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# PERIOD ANNULI IN THE LIÉNARD TYPE EQUATION

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**Abstract:** We consider the equation

$$x'' + \frac{2x}{1+x^2}x'^2 + g(x) = 0, \qquad (i)$$

where $g(x) = -x(x^2 - p^2)(x^2 - q^2)$. Comparison of phase portraits for equations $(i)$ and

$$x'' + g(x) = 0 \qquad (ii)$$

is made. We describe decomposition of the first quadrant of the $(q, p)$-plane into regions where equations $(i)$ and $(ii)$ have or have not a period annulus, that is, a set of concentric cycles enclosing several critical points of equivalent planar systems.

**AMS Subject Classification:** 34C05, 58F14
**Key Words:** period annulus, phase portrait, Liénard equation

## 1. Introduction

In this paper we consider the specific Liénard type equation of the form

$$x'' + f(x)x'^2 + g(x) = 0, \qquad (1)$$

where $g(x)$ is a polynomial. We are looking for the so called *period annuli*. Let

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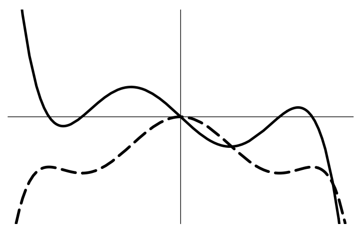

Figure 1: Functions $g(x)$ (solid) and the primitive $G(x)$ (dashed)

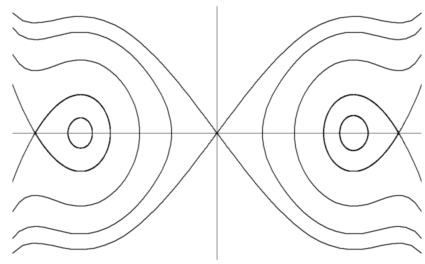

Figure 2: The phase plane

equation (1) be written as a planar system

$$x' = y, \quad y' = -f(x)y^2 - g(x). \tag{2}$$

Critical points of this system are points $(p_i, 0)$, where $p_i$ are zeros of $g(x)$. If all zeros of $g(x)$ are simple (in the meaning that $g'(p_i) \neq 0$) then only two types of critical points are possible, namely centers and saddle points.

Recall that a critical point O of (2) is a center if it has a punctured neighborhood covered with nontrivial cycles. Due to terminology in Sabatini [4], the largest connected region covered with cycles surrounding O is called *central region*. Every connected region covered with nontrivial concentric cycles is usually called *a period annulus*.

## 2. Conservative Equation

Consider equation

$$x'' + g(x) = 0, \tag{3}$$

where $g(x)$ is a polynomial of order 5 with five simple zeros. A sample of $g(x)$ is depicted in Figure 1 together with the primitive $G(x) = \int_0^x g(s)\, ds$. Zeros of $g(x)$ are $p_1 < p_2 < p_3 < p_4 < p_5$.

The equivalent system has three saddle points at $(p_1, 0)$, $(p_3, 0)$, $(p_5, 0)$ and centers at $(p_2, 0)$ and $(p_4, 0)$.

One of the typical phase portrait is given in Figure 2. There are two central regions filled with "small" amplitude periodic solutions located in neighborhoods of $(p_2, 0)$ and $(p_4, 0)$.

No other nontrivial periodic solutions exist if three local maxima of the

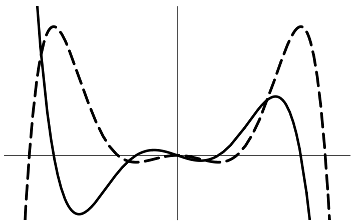

Figure 3: Functions $g(x)$ (solid) and the primitive $G(x)$ (dashed)

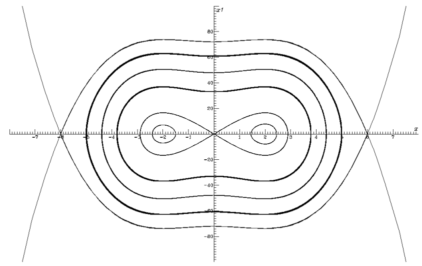

Figure 4: The phase plane for the case $G(p_3) < G(p_1) = G(p_5)$.

primitive $G(x)$ are such that either $G(p_1) > G(p_3) > G(p_5)$ or $G(p_1) < G(p_3) < G(p_5)$ or $G(p_5) < G(p_1) < G(p_3)$ or $G(p_1) < G(p_5) < G(p_3)$.

Situation is quite different if $G(p_3)$ is less (strictly) that $G(p_1)$ and $G(p_5)$. Then appear period annulus like in Figure 4.

**Theorem 1.** (see [2]) *If $G(p_3)$ is less (strictly) than $G(p_1)$ and $G(p_5)$, then equation (3) has "large"-amplitude periodic solutions (a period annulus), that is, solutions enclosing the critical points $(p_2; 0)$ and $(p_4; 0)$.*

**Remark.** If the inequalities $G(p_1) > G(p_3) > G(p_5)$ or $G(p_1) < G(p_3) < G(p_5)$ or $G(p_1) < G(p_3)$, $G(p_5) < G(p_3)$ hold then "large"-amplitude periodic solutions do not exist. This can be shown (see Figure 2, which corresponds to the latter case).

## 3. Reduction to a Conservative Equation

It was shown by Sabatini [3] that equation (1) can be reduced to the form $u'' + h(u) = 0$ by the following transformation. Let $F(x) = \int_0^x f(s)\,ds$ and $G(x) = \int_0^x g(s)\,ds$. Introduce

$$u := \Phi(x) = \int_0^x e^{F(s)}\,ds. \tag{4}$$

Since $\frac{du}{dx} > 0$, this is one-to-one transformation and the inverse $x = x(u)$ is well defined.

**Lemma 1.** (see [3], Lemma 1) *The function $x(t)$ is a solution to (1) if and only if $u(t) = \Phi(x(t))$ is a solution to*

$$u'' + g(x(u))e^{F(x(u))} = 0. \tag{5}$$

Denote $H(u) = \int_0^u g(x(s))e^{F(x(s))}\, ds$. The existence of periodic solutions depends entirely on properties of the primitive $H$.

Let us state some easy assertions (see [2], [1]) about equation (1), the equivalent system

$$x' = y, \quad y' = -f(x)y^2 - g(x) \tag{6}$$

and the system

$$x' = y, \quad y' = -g(x). \tag{7}$$

**Proposition 1.** *Critical points and their character are the same for systems (7) and (6).*

Consider a system

$$u' = v, \quad v' = -g(x(u))e^{F(x(u)}, \tag{8}$$

which is equivalent to equation (5).

**Proposition 2.** *Critical points $(x, 0)$ and $(u(x), 0)$ of systems (7) and (8) respectively are in 1-to-1 correspondence and their characters are the same.*

**Proposition 3.** *Periodic solutions $x(t)$ of equation (1) turn to periodic solutions $u(t) = \Phi(x(t))$ of equation (5) by transformation (4).*

**Proposition 4.** *Homoclinic solutions of equation (1) turn to homoclinic solutions of equation (5) by transformation (4).*

**Proposition 5.** *Let $p_i$ be a zero of $g(x)$. The equality*

$$g_x(p_i) = g_u(x(u))e^{F(x(u))}|_{u=p_i} \tag{9}$$

*is valid.*

*Proof.* By calculation of the derivative. □

## 4. Period Annuli

Consider

$$x'' + \frac{2x}{1+x^2}x'^2 - x(x^2 - p^2)(x^2 - q^2) = 0. \tag{10}$$

One has that

$$H(u) = \int_0^{x(u)} g(s)(1+s^2)^2\, ds, \qquad g(x) = -x(x^2 - p^2)(x^2 - q^2),$$

$$G(x) = \int_0^x g(s)ds = -\frac{1}{2}p^2q^2x^2 + \frac{1}{4}p^2x^4 + \frac{1}{4}q^2x^4 - \frac{1}{6}x^6.$$

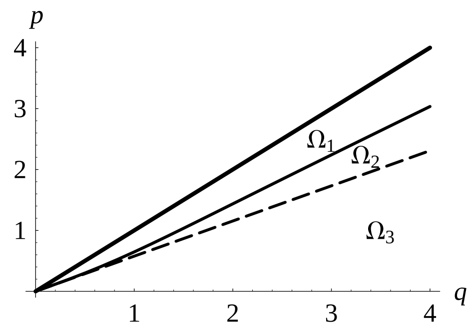


Figure 5: Regions of existence and nonexistence of a period annulus. Thick line is a bisectrix $p = q$. Dashed line is $p = \frac{1}{\sqrt{3}}q$.

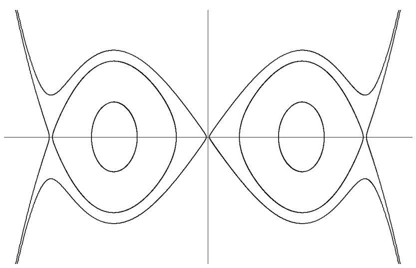

Figure 6: The phase plane (equation (11))

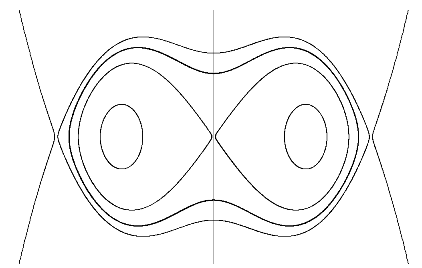

Figure 7: The phase plane (equation (10))

Since $G(x)$ is an even function we may consider only positive values of $x$. The primitive $G$ has local minimum at $x = p$ and local maximum at $x = q$. Since the existence of a period annulus in the equation

$$x'' - x(x^2 - p^2)(x^2 - q^2) = 0 \tag{11}$$

depends on the value $G(q)$ we compute it and got the result

$$G(q) = -\frac{1}{12}(3p^2q^4 - q^6).$$

If $G(q) > 0$ then equation (11) has a period annulus. We arrive thus to the following assertion.

**Proposition 7.** *Equation (11), where $0 < p < q$, has a period annulus, if $p < \frac{1}{\sqrt{3}}q$.*

One has that

$$F(x) = \int_0^x \frac{2s}{1+s^2}ds = \ln(1+x^2), \quad e^{F(x)} = 1 + x^2,$$

$$u = \int_0^x e^{F(s)}\,ds = \int_0^x (1+s^2)\,ds = x + \frac{1}{3}x^3.$$

Evidently the inverse function $x = x(u)$ exists. Equation (1) turns to

$$u'' + h(u) = 0, \tag{12}$$

where

$$h(u) = g(x(u))e^{F(x(u))} = -x(u)(x^2(u) - p^2)(x^2(u) - q^2)(1 + x^2(u)).$$

Standard computation gives

$$H(u) = \int_0^u h(s)\,ds = \int_0^{x(u)} g(\xi)e^{2F(\xi)}\,d\xi$$
$$= -\int_0^{x(u)} -\xi(\xi^2 - p^2)(\xi^2 - q^2)(1+\xi^2)^2\,d\xi.$$

The primitive $H(u)$ is a 10-th order polynomial with a unique positive point of minimum at $u(p)$ and a unique positive point of maximum at $u(q)$. Introduce

$$I(p,q,x) := \int_0^x -\xi(\xi^2 - p^2)(\xi^2 - q^2)(1+\xi^2)^2\,d\xi.$$

Integrating one gets that

$$H(u)|_{u(q)} = I(p,q,q) = \frac{1}{120}q^4(-30p^2 + 10q^2 - 20p^2q^2 + 10q^4 - 5p^2q^4 + 3q^6).$$

Therefore equations (12) and (10) have a period annulus if

$$-30p^2 + 10q^2 - 20p^2q^2 + 10q^4 - 5p^2q^4 + 3q^6 > 0, \quad 0 < p < q.$$

Equations (12) and (10) do not have period annuli if

$$-30p^2 + 10q^2 - 20p^2q^2 + 10q^4 - 5p^2q^4 + 3q^6 < 0, \quad 0 < p < q.$$

We arrive to the following result.

**Theorem 2.** *Equation (1), where $0 < p < q$, has a period annulus if*

$$p < q\sqrt{\frac{3q^4 + 10q^2 + 10}{5q^4 + 20q^2 + 30}}.$$

The line between solid and dashed lines shows the graph of

$$p = q\sqrt{\frac{3q^4 + 10q^2 + 10}{5q^4 + 20q^2 + 30}}.$$

The derivative of this function at $q = 0$ is $\frac{1}{\sqrt{3}}$ and at infinity it tends to the straight line $p = \sqrt{\frac{3}{5}}\,q$. Therefore we got three cases:

— $(q,p) \in \Omega_1$ – equations (11) and (10) do not have period annuli.

— $(q, p) \in \Omega_2$ – equation (11) does not have and equation (10) has a period annulus (see Figures 6 and 7).

— $(q, p) \in \Omega_3$ – equations (11) and (10) both have period annuli.

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