International Journal of Pure and Applied Mathematics
Volume 44 No. 1 2008, 23-40

POINTWISE WELL-POSEDNESS IN VECTOR OPTIMIZATION AND VARIATIONAL INEQUALITIES

Rosa Ferrentino
Faculty of Economics
University of Salerno
via Ponte Don Melillo, Fisciano, 84084, ITALY
e-mail: rferrentino@unisa.it

Abstract: In this note we consider some notions of well-posedness for scalar and vector variational inequalities and we recall their connections with optimization problems. Subsequently, we investigate similar connections between well-posedness of a vector optimization problem and a related variational inequality problem and we present a result obtained with scalar characterizations of vector optimality concepts.

AMS Subject Classification: 90C29, 90C31, 49K40
Key Words: well-posedness, vector optimization, vector variational inequalities

1. Introduction

The notion of well-posedness is significant for several mathematical problems and in particular it plays a crucial role in the stability theory for optimization problems. It is also important in establishing convergence of algorithms for solving scalar optimization problems and, in fact, it has been studied in different areas of scalar optimization, such as mathematical programming, calculus of variations and optimal control.

Two different concepts of well-posedness are known. The first, due to J. Hadamard, requires existence and uniqueness of the optimal solution and studies its dependence from the data of the considered optimization problem. The second approach, introduced by A.N. Tykhonov, in 1966, requires, instead,

Received: April 5, 2007

besides existence and uniqueness of the optimal solution, the convergence of every minimizing sequence to the unique minimum point. The two concepts (well-posedness in the sense of Hadamard and of Tykhonov) are equivalent at least for continuous objective functions [2]. The links between Hadamard and Tykhonov well-posedness have been studied in [22], [24] and in [25], [26]. There, besides uniqueness, additional structures are involved: in [22], for example, basic ingredient is convexity. We will deal with well-posedness of Tykhonov type.

The notion of well-posedness for a vector optimization problem is, instead, less developed; there is not a commonly accepted definition of well-posed problem, in vector optimization. Some attempts in this direction have been already done (see [3], [21]) and have been made some comparisons with their scalar counterparts.

In the present paper we recall some basic aspects of the mathematical theory of well-posedness in scalar optimization and, subsequently, in vector optimization. In particular, among the various vector well-posedness notions known in the literature, the attention is focused on the concept of pointwise well-posedness, introduced in [9].

Moreover, in this note, we establish basic well-posedness results for scalar and vector variational inequalities. The paper is organized as follows.

In Section 2 we review, first, some results on well-posedness for a scalar optimization problem, and, then, for a scalar variational inequality of differential type. In Section 3 we present and investigate the notion of well-posedness in vector optimization (in particular a type of pointwise well-posedness and strong pointwise well-posedness for vector optimization problems) and, subsequently, the notion of well-posedness for a vector variational inequality. Finally, Section 4, is devoted to the main results of paper obtained by means of scalar characterizations of vector optimality concepts. The notion of pontwise well-posedness is linked to well-posedness of a suitable scalar variational inequality of a differential type whose construction represents an interesting application of the so-called "oriented distance function".

2. Tykhonov Well-Posedness of Scalar Optimization Problems and Variational Inequalities

To study the well-posedness of an optimization problem means to investigate the behaviour of the variable when the corresponding objective function value is close to the optimal value. In this section we give a characterization of

Tykhonov well-posedness for the problem of minimizing a function f on a closed, convex set K and we summarize some known results. Let $f : R^n \to R$ be a real-valued function and let K be a nonempty, closed and convex subset of R^n. We consider the scalar optimization problem:

$$\min_{x \in K} f(x) \ ,$$

which we denote by$P(f, K)$ and which consists in finding $x^* \in K$ such that

$$f(x^*) = \inf\{f(x),\ x \in K\} = \inf_K f(x) \, .$$

In the theory of optimization the following properties of the minimization problem $P(f, K)$ are interesting.

a) Existence of the solution (i.e. $P(f, K)$ has a solution).

b) Uniqueness of the solution (i.e. the solution set for $P(f, K)$ is a singleton).

c) x^* is a good approximation of the solution of $P(f, K)$, if $f(x^*)$ is close to $\inf_K f(x)$.

If the problem $P(f, K)$ satisfies together the properties a) and c) is said well-posed. More precisely:

The problem $P(f, K)$ is said Tykhonov well-posed if there exist exactly one $x^* \in K$ such that $f(x^*) \leq f(x)$ for all, and if $x_n \to x^*$ for any sequence $\{x_n\} \subset K$ such that $f(x_n) \to \ f(x^*)$ (i.e. $f(x_n) \to \inf_K \ f(x)$).

Recalling that a *sequence $\{x_n\} \subseteq K$ is a minimizing sequence for problem $P(f, K)$ when $f(x_n) \to \inf_K f(x)$ as $n \to +\infty$*, the previous definition can be rephrased, in equivalent way, so:

Definition 2.1. The problem $P(f, K)$ is said Tykhonov well-posed if f has, on K, a unique global minimum point, x^*, and every minimizing sequence for $P(f, K)$ converges to x^*.

Definition 2.1 is motivated by since usually every numerical method for solving $P(f, K)$ provides iteratively some minimizing sequences $\{x_n\}$ for $P(f, K)$ and therefore one wants to be sure that the approximate solutions x_n are not far from the (unique) minimum x^*. The idea of the behaviour of the minimizing sequences was used by different authors to extend this concept in two directions: first, introducing strengthened notions and, second, considering the case in which the optimal solutions are not unique.

Remarks 2.1. a) It is easy to see that if K is compact and f is lower semicontinuous, then $P(f, K)$ is Tykhonov well-posed if it has a unique solution.

b) For a continuous function f Tykhonov well-posedness of $P(f, K)$ simply means that every minimizing sequence is convergent.

c) When K is compact, the uniqueness of the solution of a minimization problem is enough to guarantee its well-posedness but there are however simple examples when uniqueness of the solution of $P(f,K)$ is not enough to guarantee its Tykhonov well-posedness even for continuous function. Take e.g. $K = R$ and $f(x) = x^2/x^4 + 1$. Obviously, $P(f,K)$ has a unique solution at zero while $\{x_n = n\}$ provides a minimizing sequence which does not converge to this unique solution. Hence $P(f,K)$ is not Tykhonov well-posed.

Tykhonov well-posedness of $P(f,K)$ is often stated, equivalently, as *strong uniqueness* of $\arg\min(f,K)$, where with $\arg\min(f,K)$ let us denote the set, possible empty, of solutions of the minimization problem $P(f,K)$.

Problems which are non well-posed will be called *ill-posed*. Sometimes they are referred to as *improperly posed*.

The well-posedness of the minimization problem $P(f,K)$ in the sense of Tykhonov concerns the behaviour of the function$f$ in the set K but it does not take into account the behaviour of f outside K, see [26]. Of course, often, one can come across with minimizing sequences that do not lie necessarily in K and one wants to control the behaviour of these minimizing sequences, as well. Levitin and Polyak in [18] considered such kind of sequences.

Definition 2.2. The sequence $\{x_n\}$ is said a Levitin-Polyak minimizing sequence for minimization problem $P(f,K)$ if $f(x_n) \to \inf_K f$and, moreover, $d(x_n,K) \to 0$, where $d(x_n,K) = \inf_{y\in K} \|x_n - y\|$.

Definition 2.3. The problem $P(f,K)$ is called Levitin-Polyak well-posed if it has unique solution $x^* \in K$ and, moreover, every Levitin-Polyak minimizing sequence for $P(f,K)$ converges to x^*.

This definition is stronger than the Tykhonov one since requires the convergence to the unique solution of each sequence belonging to a larger set of minimizing sequence than in the Tykhonov case.

In the above definitions is required the existences and the uniqueness of solution towards which every minimizing sequence converges. They, however, admit generalizations which do not require uniqueness of the solution. In other words, the uniqueness requirement can be relaxed and well-posed minimum problems with several solutions can be considered. Therefore, unlike the requirements of existence and stability a), c), the uniqueness is a condition more debatable. In fact, many problems in linear and quadratic programming or many optimization problems are usually considered as well-posed problems, although uniqueness is usually not satisfied [2]. In order to weaken the requirement of uniqueness of the solution, other more general notions of well-posedness have been introduced, depending on the hypotheses made on f (and K). We recall the concept of well-setness introduced in [2].

Definition 2.4. Problem $P(f,K)$ is said to be well-set when, for every minimizing sequence $\{x_n\} \subseteq K$, we have $d(x_n, \arg\ \min(f,K)) \to 0$, where $d(x,K)$ is the distance of the point x from the set K.

The following definition gives a generalized version of the Tykhonov well-posedness where the uniqueness of the solution is dropped.

Definition 2.5. Problem $P(f,K)$ is said Tykhonov well-posed in the generalized sense when every minimizing sequence for $P(f,K)$ has some subsequence that converges to solution of $P(f,K)$, i.e. to an element of $\arg\min(f, K)$.

From the definition it follows that if $P(f,K)$ is well-posed in the generalized Tykhonov sense, then $\arg\min(f,K)$ is nonempty and compact. Moreover, when $P(f,K)$ is well-posed in the generalized sense and $\arg\min(f,K)$ is a singleton (i.e. its solution is unique), then$P(f,K)$ is Tykhonov well-posed.

The corresponding generalization of Levitin-Polyak well-posedness follows:

Definition 2.6. The minimization problem $P(f,K)$ is called generalized Levitin-Polyak well-posed if every Levitin-Polyak minimizing sequence $\{x_n\}$ for $P(f,K)$ has a subsequence converging to a solution of $P(f,K)$.

Of course, if at any of the notions of generalized well-posedness is added the uniqueness of the solution, one obtains the corresponding non generalized notion.

Different characterizations of Tykhonov well-posedness for minimization problems determined by convex functions in Banach spaces can be found in [23].

We, now, recall the next results.

Proposition 2.1. (see [6]) *Let $f : K \subseteq R^n \to R$ be a convex function and let K be a convex set. If f has a unique global minimizer on K, then $P(f,K)$ is Tykhonov well-posed.*

Proposition 2.2. (see [10]) *If K is closed, $f$is lower semicontinuous, bounded from below and uniformly quasiconvex on K, then $P(f,K)$ is Tykhonov well-posed.*

The function $f : K \in R^n \to R$ is *uniformly quasiconvex* (see [10]) if there exists an increasing function $g : [0,+\infty[\to [0,+\infty[$ such that $g(0) = 0, \quad g(t) > 0$ whenever $t > 0$ and $f\left(\frac{x+y}{2}\right) \leq \max\{f(x), f(y)\} - g(\|x-y\|)\ \forall x,y \in K$.

The following theorem gives an alternative characterization of Tykhonov well-posedness: it uses the set of ε −optimal solutions.

Theorem 2.1. (see [10]) *If $P(f,K)$ is Tykhonov well-posed, then:*

$$diam\,\{\varepsilon\ -\arg\ \min(f,K)\} \to 0 \ \ \text{for} \ \ \varepsilon \to 0\,, \tag{1}$$

where $\varepsilon\ -\arg\min(f,K)\ := \{x \in K : f(x) \leq \varepsilon + \inf_K f(x)\}$ is the set of ε −minimizers of f over K and diam denotes the diameter of given set.

Conversely, if f is lower semicontinuous and bounded from below on K, then condition (1) implies Tykhonov well-posedness of $P(f,K)$.

When K is closed and f is lower semicontinuous, we can use the sets:

$$L_{K,f}(\varepsilon) = \{x \in R : f(x) \leq \inf(f,K) + \varepsilon \text{ and } d(x,K) \leq \varepsilon\}, \quad \varepsilon > 0,$$

to introduce the equivalent notion of well-posedness of $P(f,K)$.

Definition 2.7. Let K be closed and let $f : K \to R$ be lower semicontinuous. The minimization problem $P(f,K)$ is said to well-posed if:

$$\inf\{\ diam\ L_{K,f}(\varepsilon)\ , \quad \varepsilon > 0\} = 0.$$

It is well-known that there is a very close connection between optimization problems and variational inequalities. In other words, the well-posedness of a scalar minimization problem is linked to that of a scalar variational inequality and, in particular, to a variational inequality of differential type (i.e. in which the operator is the gradient of a given function). The links between variational inequalities of differential type and corresponding optimization problems have been studied in [17]. Furthermore, by means of Ekeland's variational principle [11], that, as it is well-known, is an important tool to prove some results in well-posedness for optimization, a notion of well-posed scalar variational inequality has been introduced and its links with the concept of well-posed optimization problem have been investigated [23].

We shall deal with variational inequalities of Stampacchia type.

We recall that a point $x^* \in K$ is *a solution of a variational inequality of Stampacchia type* when:

$$SVI(F,K) < F(x^*), y - x^* >\geq 0\,, \quad \forall y \in K\,,$$

where $F : K \subseteq R^n \to R^n$, while K is closed and convex.

This problem, introduced by G. Stampacchia, has been recently studied by many authors (see for example [1]), since it describes many economic or engineering problems and it is an efficient tool for investigating vector optimization problems. In many applications, the problems $SVI(F,K)$ do not always have a unique solution. It is well known that it has a unique solution if the operator

F is strongly monotone and hemicontinuous.

If $f : R^n \to R$ is differentiable an a open set containing the convex set K, we can consider the variational inequality of differential type, $SVI(f', K)$, where f' denotes the gradient of f (f is a primitive of F, i.e. $F = f'$). It is known that, under these hypotheses, $SVI(f', K)$ is a necessary optimality condition for problem $P(f, K)$. The following definition gives the notion of well-posed variational inequality of differential type [10].

Definition 2.8. The variational inequality $SVI(f', K)$ is well-posed when:

i) $T(\varepsilon) \neq \emptyset \quad \forall \varepsilon > 0$.

ii) diam $T(\varepsilon) \to 0$ if $\varepsilon \to 0$.

where $T(\varepsilon) : \{x \in K :< f'(x) \ , x - y > \leq \varepsilon \|y - x\| \quad \forall y \in K\}$ is the approximate solutions set of $P(f, K)$.

We can see that $T(\varepsilon)$ is closed for every ε. It is obvious that the set of solutions of SVI is exactly $\bigcap\limits_{\varepsilon>0} T(\varepsilon)$. Then, if the variational inequality is well-posed, $diam\, T(\varepsilon) \to 0$ and so the intersection of $T(\varepsilon)$ is nonempty and shrinks to a single point. Therefore:

Proposition 2.3. (see [10], [23]) *If f is hemicontinuous and lower semicontinuous and if the variational inequality $SVI(f', K)$ is well-posed, then $SVI(f', K)$ has an unique solution.*

The converse of Proposition 2.3 holds under monotonicity assumptions on f.

Proposition 2.4. (see [10]) *If f is hemicontinuous and monotone on K and if the variational inequality has exactly one solution, then it is well-posed.*

The following theorem states that the boundedness of $T(\varepsilon)$ gives an existence result for variational inequalities [10].

Theorem 2.2. *If f hemicontinuous on K and $T(\varepsilon) \neq 0$ for all ε and bounded for at least some ε, then the variational inequality $SVI(f', K)$ has solutions.*

The next theorem gives, instead, the link between Tykhonov well-posedness of $P(f, K)$ and well-posedness of $SVI(f', K)$.

Theorem 2.3. (see [10], [23]) *Let f be bounded from below and differentiable on an open set containing K. If $SVI(f', K)$ is well-posed, then problem $P(f, K)$ is Tykhonov well-posed. The converse is true if f is convex.*

3. Well-Posedness of Vector Optimization Problems and of Vector Variational Inequalities

In scalar optimization the different notions of well-posedness are based either on the behaviour of " appropriate" minimizing sequences or on the dependence of optimal solution with respect to the data of optimization problems.

In vector optimization, instead, there is not a commonly accepted definition of well-posedness but there are different notions of well-posedness of vector optimization problems. For a detailed survey on these problems it is possible to refer to [2], [3], [9], [20], [21]. In this section, we propose some of these definitions of well-posedness for a vector optimization problem.

We consider the vector optimization problem:

$$VP\,(f,K)\min_{C} f\,(x)\ ,\quad x \in K\,,$$

where K is a nonempty, closed, convex subset of R^n, $f : K \subseteq R^n \to R^l$ is a continuous function and $C \subseteq R^l$ is a closed, convex, pointed cone and with nonempty interior. Denote by int C the interior of C.

The cone C gives an order relation on R^l, in the following way

$$y_1 \leq_C y_2 \Leftrightarrow y_2 - y_1 \in C\,,$$

$$y_1 <_C y_2 \Leftrightarrow y_2 - y_1 \in \ int\ C\,.$$

We recall that a point $x^* \in K$ is said to be an *efficient solution* or *minimal solution* of problem $VP\,(f,K)$ when:

$$f\,(x) - f\,(x^*) \notin -C \setminus \{0\}\ ,\quad \forall x \in K\,.$$

If, in the above definitions, instead of the cone C we use the cone $\tilde{C} = \{0\} \cup int\,C$, x^* is said *weak minimal solution*. Then, a point $x^* \in K$ is said to be a *weakly efficient solution* or *weak minimal solution* of problem $VP\,(f,K)$ when:

$$f\,(x) - f\,(x^*) \notin -int\,C\,.$$

We denote by $Eff\,(f,K)$ the set of all efficient solutions (minimal solutions) of the problem $VP\,(f,K)$ while by $WEff\,(f,K)$ the set of weakly efficient solutions (weak minimal solutions) of $VP\,(f,K)$. Moreover, every minimal is also a weak minimal solution but the converse is not generally true.

The image of the set $Eff\,(f,K)$, under the function f, is denoted by $Min\,(f,K)$ and its elements are called *minimal values* of $VP\,(f,K)$. Therefore

$$Min\,(f,K) = f\,[Eff\,(f,K)]\,.$$

Unlike the scalar case, in vector optimization one can hardly expect that the set $Min\,(f, K)$ be a singleton.

Analogously: $WMin\,(f, K) = f\,[WEff\,(f, K)]$.

In this section we recall a notion of well-posedness that considers a single point (a fixed efficient solution) and not the whole solution set: a particular type of pointwise well-posedness and strong pointwise well-posedness for vector optimization problems. This definition can be introduced considering, as in the scalar case, the diameter of the level sets of the function f.

Generalizing Tykhonov's definition of well-posedness for a scalar optimization problem, in [9] are introduced the notions of well-posedness and of strong well-posedness of vector optimization problem$VP\,(f, K)$ at a point $x^* \in Eff(f, K)$ and are provided, also, some conditions to guarantee well-posedness according to these definitions.

Definition 3.1. The vector optimization problem $VP\,(f, K)$ is said to be pointwise well-posed at the efficient solution $x^* \in K$ or Tykhonov well-posed at $x^* \in Eff\,(f, K)$, if:

$$\inf\ diam\ L\,(x^*, k, \alpha\,) = 0\,, \quad \forall k \in C\,,\ \forall \alpha > 0\,,$$

where:

$$\begin{aligned} L\,(x^*, k, \alpha) &= \{\,x \in K : f\,(x) \leq_C f\,(x^*) + \alpha k\} \\ &= \{\,x \in K : f\,(x) \in f\,(x^*) + \alpha k - C\}\ . \end{aligned}$$

Definition 3.2. The vector optimization problem $VP\,(f, K)$ is said to be strongly pointwise well-posed at the efficient solution x^*, or Tykhonov strongly well-posed at $x^* \in Eff\,(f, K)$, if :

$$\inf\ diam\ L_s\,(x^*, k, \alpha\,) = 0\,, \quad \forall\, k \in C\,,$$

where:

$$L_s\,(x^*, k, \alpha) = \{\,x \in K : f\,(x) \leq_C f\,(x^*) + \alpha k \quad and\ \ d\,(x, K) \leq \alpha\}\ .$$

For the sake of completeness, we recall that it is also possible to introduce another type of well-posedness of the vector optimization problem $VP\,(f, K)$ at a point $x^* \in Eff\,(f, K)$, see [16].

Definition 3.3. The vector optimization problem $VP\,(f, K)$ is said to be H-well-posed at a point $x^* \in Eff\,(f, K)$ if $x_n \to x^*$ for any sequence $\{\,x_n\} \subseteq K$, such that $f\,(x_n) \to f\,(x^*)\,.$

Definition 3.4. The vector optimization problem $VP\,(f, K)$ is said to be strongly H-well-posed at a point $x^* \in Eff\,(f, K)$ if $x_n \to x^*$for any sequence $\{\,x_n\}$ such that $f\,(x_n) \to f\,(x^*)$ with $d\,(x_n, K) \to 0$.

Remark 3.1. If $int\,C \neq 0$, then well-posedness at a point $x^* \in Eff\,(f, K)$ of the vector optimization problem $VP\,(f, K)$ according to Definition 3.1 (resp. to Definition 3.2) implies well-posedness according to Definition 3.3 (resp. to Definition 3.4).

It is easy realize that the pointwise well-posedness of type 3.1 is weaker than pointwise well-posedness of type 3.3, see [16].

An useful tool in the study of vector optimization problems is provided by the vector variational inequalities. They, introduced first by Giannessi in 1980 [12], have been studied intensively because they can be efficient tools for investigating vector optimization problems and also because they provide a mathematical model for equilibrium problems. In this section we deal with vector variational inequalities of differential type.

Let $f : R^n \to R^l$ be a function differentiable on an open set containing the closed convex set $K \subseteq R^n$. The *vector variational inequality problem of differential type* consists in finding a point $x^* \in K$ such that:

$$SVVI\left(f', K\right) \qquad < f'\left(x^*\right), y - x^* >_l \notin -int\,C\,, \quad \forall\, y \in K\,,$$

where f' denotes the Jacobian of f and $< f'\left(x^*\right), y - x^* >_l$ is the vector whose components are the l inner products $< f_i'\left(x^*\right), y - x^* >$.

It is well-known that $SVVI\,(f', K)$ provides a necessary condition for x^* to be an efficient solution of $VP\,(f, K)$. It is, instead, a sufficient condition for x^* to be an efficient solution of $VP\,(f, K)$ if f is *int* $C-$convex while, if f is C-convex, $SVVI\,(f', K)$ is a sufficient condition for x^* to be an weakly efficient solution of $VP\,(f, K)$. These remarks underline the links between optimization problems and variational inequalities also for vector case. This is a further reason for a suitable definition of well-posedness for a vector variational inequality which could be compared and related to the given definition for vector optimization. We take the condition $T(\varepsilon) \to 0$ for $\varepsilon \to 0$ as an expression of the convergence of the set $T(\varepsilon)$ to the set of solutions of the variational inequality (and consequently of the minimum problem). Then, we introduce a notion of well-posedness for the vector variational inequality problem $SVVI\,(f', K)$, obtained by generalizing Definition 2.8. We define the following set:

$$T_{c^0}(\varepsilon) := \left\{ x \in K : < f'\left(x\right), \, y - x >_l \notin -\sqrt{\varepsilon}\, \|y - x\|\; c^0 - int\; C, \;\; \forall\, y \in K \right\}\,,$$

where $\varepsilon > 0$ and $c^0 \in \;\; int\,C$. $T_{c^0}(\varepsilon)$ is a directional generalization of the set $T(\varepsilon)$ of Section 2.

Definition 3.5. The variational inequality $SVVI\,(f', K)$ is well-posed if, for every $c^0 \in \;\; int\,C$, $e(T_{c^0}\left(\varepsilon\right)$**,** $WEff\,(f, K)) \to 0$, where $e\left(A_K, A\right) =$

$\sup_{Ai \subset A_K} d(A_i, A)$.

The following result states the relationship between well-posed optimization problem and a well-posed variational inequality, in the vector case [27].

Theorem 3.2. *If the variational inequality $SVVI(f', K)$ is well-posed, then problem $VP(f, K)$ is well-posed at x^*.*

For C-convex functions, in particular, well-posedness of $VP(f, K)$ and $SVVI(f', K)$ substantially coincide. To show that, we assume that f is differentiable on an open set containing K and we observe that:

Lemma 1. (see [6]) *If $f : R^n \to R^l$is $C-$convex, then:*

$T_{c^0}(\varepsilon) := Z_{c^0}(\varepsilon) = \left\{ x \in K : f(y) - f(x) \notin -\sqrt{\varepsilon}\, \|y - x\|\, c^0 - int\, C \right\}$.

Definition 3.6. The function $f : K \subseteq R^n \to R^l$ is said to be $C-$convex when:

$$f(\lambda x + (1-\lambda) y) - [\lambda f(x) + (1-\lambda) f(y)] \in -C\,, \quad \forall x, y \in K, \quad \forall \lambda \in [0, 1]\ .$$

Theorem 3.3. (see [6]) *Let f be a $C-$convex function. Assume that $c^0 \in\ int\, C$, and that $T_{c^0}(\varepsilon)$ is bounded for some $\varepsilon > 0$. Then $SVVI(f', K)$ is well-posed.*

Therefore, if f is a $C-$convex function, the well-posedness of $SVVI(f', K)$ is ensured and, that is, by Theorem 3.2, substantially coincide with well-posedness of $VP(f, K)$.

4. Main Results

In this section we introduce a suitable scalar generalized variational inequality of differential type problem and we link the well-posedness of this problem to pointwise well-posedness of $VP(f, K)$. Throughout this section we assume that $f : R^n \to R^l$ is differentiable on an open set containing the closed convex set $K \subseteq R^n$.

Definition 4.1. For a set $A \subseteq R^l$, let $\Delta_A : R^l \to R \cup \{\pm\infty\}$ be defined as:

$$\Delta_A(y) = d(y, A) - d(y, A^c)\ ,$$

where $d_A(y) = \inf_{a \in A} \|y - a\|$ is the distance from the point y to the set A.

Function $\Delta_A(y)$ is called the *oriented distance function* from the point y to the set A and it has been introduced in the framework of nonsmooth scalar optimization. While $d_A(y) = 0$ when $y \in cl\ A$(the closure of A) and positive elsewhere, $\Delta_A(y) < 0$ for $y \in\ int\, A$ (the interior of A), $\Delta_A(y) = 0$ for

$y \in bd\ A$ (the boundary of A) and positive elsewhere.

The main properties of function Δ_A are gathered in the following:

Theorem 4.1. (see [30]) *i) if* $A \neq \emptyset$ *and* $A \neq R^l$, *then* Δ_A *is real valued;*

ii) Δ_A *is 1-Lipschitzian;*

iii)$\Delta_A(y) < 0 \forall y \in intA$, $\Delta_A(y) = 0$, $\forall\, y \in bd\, A$ *and* $\Delta_A(y) > 0$, $\forall\, y \in intA^c$;

iv) if A is closed, then it holds $A = \{y : \Delta_A(y) \leq 0\}$;

v) if A is convex, then Δ_A *is convex;*

vi) if A is a cone, then Δ_A *is positively homogeneous;*

vii) if A is a closed convex cone, then Δ_A *is non increasing with respect to the ordering relation induced on* R^l, *i.e. the following is true:*

if $y_1,\ y_2 \in R^l$ *then* $y_1 - y_2 \in A \Rightarrow \Delta_A(y_1) \leq \Delta_A(y_2)$

if A has nonempty interior, then $y_1 - y_2 \in\ intA \quad \Rightarrow \quad \Delta_A(y_1) < \Delta_A(y_2)$.

The oriented distance function Δ_A, used also to obtain a scalarization of a vector optimization problem [5], [15], allows to establish a relationship between the well-posedness of the original vector problem and the well-posedness of the associate scalar problem. More precisely, in [5], [6] it is Known that one of notions of well-posedness in vector optimization can be rephrased as a suitable well-posedness of a corresponding scalar optimization problem, i.e. is linked to well-posedness of a suitable scalar variational inequality of differential type. The construction of this scalar variational inequality represents on interesting application of the "oriented distance function".

It has been proved, in [13], that when A is closed, convex, pointed cone, then we have:

$$\Delta_{-A}(y) = \max_{\xi \in A' \cap S} \langle \xi, y \rangle \ ,$$

where $A' := \left\{ x \in R^l |\, \langle x, a \rangle \geq 0,\ \forall\, a \in A \right\}$ is the positive polar of the cone of A and S the unit sphere in R^l.

We use function Δ_{-A} in order to give scalar characterizations of some notions of efficiency for problem $VP(f, K)$. Furthermore, some results characterize pointwise well-posedness of problem $VP(f, K)$ through function Δ_{-A} [9].

Given a point $x^* \in K$, we consider the function:

$$\varphi_{x^*}(x) = \max_{\xi \in C' \cap S} \langle \xi, f\ (x) - f\ (x^*) \rangle \ ,$$

where C' denotes the positive polar of C and S the unit sphere in R^l. Clearly

$$\varphi_{x^*}(x) = \Delta_{-C}\left(f(x) - f(x^*)\right) .$$

The function φ_{x^*} is directionally differentiable [8] and hence we can consider the directional derivative

$$\varphi'_{x^*}(x\,;d) = \lim_{t \to 0^+} \frac{\varphi_{x^*}(x+td) - \varphi_{x^*}(x)}{t}$$

and the associated *scalar* problem: find $x^* \in K$, such that:

$$SVI\left(\varphi'_{x^*}, K\right) , \quad \varphi'_{x^*}(x^*;\ y - x^*) \geq 0 , \quad \forall\, y \in K .$$

The solutions of problem $SVI\,(\varphi'_{x^*}, K)$ coincide with the solutions of $SVVI(f',$ $K)$.

Proposition 4.1. (see [7]) *Let K be a convex set. If $x^* \in K$ solves problem $SVI\,(\varphi'_{x^*}, K)$ for some $x^* \in K$, then x^* is a solution of $SVVI\,(f', K)$. Conversely, if $x^* \in K$ solves $SVVI\,(f', K)$, then x^* solves problem $SVI\,(\varphi'_{x^*}, K)$.*

The scalar problem associated with the vector problem $VP\,(f, K)$ is:

$$P\,(\varphi_{x^*}, K) , \quad \min \varphi_{x^*}(x) , \quad x \in K .$$

The relations among the solutions of problem $P\,(\varphi_{x^*}, K)$ and those of problem $VP\,(f, K)$ are investigated in [14], [30]. We here refer only to the characterization of weak efficient solution.

Proposition 4.2. *The point $x^* \in K$ is a weak efficient solution of $VP\,(f, K)$ if and only if x^* is a solution of $P\,(\varphi_{x^*}, K)$.*

The proof is omitted and we refer to [6] for details.

Also well-posedness of $VP\,(f, K)$ can be linked to that of $P\,(\varphi_{x^*}, K)$.

Proposition 4.3. (see [6]) *Let f be a continuous function and let $x^* \in K$ be an efficient solution of $VP\,(f, K)$. Problem $VP\,(f, K)$ is pointwise well-posed at x^*if and only if problem $P\,(\varphi_{x^*}, K)$ is Tykhonov well-posed.*

Next proposition links the well-posedness of $SVI\,(\varphi'_{x^*}, K)$ to pointwise well-posedness of $VP\,(f, K)$. We need to recall Ekeland's variational principle [4]: it say that there is a "nearby point" which actually minimizes a slightly perturbed given functional. More precisely it asserts that a particular optimization problem can be replaced by other which is near the original and has a unique solution [26]. In fact, often the mathematical model of a phenomenon is so complicated that is necessary replace it by other model which has a solution "near" the original one.

Proposition 4.4. *If $SVI\,(\varphi'_{x^*}, K)$ is pointwise well-posed at $x^* \in K$, then problem $VP\,(f, K)$ is pointwise well-posed at x^*.*

Proof. By Proposition 4.3, it is enough to prove that if $SVI\left(\varphi'_{x^*}, K\right)$ is pointwise well-posed at x^*, then problem $P\left(\varphi_{x^*}, K\right)$ is Tykhonov well-posed.

In fact, for every $\varepsilon > 0$ and $x \in \varepsilon-\arg\min\left(\varphi_{x^*}, K\right)$, by Ekeland's variational principle, there exist $\bar{x}$ such that:

$$\|\bar{x} - x\| \leq \sqrt{\varepsilon} \text{ and } \varphi_{x^*}\left(\bar{x}\right) \leq \varphi_{x^*}\left(y\right) + \sqrt{\varepsilon}\,\|\bar{x} - y\|\ , \quad \forall\, y \in K.$$

If we introduce the set

$$Z\left(\varepsilon\right) = \left\{x \in K : \varphi_{x^*}\left(x\right) \leq \varphi_{x^*}\left(y\right) + \varepsilon\,\|x - y\|\,, \quad \forall y \in K\right\}\ ,$$

then, we have

$$\varepsilon - \arg\min\left(\varphi_{x^*}, K\right) \subseteq Z\left(\varepsilon\right) + \sqrt{\varepsilon}\,B.$$

We get, then, that $\forall\, u \in \varepsilon - \arg\min\left(\varphi_{x^*}, K\right)$ there exist x such that $\|u - x\| \leq \sqrt{\varepsilon}$ and

$$\varphi_{x^*}\left(x + t(y - x)\right) \geq \varphi_{x^*}\left(x\right) - \sqrt{\varepsilon}\,t\ \|y - x\|\,, \quad 0 < t < 1,\ y \in K\,.$$

Since $\varphi'_{x^*}\left(x, y - x\right) \geq\ -\sqrt{\varepsilon}\,\|y - x\|$, it follows $x \in T_{x^*}\left(\sqrt{\varepsilon}\right)$ and so:

$$\varepsilon - \arg\min\left(\varphi_{x^*}, K\right) \subseteq T_{x^*}\left(\sqrt{\varepsilon}\right) + \sqrt{\varepsilon}\,B\,.$$

Since $\mathrm{diam}T_{x^*}\left(\sqrt{\varepsilon}\right) \to 0$ as $\varepsilon \to 0$, then we have that $P\left(\varphi_{x^*}, K\right)$ is Tykhonov well-posed. $\square$

Now we prove that the converse of the previous proposition holds under convexity assumptions i.e. is true if f is $C-$convex. Before, we need the following lemma.

Lemma 4.1. (see [7]) *If $f : R^n \to R^l$ is $C-$convex function, then the function $\varphi_{x^*}\left(x\right)$, is convex $\forall\, x \in K$.*

Proposition 4.5. *Let f be $C-$convex and assume $VP\left(f, K\right)$ is pointwise well posed at $x^* \in K$. Then $SVI\left(\varphi'_{x^*}, K\right)$ is pointwise well-posed at x^*.*

Assuming, that $SVI\left(\varphi'_{x^*}, K\right)$ is not pointwise well-posed at x^*, follows that exist $a > 0$ and $\varepsilon_n \to 0$, with *diam* $T_{x^*}\left(\varepsilon_n\right) > 2a$ and one can find some $x_n \in T_{x^*}\left(\varepsilon_n\right)$, with $\|x_n\| \geq a$.

Without loss of generality we put $x^* = 0$. Since φ_{x^*} is convex, we have:

$$\varphi_{x^*}\left(0\right) - \varphi_{x^*}\left(y_n\right) \geq \varphi'_{x^*}\left(y_n, -y_n\right)\ ,$$

where $y_n = a\frac{x_n}{\|x_n\|}$. The boundedness of y_n implies we can assume $y_n \to \bar{y} \in K$ (here we need K closed). Further, since $x_n \in T_{x^*}\left(\varepsilon_n\right)$, we have

$$\varphi'_{x^*}\left(x_n,\, -x_n\right) \geq -\varepsilon_n\,\|x_n\|\ .$$

Since

$$\varphi'_{x^*}\left(x_n,\, -x_n\right) = \lim_{t \to 0^+} \frac{\varphi_{x^*}\left(x_n - tx_n\right) - \varphi_{x^*}\left(x_n\right)}{t}$$

$$= \lim_{t\to 0^+} -\frac{\varphi_{x^*}\left(x_n+(-t)\left(x_n\right)\right)-\varphi_{x^*}\left(x_n\right)}{-t} = -\ \varphi'_{x^*}\left(x_n,x_n\right)$$

and

$$\varphi'_{x^*}\left(y_n,-y_n\right) = -\ \varphi'_{x^*}\left(y_n,y_n\right),$$

from the continuity of φ_{x^*} we have

$$\varphi'_{x^*}\left(y_n,y_n\right) = \varphi'_{x^*}\left(\frac{ax_n}{\|x_n\|},\frac{ax_n}{\|x_n\|}\right) \leq \frac{a}{\|x_n\|}\varphi'_{x^*}\left(\frac{ax_n}{\|x_n\|},x_n\right) \leq \frac{a}{\|x_n\|}\varphi'_{x^*}\left(x_n,x_n\right)$$

The last inequality follows from the convexity of φ'_{x^*} [29]. Hence

$$-\varphi'_{x^*}\left(y_n,-y_n\right) \leq \frac{a}{\|x_n\|}\varphi'_{x^*}\left(x_n,x_n\right)\ ,$$

it follows $\frac{a}{\|x_n\|}\varphi'_{x^*}\left(x_n,-x_n\right) \leq \varphi'_{x^*}\left(y_n,-y_n\right)$ and so

$$\varphi_{x^*}\left(0\right)-\varphi_{x^*}\left(y_n\right) \geq \frac{a}{\|x_n\|}\varphi'_{x^*}\left(x_n,-x_n\right) \geq -a\varepsilon_n.$$

Sending n to$+\infty$ we obtain $\varphi_{x^*}\left(0\right)-\varphi_{x^*}\left(\bar{y}\right) \geq 0$ which contradicts to Tykhonov well-posedness.

References

[1] C. Baiocchi, A. Capelo, *Variational and Quasivariational Inequalities, Application to Free Boundary Problems*, J. Wiley and Sons, New York (1984).

[2] E. Bednarczuk, J.P. Penot, On the positions of the notions of well-posed minimization problems, *Bollettino U.M.I.*, **7** (1992).

[3] E. Bednarczuk, Well posedness of vector optimization problems, In: *Recent Advances and Historical Development of Vector Optimization Problems* (Ed-s: J. Jahn, W. Krabs), Lecture Notes in Economics and Mathematical System, **294**, Springer-Verlag, Berlin (1987).

[4] Frank A. Clarke, *Optimization and Nonsmooth Analysis*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia (1990).

[5] G. Crespi, I. Ginchev, M. Rocca, Increase-along-rays property for vector function, *Journal of Nonlinear and Convex Analysis*, **7**, No. 1 (2006), 39-50.

[6] G. Crespi, A. Guerraggio, M. Rocca, Well-posedness in vector optimization problem and vector variational inequalities, *Journal of Optimization Theory and Applications*, **131**, No. 2 (2006).

[7] G. Crespi, I. Ginchev, M. Rocca, Variational inequalities in vector optimization, In: *Variational Analysis and Application, Series, Nonconvex Optimization and its Application* (Ed-s: F. Giannessi, A. Maugeri), Part II, **79** (2005).

[8] V.F. Demyanov, A.M. Rubinov, Constructive nonsmooth analysis, *Approximation and Optimization*, **7**, Peter Lang, Frankfurt am Main (1995).

[9] D. Dentcheva, S. Helbig, On variational principles, level sets, well-posedness and ε-solutions in vector optimization, *Journal of Optimization Theory and Applications*, **89** (1996).

[10] A.L. Dontchev, T. Zolezzi, *Well-Posed Optimization Problems*, Lecture Notes in Mathematics, **1543**, Springer Verlag, Berlin (1993).

[11] I. Ekeland, R. Téman, *Convex Analysis and Variational Problems*, Translated from the French. Corrected reprint of the 1976 English edition. Classics in Applied Mathematics, 28. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA (1999).

[12] F. Giannessi, *Theorems of the Alternative Quadratic Programs and Complementary Problems*, Variational Inequalities and Complementarity Problem. Theory and Applications, Wiley, New York (1980).

[13] I. Ginchev, A. Hoffmann, Approximation of set-valued functions by single-valued one, *Discuss. Math. Differ. Incl. Control Optim.*, **22** (2002).

[14] I. Ginchev, A. Guerraggio, M. Rocca, First-order conditions for $C^{0,1}$ constrained vector optimization, In: *Variational Analysis and Application, Series, Nonconvex Optimization and its Application* (Ed-s: F. Giannessi, A. Maugeri), Parte II, **79** (2005).

[15] V.V. Gorohovik, *Convex and Nonmooth Optimization*, Nauka I Tèkhnika, **240**, Minsk (1990), In Russian.

[16] X.X. Huang, Pointwise well-posedness of perturbed vector optimization problems in a vector-valued variational principle, *Journal of Optimization Theory and Applications*, **108** (2001).

[17] D. Kinderlehrer, G. Stampacchia, An introduction to variational inequalities and their applications, *Pure and Applied Math.*, Academ. Press, **88** (1980).

[18] E.S. Levitin, B.T. Polya, Convergence of minimizing sequences in conditional extremum problems, *Soviet Math. Dokl.*, **7** (1996).

[19] G.M. Lee, *On Relations between Vector Variational Inequality and Vector Optimization Problem*, Progress in Optimization, Kluwer Academic Publishers, Dordrecht, Netherlands (2000).

[20] P. Loridan, Well-posedness in vector optimization, In: *Recent Developments in Well-Posed Variational Problems* (Ed-s: R. Lucchetti, J. Revolskyi, M.I.A. Serie), **331**, Kluwer Academic Publishers, Dordrecht (1995).

[21] R. Lucchetti, Well-posedness towards vector optimization, *Lecture Notes in Economics and Mathematical Systems*, **294**, Springer-Verlag, Berlin (1987).

[22] R. Lucchetti, F. Patrone, Hadamard and Tykhonov well-posedness of a certain class of convex function, *Journal of Mathematical Analysis and Applications*, **88** (1982).

[23] R. Lucchetti, F. Patrone, A characterization of Tykhonov well-posedness for minimum problems, with applications to variational inequalities, *Numerical Functional Analysis and Optimization*, **3** (1981).

[24] R. Lucchetti, F. Patrone, Some aspects of the connection between Hadamard and Tykhonov well-posedness of convex problems, *Bollettino U.M.I. sez. C*, **6** (1982).

[25] M. Pappalardo, Vector optimization, In: *Encyclopedia of Optimization* (Ed-s: P. Pardalos, C. Floudas) (2001), 1000-1006.

[26] J.P. Revalski, *Generic Well-Posedness in Some Classes of Optimization Problems*, Acta University Carolinae, Mathematica et Physica, **28** (1987).

[27] J.P. Revalski, *Well-Posedness of Optimization Problem, A Survey*, Functional Analysis and Approximation (Ed-s: P.L. Papini, Bagni di Lucca) (1988).

[28] J.P. Revalski, *Various Aspects of Well-Posedness of Optimization Problems*, Recent Developments in Well-Posed Variational Problems (Ed-s: R. Lucchetti, J. Revalski), Kluwer Academic Publishers, Netherlands (1995).

[29] R.T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton (1997).

[30] A. Zaffaroni, Degrees of efficiency and degrees of minimality, *S.I.A.M. Journal Control of Optimization*, **42** (2003).

[31] A.J. Zaslavski, Generic well-posedness of Optimal Control problems without convexity assumptions, *SIAM Journal on Control and Optimization*, **39** (2000).

[32] A.J. Zaslavski, Generic well-posedness of variational problems without convexity assumptions, *Journal of Mathematical Analysis and Application*, **279** (2003).

[33] T. Zolezzi, Well-posedness criteria in optimization with application to the calculus of variations, *Nonlinear Analysis, Theory, Methods and Applications*, **25** (1995).