y_n + h_{n+1}\sum_{j=1}^{7} b_j f_{n+c_j} + \sum_{j=2}^{8} h_{n+1}^{j}\gamma_{1j} f_n^{(j-1)}. \tag{3}$$

By notation y_{n+c_7} is different from y_{n+1}. One sees that the derivatives f'_n to $f_n^{(7)}$ are computed only once per step at x_n (see Appendix C).

3. Order Conditions

To simplify notation in formulae (6) and (7), we take $\gamma_{i1} = 0$ and $\gamma_{i,j} = 0$ for $j > 8$.

Forcing a Taylor expansion of the numerical solution to agree with an expansion of the true solution leads to the 39 Taylor- and Runge-Kutta-type order conditions listed in Appendix A.

As in similar searches for ODE solvers Butcher [3], Lambert [10], Nguyen-Ba et al [13]-[16], we impose the following simplifying assumptions on HBT (13)7:

$$\sum_{i=j+1}^{7} b_i a_{ij} = b_j(1-c_j), \qquad j = 2,\dots,6, \tag{4}$$

$$b_2 = 0. \tag{5}$$

$$\sum_{j=1}^{i-1} a_{ij} c_j^k + k!\gamma_{i,k+1} = \frac{1}{k+1}\, c_i^{k+1}, \qquad \begin{cases} i = 3,\dots,7, \\ k = 0,1,2,\dots,8. \end{cases} \tag{6}$$

There remain seven sets of equations to be solved:

$$\sum_{i=1}^{7} b_{1i} c_i^k + k!\gamma_{1,k+1} = \frac{1}{k+1}, \qquad k = 0,1,\dots,12, \tag{7}$$

$$\begin{aligned} b_6(1-c_6)a_{65}c_5^8(c_5-c_3)(c_5-c_4) &= 10!\left(\frac{1}{12!} - \frac{12}{13!}\right) \\ &- 9!\left(\frac{1}{11!} - \frac{11}{12!}\right)(c_3+c_4) + 8!\left(\frac{1}{10!} - \frac{10}{11!}\right)c_3c_4, \end{aligned} \tag{8}$$

$$
\begin{aligned}
b_5(1-c_5)(c_6-c_5)a_{54}c_4^8(c_4-c_3) &= 9!\left[\frac{c_6}{11!}-(1+c_6)\frac{11}{12!}+11\frac{12}{13!}\right] \\
&\quad - c_3 8!\left[\frac{c_6}{10!}-(1+c_6)\frac{10}{11!}+10\frac{11}{12!}\right], \qquad (9)
\end{aligned}
$$

$$
\begin{aligned}
& b_6(1-c_6)a_{64}c_4^8(c_4-c_3)+b_6(1-c_6)a_{65}c_5^8(c_5-c_3) \\
& \quad + b_5(1-c_5)a_{54}c_4^8(c_4-c_3) \\
& \quad = \left(\frac{9!}{11!}-8!\frac{c_3}{10!}\right)-\left(9!\frac{11}{12!}-c_3 8!\frac{10}{11!}\right), \qquad (10)
\end{aligned}
$$

$$
\sum_{i=3}^{5} b_i(1-c_i)(c_6-c_i)a_{i2}=0, \qquad (11)
$$

$$
\sum_{i=3}^{6} b_{1i}(1-c_i)a_{i2}=0, \qquad (12)
$$

$$
\sum_{i=4}^{6} b_{1i}(1-c_i)\sum_{j=3}^{i-1} a_{ij}a_{j2}=0. \qquad (13)
$$

Equations (8)-(10) are obtained from the following sums of equations given in Appendix A

$$
\begin{aligned}
& 10!(\text{Eq. }(53)-\text{Eq. }(61))-9!(\text{Eq. }(47)-\text{Eq. }(51))(c_3+c_4) \\
& \quad + 8!(\text{Eq. }(44)-\text{Eq. }(46))c_3c_4, \\
& 9![c_6\text{Eq. }(47)-(1+c_6)\text{Eq. }(51)+\text{Eq. }(59)] \\
& \quad - c_3 8![c_6\text{Eq. }(44)-(1+c_6)\text{Eq. }(46)+\text{Eq. }(50)], \\
& 9!\text{Eq. }(47)-8!\text{Eq. }(44)c_3-(9!\text{Eq. }(51)-8!\text{Eq. }(46)c_3),
\end{aligned}
$$

respectively.

The seven off-step points used in this paper are

$$
\begin{aligned}
& c_1=0, \quad c_2=0.4658152685849384, \quad c_3=0.5175725206499315, \\
& c_4=0.4237064507487538, \quad c_5=0.8621635773285519, \qquad (14) \\
& c_6=0.9234242096269732, \quad c_7=1\,.
\end{aligned}
$$

To obtain the c_4 value of (14), firstly, we write the following reduced equation

$$
b_6(1-c_6)a_{65}a_{54}c_4^8(c_4-c_3)=9!\left(\frac{1}{12!}-\frac{12}{13!}\right)-8!\left(\frac{1}{11!}-\frac{11}{12!}\right), \qquad (15)
$$

which is obtained from the following expression of equations

$$9![\text{Eq. (55)} - \text{Eq. (63)}] - 8![\text{Eq. (48)} - \text{Eq. (52)}]c_3.$$

Secondly, we write

$$\theta = c_5^8(c_5 - c_3)(c_5 - c_4)b_5(1 - c_5)(c_6 - c_5)$$

so that the product of the left hand sides of (8), (9) is the product of θ with the left hand side of (15). We have therefore

$$\begin{aligned}\Bigg[10!\left(\frac{1}{12!} - \frac{12}{13!}\right) &- 9!\left(\frac{1}{11!} - \frac{11}{12!}\right)(c_3 + c_4) \\ &+ 8!\left(\frac{1}{10!} - \frac{10}{11!}\right)c_3c_4\Bigg]\Bigg[9!\left(\frac{c_6}{11!} - (1 + c_6)\frac{11}{12!} + 11\frac{12}{13!}\right) \\ &- c_3 8!\left(\frac{c_6}{10!} - (1 + c_6)\frac{10}{11!} + 10\frac{11}{12!}\right)\Bigg] \\ &= \left[9!\left(\frac{1}{12!} - \frac{12}{13!}\right) - 8!\left(\frac{1}{11!} - \frac{11}{12!}\right)c_3\right]\theta. \quad (16)\end{aligned}$$

Setting c_i equal to the values of (14) for $i = 1, 2, 3, 5, 6, 7$, we can calculate c_4 so that (16) is satisfied.

To put it simply, this results in condition (63) being met automatically when all other order conditions of HBT(13)7 are satisfied.

4. Vandermonde-Type Formulation

4.1. Integration Formula IF

The 13-vector of the reordered coefficients of IF in (3),

$$\boldsymbol{u}^1 = [b_{17}, b_{16}, \ldots, b_{14}, b_{13}, b_{11}, \gamma_{12}, \gamma_{13}, \ldots, \gamma_{1,8}]^T,$$

is the solution of the Vandermonde-type system of order conditions:

$$M^1\boldsymbol{u}^1 = \boldsymbol{r}^1\,, \quad (17)$$

where

$$M^1 = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 \\ c_7 & c_6 & & c_3 & 0 & 1 & 0 & & 0 \\ c_7^2/2! & c_6^2/2! & & c_3^2/2! & 0 & 0 & 1 & & 0 \\ \vdots & & & & & & \ddots & & \\ c_7^7/7! & c_6^7/7! & & c_3^7/7! & 0 & 0 & 0 & & 1 \\ \vdots & & & & & & & & \vdots \\ c_7^{12}/12! & c_6^{12}/12! & \cdots & c_3^{12}/12! & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{18}$$

and $\boldsymbol{r}^1 = r_1(1:13)$ has components $r_1(i) = 1/i!$ for $i = 1, 2, \ldots, 13$.

The leading error term of IF is of order 14 with the choice of c_i, $i = 1, 3, \ldots, 7$ in (14):

$$\left[b_{17}\frac{c_7^{13}}{13!} + \cdots + b_{14}\frac{c_4^{13}}{13!} + b_{13}\frac{c_3^{13}}{13!} - \frac{1}{14!} \right] h_{n+1}^{14} y_n^{(14)}.$$

4.2. Predictor P_2

The 8-vector of the reordered coefficients of predictor P_2 in (2) with $\ell = 2$, $\boldsymbol{u}^2 = [a_{21}, \gamma_{22}, \gamma_{23}, \ldots, \gamma_{2,8}]^T$, is the solution of the system of order conditions

$$M^2 \boldsymbol{u}^2 = \boldsymbol{r}^2, \tag{19}$$

where M^2 is the 8×8 identity matrix I_8 and $\boldsymbol{r}^2 = r_2(1:8)$ has components

$$r_2(i) = \frac{c_2^i}{i!}, \qquad i = 1, 2, \ldots, 8.$$

A truncated Taylor expansion of the right-hand side of (2) with $\ell = 2$ about x_n gives

$$\sum_{j=0}^{13} S_2(j) h_{n+1}^j y_n^{(j)}$$

with coefficients

$$S_2(j) = M^2(j, 1:8)\boldsymbol{u}^2 = \frac{c_2^j}{j!}, \qquad j = 1, 2, \ldots, 8,$$
$$S_2(j) = 0, \qquad j = 9, 10, \ldots, 13.$$

We note that P_2 is of order 8 since it satisfies the order conditions

$$S_2(j) = c_2^j/j!, \qquad j = 1, 2, \ldots, 8,$$

and its leading error term is

$$\left[S_2(9) - \frac{c_2^9}{9!} \right] h_{n+1}^9 y_n^{(9)}.$$

4.3. Coefficients a_{ij} of Predictor P_3 to P_7

We obtain a_{65}, a_{54}, a_{64} from (8)-(10) and $a_{32} = (c_3^9/9!)/(c_2^8/8!)$. Then we solve the system of conditions

$$M\boldsymbol{v} = \boldsymbol{r} \tag{20}$$

for the vector of six coefficients $\boldsymbol{v} = [a_{43}, a_{42}, a_{53}, a_{52}, a_{63}, a_{62}]^T$. The matrix M is

$$M = \begin{bmatrix} c_3^8/8! & c_2^8/8! & 0 & 0 & 0 & 0 \\ 0 & 0 & c_3^8/8! & c_2^8/8! & 0 & 0 \\ 0 & 0 & 0 & 0 & c_3^8/8! & c_2^8/8! \\ 0 & g_4 & 0 & g_5 & 0 & g_6 \\ g_4 a_{32} & g_5 a_{54} + g_6 a_{64} & g_5 a_{32} & g_6 a_{65} & g_6 a_{32} & 0 \\ 0 & g_4(c_6 - c_4) & 0 & g_5(c_6 - c_5) & 0 & 0 \end{bmatrix}, \tag{21}$$

where $g_i = b_i(1 - c_i)$ and the righ-hand side $\boldsymbol{r} = r(1:6)$ has components

$$r(1) = c_4^9/9!, \qquad r(2) = c_5^9/9! - a_{54}c_4^8/8!,$$
$$r(3) = c_6^9/9! - a_{65}c_5^8/8! - a_{64}c_4^8/8!, \qquad r(4) = -b_3(1 - c_3)a_{32},$$
$$r(5) = 0, \qquad r(6) = -b_3(1 - c_3)(c_6 - c_3)a_{32}.$$

The equations for $r(4)$, $r(5)$ and $r(6)$ correspond to equations (12), (13) and (11), respectively.

The coefficients a_{i1} can then be obtained from

$$a_{i1} = c_i - \sum_{j=2}^{i-1} a_{ij}, \qquad i = 2, 3, \ldots, 6.$$

4.4. Predictor P_3

We solve the system of nine order conditions

$$M^3 \boldsymbol{u}^3 = \boldsymbol{r}^3, \tag{22}$$

for the nine reordered coefficients $\boldsymbol{u}^3 = [a_{32}, a_{31}, \gamma_{32}, \gamma_{33}, \ldots, \gamma_{3,8}]^T$, of predictor P_3 in (2) with $\ell = 3$. The matrix M^3 is

$$M^3 = \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ c_2 & 0 & 1 & 0 & & 0 \\ c_2^2/2! & 0 & 0 & 1 & & 0 \\ \vdots & & & & \ddots & \\ c_2^7/7! & 0 & 0 & 0 & & 1 \\ c_2^8/8! & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \qquad (23)$$

and the right-hand side, $\boldsymbol{r}^3 = r_3(1:9)$, has components

$$r_3(i) = c_3^i/i!, \qquad i = 1, 2, \ldots, 9.$$

A truncated Taylor expansion of the right-hand side of (2) with $\ell = 3$ about x_n gives

$$\sum_{j=0}^{13} S_3(j) h_{n+1}^j y_n^{(j)}$$

with coefficients

$$S_3(j) = M^3(j, 1:9)\boldsymbol{u}^3 = \frac{c_3^j}{j!}, \qquad j = 1, 2, \ldots, 9,$$
$$S_3(j) = a_{32} S_2(j-1), \qquad j = 10, 11, 12, 13.$$

4.5. Predictor P_4

The 10-vector of the reordered coefficients of predictor P_4 in (2) with $\ell = 4$, $\boldsymbol{u}^4 = [a_{43}, a_{42}, a_{41}, \gamma_{42}, \gamma_{43}, \ldots, \gamma_{4,8}]^T$, is the solution of the system of order conditions

$$M^4 \boldsymbol{u}^4 = \boldsymbol{r}^4, \qquad (24)$$

where the matrix M^4 is

$$M^4 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & \cdots & 0 \\ c_3 & c_2 & 0 & 1 & 0 & & 0 \\ c_3^2/2! & c_2^2/2! & 0 & 0 & 1 & & 0 \\ \vdots & & & & & \ddots & \\ c_3^7/7! & c_2^7/7! & 0 & 0 & 0 & & 1 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \qquad (25)$$

and the right-handside, $\boldsymbol{r}^4 = r_4(1:10)$, has components

$$r_4(i) = c_4^i/i!, \quad i = 1, 2, \ldots, 8, \qquad r_4(9) = a_{43}, \quad r_4(10) = a_{42}.$$

4.6. Predictor $\mathbf{P_5}$

The 11-vector of the reordered coefficients of predictor P_5 in (2) with $\ell = 5$, $\boldsymbol{u}^5 = [a_{54}, a_{53}, a_{52}, a_{51}, \gamma_{52}, \gamma_{53}, \dots, \gamma_{5,8}]^T$, is the solution of the system of order conditions

$$M^5 \boldsymbol{u}^5 = \boldsymbol{r}^5, \tag{26}$$

where

$$M^5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & \cdots & 0 \\ c_4 & c_3 & c_2 & 0 & 1 & 0 & & 0 \\ c_4^2/2! & c_3^2/2! & c_2^2/2! & 0 & 0 & 1 & & 0 \\ \vdots & & & & & & \ddots & \\ c_4^7/7! & c_3^7/7! & c_2^7/7! & 0 & 0 & 0 & & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{27}$$

and $\boldsymbol{r}^5 = r_5(1:11)$ has components

$$r_5(i) = c_5^i/i!,\ i = 1, 2, \dots, 8, \quad r_5(9) = a_{54}, \quad r_5(10) = a_{53}, \quad r_5(11) = a_{52}.$$

4.7. Predictor $\mathbf{P_6}$

The 12-vector of the reordered coefficients of predictor P_6, $\boldsymbol{u}^6 = [a_{65}, a_{64}, a_{63}, a_{62}, a_{61}, \gamma_{62}, \gamma_{63}, \dots, \gamma_{6,8}]^T$, in (2) with $\ell = 6$ is the solution of the system of order conditions

$$M^6 \boldsymbol{u}^6 = \boldsymbol{r}^6, \tag{28}$$

where

$$M^6 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & \cdots & 0 \\ c_5 & c_4 & c_3 & c_2 & 0 & 1 & 0 & & 0 \\ c_5^2/2! & c_4^2/2! & c_3^2/2! & c_2^2/2! & 0 & 0 & 1 & & 0 \\ \vdots & & & & & & & \ddots & \\ c_5^7/7! & c_4^7/7! & c_3^7/7! & c_2^7/7! & 0 & 0 & 0 & & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{29}$$

and $\boldsymbol{r}^6 = r_6(1:12)$ has components

$$r_6(i) = c_6^i/i!, \qquad i = 1, 2, \ldots, 8,$$
$$r_6(9) = a_{65}, \quad r_6(10) = a_{64}, \quad r_6(11) = a_{63}, \quad r_6(12) = a_{62}.$$

4.8. Predictor $\mathbf{P_7}$

The 13-vector of the reordered coefficients of predictor P_7, $\boldsymbol{u}^7 = [a_{76}, a_{75}, \ldots, a_{71}, \gamma_{72}, \gamma_{73}, \ldots, \gamma_{7,8}]^T$, in (2) with $\ell = 7$ is the solution of the system of order conditions

$$M^7 \boldsymbol{u}^7 = \boldsymbol{r}^7, \tag{30}$$

where

$$M^7 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & \cdots & 0 \\ c_6 & c_5 & c_4 & c_3 & c_2 & 0 & 1 & 0 & & 0 \\ c_6^2/2! & c_5^2/2! & c_4^2/2! & c_3^2/2! & c_2^2/2! & 0 & 0 & 1 & & 0 \\ \vdots & & & & & & & & \ddots & \\ c_6^7/7! & c_5^7/7! & c_4^7/7! & c_3^7/7! & c_2^7/7! & 0 & 0 & 0 & & 1 \\ b_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & b_7 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & b_7 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & b_7 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & b_7 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{31}$$

and $\boldsymbol{r}^7 = r_7(1:13)$ has components

$$\begin{aligned} r_7(i) &= c_7^i/i!, \qquad i = 1, 2, \ldots, 8, \\ r_7(9) &= b_6(1 - c_6), \\ r_7(10) &= b_5(1 - c_5) - (b_6 a_{65}), \\ r_7(11) &= b_4(1 - c_4) - (b_6 a_{64} + b_5 a_{54}), \\ r_7(12) &= b_3(1 - c_3) - (b_6 a_{63} + b_5 a_{53} + b_4 a_{43}), \\ r_7(13) &= b_2(1 - c_2) - (b_6 a_{62} + b_5 a_{52} + b_4 a_{42} + b_3 a_{32}). \end{aligned}$$

The formulae for HBT(13)7 are listed in Appendix B.

5. Region of Absolute Stability

To obtain the region of absolute stability, R, of HBT(13)7, we apply the predictors $\mathrm{P}_2, \mathrm{P}_3, \ldots, \mathrm{P}_7$ and the integration formula IF with constant h to the linear

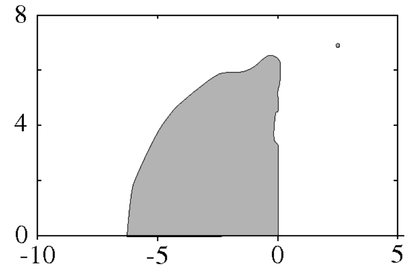


Figure 1: Region of absolute stability of HBT(13)7

test equation

$$y' = \lambda y, \qquad y_0 = 1.$$

Thus we obtain

$$y_{n+c_\ell} = y_n + \lambda h_{n+1} \sum_{j=1}^{\ell-1} a_{\ell j} y_{n+c_\ell} + \sum_{j=2}^{8} (\lambda h_{n+1})^j \gamma_{\ell j} y_n, \quad \ell = 2, 3, \ldots, 7, \tag{32}$$

and

$$y_{n+1} = y_n + \lambda h_{n+1} \sum_{j=1}^{7} b_j y_{n+c_j} + \sum_{j=2}^{8} (\lambda h_{n+1})^j \gamma_{1j} y_n. \tag{33}$$

If we replace y_{n+c_ℓ}, $\ell = 2, 3, \ldots, 7$, in (32)-(33) with the corresponding right-hand sides of (32), then (33) reduces to the following first-order difference equation and corresponding linear characteristic equation:

$$-r_s y_n + y_{n+1} = 0, \qquad -r_s + r = 0, \tag{34}$$

respectively. The root, r_s, of the characteristic equation is

$$r_s = 1 + \sum_{j=1}^{14} s_j \lambda^j h^j, \tag{35}$$

where the coefficients $\{s_j\}$ are listed in Nguyen-Ba et al [17].

A complex number λh is in R if r_s satisfies the root condition: $|r_s| \leq 1$. (see Hairer et al [6, pp. 378-380]).

The root condition is used to find the region of absolute stability of HBT(13)7 shown in grey in Figure 1, with interval of absolute stability $(\alpha, 0) = (-6.1, 0)$. It is seen that HBT(13)7 has a larger scaled interval of absolute stability than DP(8,7)13M, namely, $6.10/14 = 0.4357 > 0.3938 = 5.12/13$.

6. Controlling Stepsize

Generally, the step size h_{n+1} of the Taylor method is chosen within the tolerance TOL by the following formula (see Lara et al [11], Barrio et al [2])

$$h_{n+1} = k(\mathrm{TOL}, p)\|y^{(p)}/p!\|_{\infty}^{-1/p}\,, \tag{36}$$

where p is the order of the method and $k(\mathrm{TOL}, p)$ is such that $k^{p+1}/(1-k) = \mathrm{TOL}$.

Since HBT(13)7 does not have derivatives of order higher than 8, to determine the stepsize, for simplicity we consider the following formula

$$h_{n+1} = 1.4k(\mathrm{TOL}, 10)\left[\frac{\|y^{(6)}\|_{\infty}/6!}{\left[\|y^{(8)}\|_{\infty}/8!\right]^2}\right]^{1/10} \tag{37}$$

similar to an error estimator formula found in Barrio et al [2].

7. Numerical Results

The higher derivatives, y' to $y^{(8)}$, are calculated at each integration step by known recurrence formulae (see, for example, Hairer et al [6, pp. 46-49], Lara et al [11]).

Computations were performed in *C++* on a *Mac* with a dual 2.5 GHz PowerPC G5 and 4 GB DDR SSRAM running under *Mac OS X Version 10.4.8*.

The numerical performance of HBT(13)7, DP(8,7)13M and T13 is compared on Kepler's problem (D1-D5 Hull et al [8]) with eccentricity $\mathrm{e} = 0.1, 0.3, 0.5, 0.7, 0.9$ and $\mathrm{e} = 0.99$, Hénon-Heiles' problem, and the equatorial main problem.

7.1. Comparison Based on CPU

For each of the three methods, HBT(13)7, T13 and DP(8,7), applied to Kepler's problem, the *maximum global error* (MGE) was calculated from the uniform norm $\|y_{n+1} - y(t_{n+1})\|_{\infty}$ of the difference between the numerical solution y_{n+1} and the analytic solution $y(t_{n+1})$ at every integration step.

In Figure 2, CPU time (horizontal axis) is plotted versus $\log_{10}(|\mathrm{MGE}|)$ (vertical axis) for Kepler's problem with six values of the eccentricity. It is seen from the figure that HBT(13)7 compares favorably with both T13 and DP(8,7)13M on the basis of CPU time versus MGE at stringent tolerance.

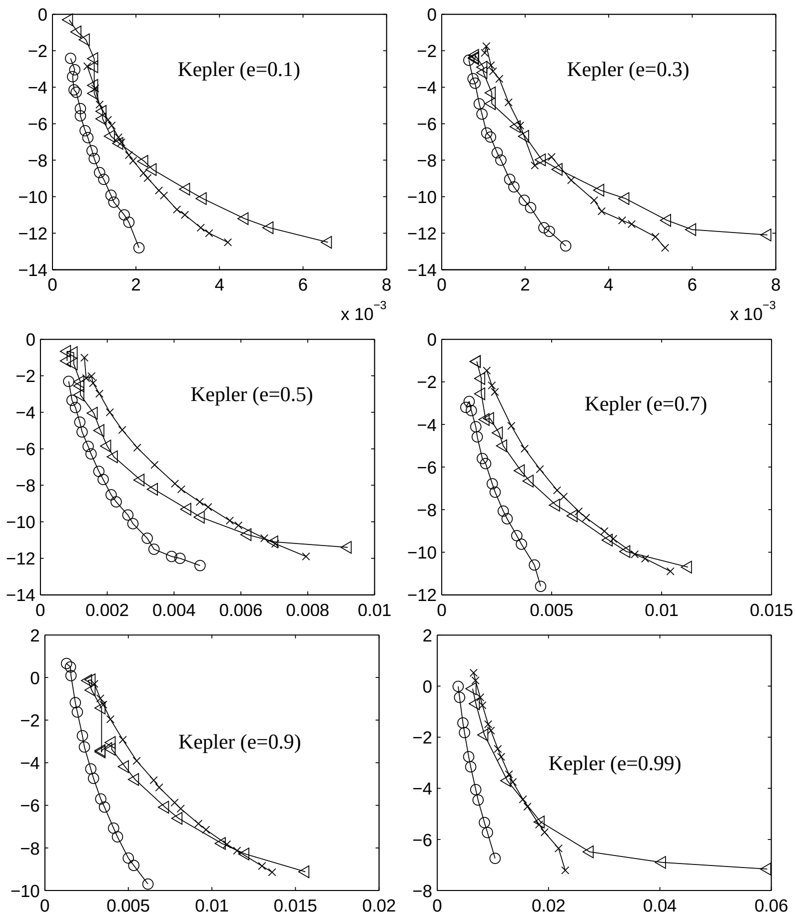
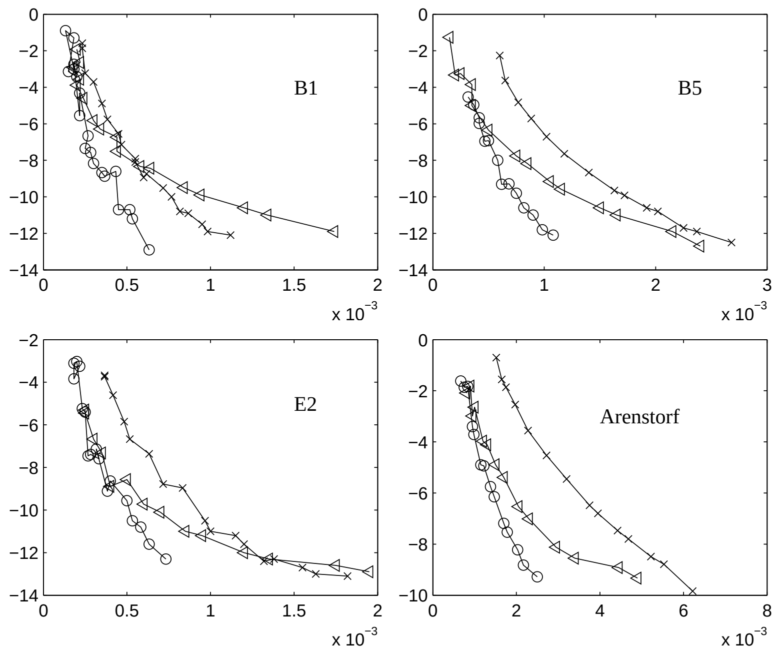


Figure 2: CPU time (horizontal axis) versus $\log_{10}(|\mathrm{MGE}|)$ (vertical axis) for Kepler's problem with varying eccentricity

In Figure 3, CPU time (horizontal axis) is plotted versus $\log_{10}(|\mathrm{MGE}|)$ (vertical axis) for the following nonstiff DETEST problems: B1, B5, E2 and Arenstorf's orbits problem. It is seen from the figure that HBT(13)7 compares favorably with both T13 and DP(8,7)13M on the basis of CPU time versus MGE at stringent tolerance.

The *CPU percentage efficiency gain* (CPU PEG) is defined by formula (cf.



HBT(13)7 ○, T13 × and DP(8,7)13M ▷

Figure 3: CPU time (horizontal axis) versus $\log_{10}(|\text{MGE}|)$ (vertical axis) for the following nonstiff DETEST problems: B1, B5, E2 and Arenstorf's orbits problem

Sharp [20]),

$$(\text{CPU PEG})_i = 100\left[\frac{\sum_j \text{CPU}_{2,ij}}{\sum_j \text{CPU}_{1,ij}} - 1\right], \tag{38}$$

where $\text{CPU}_{1,ij}$ and $\text{CPU}_{2,ij}$ are the CPU time of methods 1 and 2, respectively, associated with problem i, and $j = -\log_{10}(|\text{MGE}|)$. The CPU time was obtained from the curves which fit, in a least-squares sense, the data $(\log_{10}(|\text{MGE}|), \log_{10}(\text{CPU}))$ by means of *Matlab's* `polyfit`.

The CPU PEG of HBT(13)7 over DP(8,7)13M and T13 for Kepler's problems with varying eccentricity is listed in Tables 1.

	CPU PEG of HBT(13)7 over:	
Problem	DP(8,7)13M	T13
Kepler (e=0.1)	131%	91%
Kepler (e=0.3)	111%	82%
Kepler (e=0.5)	78%	102%
Kepler (e=0.7)	110%	129%
Kepler (e=0.9)	124%	132%
Kepler (e=0.99)	179%	110%
Hénon-Heiles	66%	84%
Arenstorf	48%	143%
B1	97%	70%
B5	67%	129%
E2	51%	93%

Table 1: CPU PEG of HBT(13)7 over DP(8,7)13M and T13 for Kepler's equation with varying eccentricity e, for Hénon-Heiles, Arenstorf problem and the nonstiff DETEST problems: B1, B5, E2

7.2. Comparison Based on the Number of Steps

The *maximum global energy error* (MGEE) was obtained from the maximum of the absolute value of the relative error $H/H_0 - 1$ at every integration step where H is the value of the Hamiltonian at t_{n+1} and H_0 is the value of the Hamiltonian at t_0.

The Hamiltonians of Kepler's problem, Hénon-Heiles' problem and the equatorial main problem are

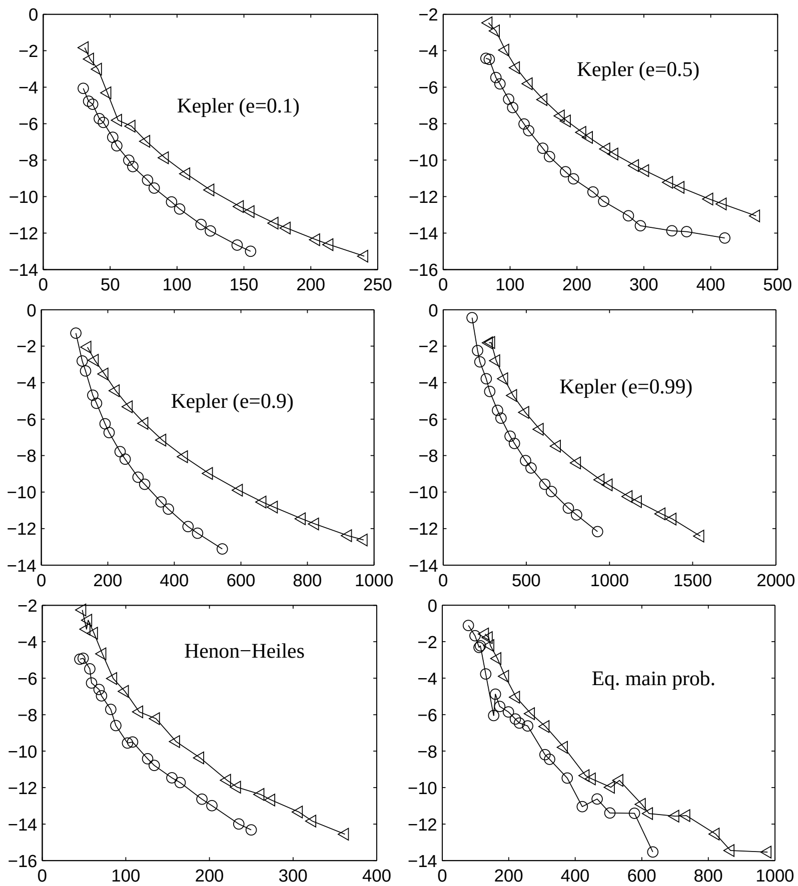
$$H_{\text{Kepler}} = \frac{1}{2}\left(y_3^2 + y_4^2\right) - 1 \Big/ \sqrt{y_1^2 + y_2^2}, \tag{39}$$

$$H_{\text{Hénon-Heiles}} = \frac{1}{2}\left(X^2 + Y^2\right) + \frac{1}{2}\left(x^2 + y^2\right) + \epsilon y \left(x^2 - \frac{1}{3}\, y^2\right), \tag{40}$$

$$H_{\text{eq. main prob.}} = \frac{1}{2}\left(P^2 + \frac{\Lambda^2}{\rho^2} + Z^2\right) + \frac{\mu}{r} + \frac{\alpha^2 J_2 \mu P_2(u)}{r^3}, \tag{41}$$

respectively, where, in (41), $u = z/r$, $r = \sqrt{\rho^2 + z^2}$ and $P_2(x) = (3x^2 - 1)/2$ is the Legendre polynomial of degree 2.

The *number ot step percentage efficiency gain* (NS PEG) is defined by the



HBT(13)7 ○, T13 ▷

Figure 4: Number of steps (horizontal axis) versus $\log_{10}(|\text{MGEE}|)$ (vertical axis) for Kepler's problem with eccentricity=0.1, 0.5, 0.9 and 0.99, Hénon-Heiles' problem, and the equatorial main problem

formula

$$(\text{NS PEG})_i = 100\left[\frac{\sum_j \text{NS}_{2,ij}}{\sum_j \text{NS}_{1,ij}} - 1\right], \tag{42}$$

where $\text{NS}_{1,ij}$ and $\text{NS}_{2,ij}$ are the number of steps of methods 1 and 2, respectively, associated with problem i, and $j = -\log_{10}(|\text{MGEE}|)$.

In Figure 4, NS (horizontal axis) is plotted versus $\log_{10}(|\text{MGEE}|)$ (vertical axis) for the problems in hand. NS was obtained from the curve which fit the

Problem	NS PEG
Kepler (e = 0.1)	47%
Kepler (e = 0.3)	60%
Kepler (e = 0.5)	60%
Kepler (e = 0.7)	79%
Kepler (e = 0.9)	78%
Kepler (e = 0.99)	61%
Hénon-Heiles	47%
Eq. main prob.	31%

Table 2: PEG of number of steps, NS, of HBT(13)7 over the Taylor method of order 13 of Lara et al [11] for the listed problems

data: $(\log_{10}(|\text{MGEE}|), \log_{10}(\text{NS}))$ in the least squares sense. It is observed that HBT(13)7 compares favorably with the Taylor method of order 13 of Lara et al[11] on the basis of the number of steps versus MGEE.

The NS PEG listed in Table 2 shows that HBT(13)7 performs better than DP(8,7)13M and T13 on the problems in hand.

It is to be noted that HBT(13)7 uses eight derivatives of y as compared to thirteen for T13.

The numerical results show that a combination of high order derivatives with a Runge-Kutta method achieves a high degree of accuracy.

8. Conclusion

An one-step 7-stage Hermite-Birkhoff-Taylor method of order 13, called HBT (13)7, was constructed by solving Vandermonde-type systems satisfying Taylor- and Runge-Kutta-type order conditions. By construction, HBT(13)7 uses lower order derivatives than the traditional Taylor method of order 13. The stability region of HBT(13)7 has a remarkably good shape. The stepsize is controlled by a formula which uses two high order derivatives.

On the basis of CPU time versus the maximum global error, HBT(13)7 won over DP(8,7)13M and T13 in solving Kepler's problem with varying eccentricity.

On the basis of the number of steps versus the maximum global energy error, HBT(13)7 won over T13 of Lara et al [11] in solving Kepler's problem with varying eccentricity, Hénon-Heiles' problem and the equatorial main problem.

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Appendix A. The 39 Order Conditions

Order 1 to 9:

$$\sum b_i = 1, \qquad \sum b_i c_i + \gamma_{12} = \frac{1}{2} \qquad \sum b_i c_i^2 + 2!\gamma_{13} = \frac{1}{3}\,,$$
$$\sum b_i c_i^3 + 3!\gamma_{14} = \frac{1}{4} \qquad \sum b_i c_i^4 + 4!\gamma_{15} = \frac{1}{5}, \qquad \sum b_i c_i^5 + 5!\gamma_{16} = \frac{1}{6}\,,$$
$$\sum b_i c_i^6 + 6!\gamma_{17} = \frac{1}{7}, \qquad \sum b_i c_i^7 + 7!\gamma_{18} = \frac{1}{8}, \qquad \sum b_i c_i^8 = \frac{1}{9}\,.$$

Order 10:

$$\sum b_i c_i^9 = \frac{1}{10}\,, \tag{43}$$
$$\sum b_i \left[\sum a_{ij}\,\frac{c_j^8}{8!}\right] = \frac{1}{10!}\,. \tag{44}$$

Order 11:

$$\sum b_i c_i^{10} = \frac{1}{11}\,, \tag{45}$$
$$\sum b_i \frac{c_i}{10}\left[\sum a_{ij}\,\frac{c_j^8}{8!}\right] = \frac{1}{11!}\,, \tag{46}$$
$$\sum b_i \left[\sum a_{ij}\,\frac{c_j^9}{9!}\right] = \frac{1}{11!}\,, \tag{47}$$
$$\sum b_i \left[\sum a_{ij}\left(\sum a_{jk}\,\frac{c_k^8}{8!}\right)\right] = \frac{1}{11!}\,. \tag{48}$$

Order 12:

$$\sum b_i c_i^{11} = \frac{1}{12}\,, \tag{49}$$
$$\sum b_i \frac{c_i^2}{10\times 11}\left[\sum a_{ij}\,\frac{c_j^8}{8!}\right] = \frac{1}{12!}\,, \tag{50}$$
$$\sum b_i \frac{c_i}{11}\left[\sum a_{ij}\,\frac{c_j^9}{9!}\right] = \frac{1}{12!}\,, \tag{51}$$
$$\sum b_i \frac{c_i}{11}\left[\sum a_{ij}\left(\sum a_{jk}\,\frac{c_k^8}{8!}\right)\right] = \frac{1}{12!}\,, \tag{52}$$
$$\sum b_i \left[\sum a_{ij}\,\frac{c_j^{10}}{10!} + \gamma_{i,11}\right] = \frac{1}{12!}\,, \tag{53}$$

$$\sum b_i \left[\sum a_{ij} \frac{c_j}{10} \left(\sum a_{jk} \frac{c_k^8}{8!}\right)\right] = \frac{1}{12!}, \tag{54}$$

$$\sum b_i \left[\sum a_{ij} \left(\sum a_{jk} \frac{c_k^9}{9!}\right)\right] = \frac{1}{12!}, \tag{55}$$

$$\sum b_i \left[\sum a_{ij} \left(\sum a_{jk} \left(\sum a_{k\ell} \frac{c_\ell^8}{8!}\right)\right)\right] = \frac{1}{12!}. \tag{56}$$

Order 13:

$$\sum b_i c_i^{12} = \frac{1}{13}, \tag{57}$$

$$\sum b_i \frac{c_i^3}{10 \times 11 \times 12} \left[\sum a_{ij} \frac{c_j^8}{8!}\right] = \frac{1}{13!}, \tag{58}$$

$$\sum b_i \frac{c_i^2}{11 \times 12} \left[\sum a_{ij} \frac{c_j^9}{9!}\right] = \frac{1}{13!}, \tag{59}$$

$$\sum b_i \frac{c_i^2}{11 \times 12} \left[\sum a_{ij} \left(\sum a_{jk} \frac{c_k^8}{8!}\right)\right] = \frac{1}{13!}, \tag{60}$$

$$\sum b_i \frac{c_i}{12} \left[\sum a_{ij} \frac{c_j^{10}}{10!}\right] = \frac{1}{13!}, \tag{61}$$

$$\sum b_i \frac{c_i}{12} \left[\sum a_{ij} \frac{c_j}{10} \left(\sum a_{jk} \frac{c_k^8}{8!}\right)\right] = \frac{1}{13!}, \tag{62}$$

$$\sum b_i \frac{c_i}{12} \left[\sum a_{ij} \left(\sum a_{jk} \frac{c_k^9}{9!}\right)\right] = \frac{1}{13!}, \tag{63}$$

$$\sum b_i \frac{c_i}{12} \left[\sum a_{ij} \left(\sum a_{jk} \left(\sum a_{k\ell} \frac{c_\ell^8}{8!}\right)\right)\right] = \frac{1}{13!}, \tag{64}$$

$$\sum b_i \left[\sum a_{ij} \frac{c_j^{11}}{11!}\right] = \frac{1}{13!}, \tag{65}$$

$$\sum b_i \left[\sum a_{ij} \frac{c_j^2}{10 \times 11} \left(\sum a_{jk} \frac{c_k^8}{8!}\right)\right] = \frac{1}{13!}, \tag{66}$$

$$\sum b_i \left[\sum a_{ij} \frac{c_j}{11} \left(\sum a_{jk} \frac{c_k^9}{9!}\right)\right] = \frac{1}{13!}, \tag{67}$$

$$\sum b_i \left[\sum a_{ij} \frac{c_i}{11} \left(\sum a_{jk} \left(\sum a_{k\ell} \frac{c_\ell^8}{8!}\right)\right)\right] = \frac{1}{13!}, \tag{68}$$

$$\sum b_i \left[\sum a_{ij} \left(\sum a_{jk} \frac{c_k^{10}}{10!}\right)\right] = \frac{1}{13!}, \tag{69}$$

$$\sum b_i \left[\sum a_{ij} \left(\sum a_{jk} \frac{c_k}{10} \left(\sum a_{k\ell} \frac{c_\ell^8}{8!}\right)\right)\right] = \frac{1}{13!}, \tag{70}$$

$$\sum b_i \left[\sum a_{ij} \left(\sum a_{jk} \left(\sum a_{k\ell} \frac{c_\ell^9}{9!}\right)\right)\right] = \frac{1}{13!}, \tag{71}$$

$$\sum b_i \left[\sum a_{ij} \left(\sum a_{jk} \left(\sum a_{k\ell} \left(\sum a_{\ell,m} \frac{c_m^8}{8!}\right)\right)\right)\right] = \frac{1}{13!}. \tag{72}$$

Appendix B. Formulae for HBT(13)7

The formulae for the one-step HBT(13)7 listed in this appendix use the offstep points listed in (14).

$$y_{n+c_2} = y_n + h_{n+1}\,(4.65815268584938435570\,\text{e-01} f_n)\ ,$$
$$+\,1.08491932223429163318\,\text{e-01}\, h_{n+1}^2 y_n'' + 1.68457328493185308493\,\text{e-02}\, h_{n+1}^3 y_n''' \,,$$
$$+\,1.96174989292885806891\,\text{e-03}\, h_{n+1}^4 y_n^{(4)} + 1.82762610654226047508\,\text{e-04}\, h_{n+1}^5 y_n^{(5)} \,,$$
$$+\,1.41889357615304732564\,\text{e-05}\, h_{n+1}^6 y_n^{(6)} + 9.44203274670250728208\,\text{e-07}\, h_{n+1}^7 y_n^{(7)} \,,$$
$$+\,5.49780377486626563905\,\text{e-08}\, h_{n+1}^8 y_n^{(8)},$$
$$y_{n+c_3} = y_n + h_{n+1}(1.33594514318505996897\,\text{e-01}\, f_{n+c_2}\,,$$
$$+\,3.83978006331425536501\,\text{e-01} f_n) + 7.17102924972126132275\,\text{e-02}\, h_{n+1}^2 y_n'' \,,$$
$$+\,8.61407417218069439147\,\text{e-03}\, h_{n+1}^3 y_n''' + 7.39519104199590042448\,\text{e-04}\, h_{n+1}^4 y_n^{(4)} \,,$$
$$+\,4.74310617925152917286\,\text{e-05}\, h_{n+1}^5 y_n^{(5)} + 2.28290368625479409286\,\text{e-06}\, h_{n+1}^6 y_n^{(6)} \,,$$
$$+\,7.85305078156948344504\,\text{e-08}\, h_{n+1}^7 y_n^{(7)} + 1.57675472371894850353\,\text{e-09}\, h_{n+1}^8 y_n^{(8)},$$
$$y_{n+c_4} = y_n + h_{n+1}(-1.55007754337264901101\,\text{e-02}\, f_{n+c_3}\,,$$
$$+\,5.80706073475723005828\,\text{e-02}\, f_{n+c_2} + 3.81136618834907958675\,\text{e-01}\, f_n)\,,$$
$$+\,7.07361780578155130428\,\text{e-02}\, h_{n+1}^2 y_n'' + 8.45379402574033579632\,\text{e-03}\, h_{n+1}^3 y_n''' \,,$$
$$+\,7.22866660976061604221\,\text{e-04}\, h_{n+1}^4 y_n^{(4)} + 4.62280586516011727984\,\text{e-05}\, h_{n+1}^5 y_n^{(5)} \,,$$
$$+\,2.22084418949858400258\,\text{e-06}\, h_{n+1}^6 y_n^{(6)} + 7.63300686896305395857\,\text{e-08}\, h_{n+1}^7 y_n^{(7)} \,,$$
$$+\,1.53275433381131921113\,\text{e-09}\, h_{n+1}^8 y_n^{(8)},$$
$$y_{n+c_5} = y_n + h_{n+1}(-7.63150520705699904056\,\text{e+01}\, f_{n+c_4}\,,$$
$$+\,2.15060688532934491946\,\text{e+01}\, f_{n+c_3} - 1.00455109487621419007\,\text{e+00}\, f_{n+c_2}\,,$$
$$+\,5.66756978894813059355\,\text{e+01}\, f_n) + 2.20438278409605104002\,\text{e+01}\, h_{n+1}^2 y_n'' \,,$$
$$+\,4.18557227956716992878\,\text{e+00}\, h_{n+1}^3 y_n''' + 5.10489516736485349213\,\text{e-01}\, h_{n+1}^4 y_n^{(4)} \,,$$
$$+\,4.41217162195926548840\,\text{e-02}\, h_{n+1}^5 y_n^{(5)} + 2.78237369209650454993\,\text{e-03}\, h_{n+1}^6 y_n^{(6)} \,,$$
$$+\,1.23614727138093957910\,\text{e-04}\, h_{n+1}^7 y_n^{(7)} + 3.18757633925890538143\,\text{e-06}\, h_{n+1}^8 y_n^{(8)},$$
$$y_{n+c_6} = y_n + h_{n+1}(7.11506013606492349055\,\text{e-01}\, f_{n+c_5}\,,$$

$$+ 5.9025898156544053563 3\,\text{e+02}\, f_{n+c_4} - 1.5180801543336963277 4\,\text{e+02}\, f_{n+c_3}\,,$$
$$+ 2.5383187580719321907 5\,\text{e+00}\, f_{n+c_2} - 4.4077736669412234960 0\,\text{e+02} f_n)\,,$$
$$- 1.7289434696751871456 4\,\text{e+02}\, h_{n+1}^2 y_n'' - 3.3059084635960708453 7\,\text{e+01}\, h_{n+1}^3 y_n'''\,,$$
$$- 4.0636671495579710367 0\,\text{e+00}\, h_{n+1}^4 y_n^{(4)} - 3.5452481710828420702 3\,\text{e-01}\, h_{n+1}^5 y_n^{(5)}\,,$$
$$- 2.2612943669976279248 4\,\text{e-02}\, h_{n+1}^6 y_n^{(6)} - 1.0186814608197911936 9\,\text{e-03}\, h_{n+1}^7 y_n^{(7)}\,,$$
$$- 2.6712595901721763666 5\,\text{e-05}\, h_{n+1}^8 y_n^{(8)},$$
$$y_{n+c_7} = y_n + h_{n+1}(8.7169955972910467823 0\,\text{e-02}\, f_{n+c_6}\,,$$
$$- 2.2583013754278846718 6\,\text{e-02}\, f_{n+c_5} - 2.6951530707422028854 1\,\text{e+02}\, f_{n+c_4}\,,$$
$$+ 6.7897855437293799241 2\,\text{e+01}\, f_{n+c_3} + 1.0105901684342197821 6\,\text{e+00}\, f_{n+c_2}\,,$$
$$+ 2.0154227452627364414 2\,\text{e+02}\, f_n) + 7.9021537070962821758 3\,\text{e+01}\, h_{n+1}^2 y_n''\,,$$
$$+ 1.5126628584911667729 8\,\text{e+01}\, h_{n+1}^3 y_n''' + 1.8634929125452137643 7\,\text{e+00}\, h_{n+1}^4 y_n^{(4)}\,,$$
$$+ 1.6315064845169244267 5\,\text{e-01}\, h_{n+1}^5 y_n^{(5)} + 1.0461999678644702538 3\,\text{e-02}\, h_{n+1}^6 y_n^{(6)}\,,$$
$$+ 4.7500091325731097404 6\,\text{e-04}\, h_{n+1}^7 y_n^{(7)} + 1.2596472255056101805 1\,\text{e-05}\, h_{n+1}^8 y_n^{(8)},$$
$$y_{n+1} = y_n + h_{n+1}\big(3.1869778222346684115 6\,\text{e-02}\, f_{n+1}\,,$$
$$+ 3.6278922502469418409 6\,\text{e-02}\, f_{n+c_6} + 1.8204880394743189198 1\,\text{e-01}\, f_{n+c_5}\,,$$
$$- 1.8540849006646500374 3\,\text{e+00}\, f_{n+c_4} + 1.2444403752474231250 6\,\text{e+00}\, f_{n+c_3}\,,$$
$$+ 0.0\,\text{e+00}\, f_{n+c_2} + 1.3594470207449789178 6\,\text{e+00}\, f_n\big)\,,$$
$$+ 4.1917312921210925225 1\,\text{e-01}\, h_{n+1}^2 y_n'' + 6.7351362071557591138 0\,\text{e-02}\, h_{n+1}^3 y_n'''\,,$$
$$+ 6.8982390399240905293 2\,\text{e-03}\, h_{n+1}^4 y_n^{(4)} + 4.8411231460696447885 5\,\text{e-04}\, h_{n+1}^5 y_n^{(5)}\,,$$
$$+ 2.3449274667622798556 2\,\text{e-05}\, h_{n+1}^6 y_n^{(6)} + 7.3568421146883380464 9\,\text{e-7}\, h_{n+1}^7 y_n^{(7)}\,,$$
$$+ 1.1757361220578129431 3\,\text{e-08}\, h_{n+1}^8 y_n^{(8)}.$$

Appendix C. Recurrent Computation of Derivatives

To advance integration from x_n to x_{n+1}, once y_{n+1} is obtained by formula (3), the function g, $[f_{n+1}, f'_{n+1}, \dots, f^{(7)}_{n+1}] = g(x_{n+1}, y_{n+1})$, with input (x_{n+1}, y_{n+1}) will output f_{n+1} and f'_{n+1} to $f^{(7)}_{n+1}$ by means of the recurrent power series method. In adding, multiplying or taking powers of input power series, this method computes, in a recurrent way, the kth term of the output power series as a combination of the preceding terms of the input series. For precision and efficiency, Horner's scheme is used to evaluate the second summations in (2) and (3) as nested polynomials in h_{n+1}.