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GLOBAL STABILITY OF A CLASS OF RECURSIVE SEQUENCE

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Abstract: We investigate the global asymptotic stability, the invariant interval and the semicycle character of solutions of the equation

$$x_{n+1} = \frac{\alpha + \beta x_{n-k+1}}{Bx_{n-k+1} + Cx_{n-2k+1}}, \quad n = 0, 1, 2, \dots ,$$

for all admissible non-negative values of the parameters α, β, B, C and the initial conditions $x_{-2k+1}, \cdots, x_1, x_0$. The results generalize some known results in recent literatures.

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1. Introduction

Recently, there has been a great interest in studying properties of rational difference equations. Some basics on global asymptotic stability, the invariant interval and the semicycle character of solutions of second-order difference equations can be found in the classical book [6]. Those equations which model

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some situations in population biology and ecology are particular interest.

In [1], the authors studied the global asymptotic stability, the invariant interval and the semicycle character of solutions of the following equation

$$x_{n+1} = \frac{\alpha + \beta x_n}{Bx_n + Cx_{n-1}}, \tag{1}$$

where the initial conditions x_{-1}, x_0 are positive and the parameters α, β, B, C are also positive. They showed that under some conditions, the positive equilibrium point is globally asymptotically stable. Equation (1) is the particular case of (2,2)-type rational difference equation of the form

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}}, \quad n = 0, 1, 2, \ldots, \tag{2}$$

which was studied by Kulenović and Ladas in [6]. Here the parameters α, β, γ, A, B, C are non-negative real numbers and the initial conditions x_{-1}, x_0 are arbitrary non-negative real numbers such that

$$A + Bx_n + Cx_{n-1} > 0.$$

The work of this paper was motivated by some recent works (see [2, 3, 4, 5]). For example, the rational difference equation

$$x_{n+1} = \frac{\beta x_{n-k+1} + \gamma x_{n-2k+1}}{A + Bx_{n-k+1}}, \quad n = 0, 1, 2, \ldots ,$$

was studied in [3] and

$$x_{n+1} = \frac{\alpha + \gamma x_{n-2k+1}}{Bx_{n-k+1} + Cx_{n-2k+1}}, \quad n = 0, 1, 2, \ldots ,$$

was studied in [4].

The aim of this work is to consider the qualitative manner of positive solutions of the family of nonlinear difference equations

$$x_{n+1} = \frac{\alpha + \beta x_{n-k+1}}{Bx_{n-k+1} + Cx_{n-2k+1}}, \quad n = 0, 1, 2, \ldots , \tag{3}$$

where the initial conditions $x_{-2k+1}, \cdots, x_1, x_0$ are positive, $k \in \{1, 2, 3, \ldots\}$, and the parameters α, β, B, C are also positive. It is easy to see that equation (1) is the special case of equation (3). Using the change of variable

$$x_n = \frac{\beta}{B} y_n,$$

we reduce equation (3) to the easy-working difference equation

$$y_{n+1} = \frac{y_{n-k+1} + p}{y_{n-k+1} + qy_{n-2k+1}}, \quad n = 0, 1, 2, \ldots, \tag{4}$$

where

$$p = \frac{\alpha B}{\beta^2} \quad \text{and} \quad q = \frac{C}{B}.$$

When $k = 1$, the equation (3) is

$$y_{n+1} = \frac{y_n + p}{y_n + q y_{n-1}}, \quad n = 0, 1, 2, \dots, \tag{5}$$

which was studied in [1].

In this paper we study the global asymptotic stability, the invariant interval and the semicycle character of solutions of equation (4). Among other results in this paper, we prove that the equilibrium point of equation (4) is globally asymptotically stable when $q \le 4p + 1$. The results that we obtained in this paper generalize some results of [1, 6].

Let I be some interval of real numbers and let $f \in C^1[I \times I, I]$. Let $\overline{x} \in I$ be an equilibrium point of the difference equations

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \dots, \tag{6}$$

that is, $\overline{x} = f(\overline{x}, \overline{x})$. Let

$$r = \frac{\partial f}{\partial u}(\overline{x}, \overline{x}) \quad \text{and} \quad s = \frac{\partial f}{\partial v}(\overline{x}, \overline{x})$$

denote the partial derivatives of $f(u, v)$ evaluated at an equilibrium $\overline{x}$ of equation (6). Then the equation

$$y_{n+1} = r y_n + s y_{n-1}, \quad n = 0, 1, \dots,$$

is called the linearized equation associated with equation (6) about the equilibrium point $\overline{x}$.

An invariant interval for equation (4) is an interval I with the property that if $2k$ consecutive terms of the solution fall in I, then all subsequent terms of the solution also belong to I. In other words, I is an invariant interval for equation (4) in the sense that if $y_N, y_{N+1}, \cdots, y_{N+2k-1} \in I$ for some $N \ge 0$, then $y_n \in I$ for every $n > N + 2k - 1$.

Similarly, for a subsequence $\{y_{n_m}\}_{m=-2}^{\infty}$ of a solution $\{y_n\}_{n=-2k+1}^{\infty}$, an invariant interval is an interval I with the property that if two consecutive terms of $\{y_{n_m}\}_{m=-2}^{\infty}$ fall in I then all subsequent terms also belong to I. So I is an invariant interval in the sense that if $y_{n_M}, y_{n_{M+1}} \in I$ for some $M \ge 0$, then $y_{n_m} \in I$ for every $m > M$.

2. Local Asymptotic Stability and Invariant Interval

In this section, we will study the local asymptotic stability of the higher order rational difference equation (4). Moreover we will verify that every solution of equation (4) will be trapped in some intervals. The following results are main results in this section.

Theorem 2.1. *The unique equilibrium point*

$$\bar{y} = \frac{1 + \sqrt{1 + 4p(1 + q)}}{2(1 + q)}$$

of equation (4) is locally asymptotically stable for all values of the parameters p and q.

Theorem 2.2. *Let $\{y_{mk+i+1}\}_{m=-2}^{\infty}$ be the subsequences of a solution $\{y_n\}_{n=-2k+1}^{\infty}$ of equation (4) for all $i \in \{0, 1, \ldots, k-1\}$. Then the following statements are true:*

(i) Suppose $p < q$, assume that for some $M \geq 0$ and some $i_0 \in \{0, 1, \ldots, k-1\}$,

$$y_{(M-1)k+i_0+1}, \quad y_{Mk+i_0+1} \in \big[\frac{p}{q}, 1\big].$$

Then

$$y_{mk+i_0+1} \in \big[\frac{p}{q}, 1\big] \quad \text{for all} \quad m > M.$$

(ii) Suppose $p < q$ and assume that for some $N \geq 0$,

$$y_{N-k+1}, \; y_{N-k+2}, \; \cdots, \; y_{N+k-1} \in \big[\frac{p}{q}, 1\big].$$

Then

$$y_n \in \big[\frac{p}{q}, 1\big] \quad \text{for all} \quad n > N.$$

(iii) Suppose $p > q$, assume that for some $M \geq 0$ and some $i_0 \in \{0, 1, \ldots, k-1\}$,

$$y_{(M-1)k+i_0+1}, \quad y_{Mk+i_0+1} \in \big[1, \frac{p}{q}\big].$$

Then

$$y_{mk+i_0+1} \in \big[1, \frac{p}{q}\big] \quad \text{for all} \quad m > M.$$

(iv) Suppose $p > q$, assume that for some $N \geq 0$,

$$y_{N-k+1}, \; y_{N-k+2}, \; \cdots, \; y_{N+k-1} \in \big[1, \frac{p}{q}\big].$$

Then

$$y_n \in \big[1, \frac{p}{q}\big] \quad \textit{for all} \quad n > N.$$

Remark 1. In [1], the authors proved that the equilibrium point $\bar{y} = \frac{1+\sqrt{1+4p(1+q)}}{2(1+q)}$ of equation (5) is locally asymptotically stable for all values of the parameters p and q. When $k = 1$, Theorem 2.2 was first obtained also in [1].

Lemma 2.1. (see [1] or [6]) *Let the equation*

$$y_{n+1} = ry_n + sy_{n-1}, \quad n = 0, 1, \ldots,$$

be the linearized equation associated with equation (6) about the equilibrium point $\overline{x}$. If both roots of the quadratic equation

$$\lambda^2 - r\lambda - s = 0$$

lie in the open unit disk $|\lambda| < 1$, then the equilibrium $\overline{x}$ of equation (6) is locally asymptotically stable.

Proof of Theorem 2.1. It is easy to see that the only equilibrium point of equation (4) is

$$\bar{y} = \frac{1 + \sqrt{1 + 4p(1 + q)}}{2(1 + q)}. \tag{7}$$

Set

$$f(x, y) = \frac{x + p}{x + qy}.$$

Observing that

$$\frac{\partial f(\overline{y}, \overline{y})}{\partial x} = \frac{q\bar{y} - p}{(\bar{y} + p)(1 + q)}$$

and

$$\frac{\partial f(\overline{y}, \overline{y})}{\partial y} = -\frac{q}{1 + q},$$

we see that the linearized equation associated with equation (4) about the positive equilibrium point $\overline{y}$ is

$$z_{n+1} - \frac{\partial f(\overline{y}, \overline{y})}{\partial x} z_{n-k+1} - \frac{\partial f(\overline{y}, \overline{y})}{\partial y} z_{n-2k+1}$$
$$= z_{n+1} - \frac{q\bar{y} - p}{(\bar{y} + p)(1 + q)} z_{n-k+1} + \frac{q}{1 + q} z_{n-2k+1} = 0.$$

Hence the characteristic equation of the above equation is

$$\lambda^{2k} - \frac{q\bar{y} - p}{(\bar{y} + p)(1 + q)} \lambda^k + \frac{q}{1 + q} = 0. \tag{8}$$

Substituting ρ for λ^k in equation (8), this gives

$$\rho^2 - \frac{q\bar{y} - p}{(\bar{y} + p)(1 + q)}\rho + \frac{q}{1 + q} = 0. \tag{9}$$

After some computation, we see that all roots of equation (9) and hence all roots of equation (8) lie inside the open unit disk $|\lambda| < 1$. Applying Lemma 2.1, we see that the equilibrium point $\overline{y}$ is locally asymptotically stable. The proof of the theorem is finished. □

Proof of Theorem 2.2. Let $\{y_{mk+i+1}\}_{m=-2}^{\infty}$ be the subsequences of a solution $\{y_n\}_{n=-2k+1}^{\infty}$ of equation (4) for all $i \in \{0, 1, \ldots, k-1\}$.

(i) For all $i \in \{0, 1, \ldots, k-1\}$, define

$$y_m^i = y_{mk+i+1}, \quad m = 0, 1, 2, \ldots .$$

Then, by equations (4) and (7) and for all $i \in \{0, 1, \ldots, k-1\}$, we obtain the following equation:

$$y_{m+1}^i = f(y_m^i, y_{m-1}^i) = \frac{p + y_m^i}{y_m^i + q y_{m-1}^i}. \tag{10}$$

It is worth noting that we can consider equation (10) as a second order rational difference equation which was investigated by some authors (see [1] and [6]). Then the result follows by the result of [1].

(ii) By using (i) and a straightforward calculation, we get the required result.

(iii) The proof is similar to (i) and will be omitted.

(iv) The proof follows in exactly the same way as (ii) and (iii) of the theorem. The proof of the theorem is finished. □

3. Semicycle Analysis

The goal of this section is to interpret analytically the attitude of semicycles of solutions of equation (4) relative to the equilibrium point $\overline{y}$. We first present the definitions of semicyle and oscillation.

Definition 3.1. (see [3]) Let $\{y_n\}_{n=-2k+1}^{\infty}$ be a positive solution of the equation

$$x_{n+1} = f(x_{n-k+1}, x_{n-2k+1}), \quad n = 0, 1, 2, \ldots . \tag{11}$$

A positive semicycle of the solution $\{y_n\}_{n=-2k+1}^{\infty}$ of equation (11) consists of a "string" of terms $\{y_l, y_{l+1}, \cdots, y_m\}$, all greater than or equal to the equilibrium point, with $l \geq -2k+1$ and $m \leq \infty$ such that

$$\text{either} \;\; l = -2k+1, \;\text{ or }\; l > -2k+1 \;\text{ and }\; y_{l-1} < \overline{y},$$

and

$$\text{either}\ \ m=\infty,\ \ \text{or}\ \ m<\infty\ \ \text{and}\ \ y_{m+1}<\overline{y}.$$

Definition 3.2. (see [3]) Let $\{y_{n_p}\}_{p=-2}^{\infty}$ be a subsequence of a positive solution $\{y_n\}_{n=-2k+1}^{\infty}$ of equation (11). A positive semicycle of the subsequence $\{y_{n_p}\}_{p=-2}^{\infty}$ for equation (11) consists of a "string" of terms $\{y_{n_l}, y_{n_{l+1}}, \cdots, y_{n_m}\}$, all greater than or equal to the equilibrium point, with $l \geq -2$ and $m \leq \infty$ such that

$$\text{either}\ \ l=-2,\ \ \text{or}\ \ l>-2\ \ \text{and}\ \ y_{l-1}<\overline{y},$$

and

$$\text{either}\ \ m=\infty,\ \ \text{or}\ \ m<\infty\ \ \text{and}\ \ y_{m+1}<\overline{y}.$$

Definition 3.3. (see [3]) Let $\{y_n\}_{n=-2k+1}^{\infty}$ be a positive solution of equation (11). A negative semicycle of the solution $\{y_n\}_{n=-2k+1}^{\infty}$ of equation (11) consists of a "string" of terms $\{y_l, y_{l+1}, \cdots, y_m\}$, all less than the equilibrium point, with $l \geq -2k+1$ and $m \leq \infty$ such that

$$\text{either}\ \ l=-2k+1,\ \ \text{or}\ \ l>-2k+1\ \ \text{and}\ \ y_{l-1}\geq\overline{y},$$

and

$$\text{either}\ \ m=\infty,\ \ \text{or}\ \ m<\infty\ \ \text{and}\ \ y_{m+1}\geq\overline{y}.$$

Definition 3.4. (see [3]) Let $\{y_{n_p}\}_{n=-2}^{\infty}$ be a subsequence of $\{y_n\}_{n=-2k+1}^{\infty}$ a positive solution of equation (11). A negative semicycle of the subsequence $\{y_{n_p}\}_{n=-2}^{\infty}$ for equation (11) consists of a "string" of terms $\{y_{n_l}, y_{n_{l+1}}, \cdots, y_{n_m}\}$, all less than the equilibrium point, with $l \geq -2$ and $m \geq \infty$ such that

$$\text{either}\ \ l=-2,\ \ \text{or}\ \ l>-2\ \ \text{and}\ \ y_{l-1}\geq\overline{y},$$

and

$$\text{either}\ \ m=\infty,\ \ \text{or}\ \ m<\infty\ \ \text{and}\ \ y_{m+1}\geq\overline{y}.$$

Definition 3.5. (see [3]) A solution $\{y_n\}_{n=-2k+1}^{\infty}$ of equation (11) is called non-oscillatory about $\bar{y}$, if there exists $N \geq -2k+1$ such that either

$$y_n>\overline{y},\quad \text{for all}\ \ n\geq N,$$

or

$$y_n<\overline{y},\quad \text{for all}\ \ n\geq N.$$

Note that $\{y_n\}_{n=-2k+1}^{\infty}$ is called oscillatory about $\overline{y}$, if it is not non-oscillatory.

Now we state the main results of this section as follows.

Theorem 3.1. *Assume that $p > q$ holds. Then every nontrival and oscillatory solution of equation (4) which lies in the interval $[1, \frac{p}{q}]$, oscillates about $\bar{y}$ with semicycles of length at most $2k$.*

Theorem 3.2. *Suppose that $p = q$ holds, let $\{y_n\}_{n=-2k+1}^{\infty}$ be a solution of equation (4). Then $\{y_n\}_{n=-2k+1}^{\infty}$ oscillates about the equilibrium point 1 and except possibly for the first semicycle, the following are true:*

(i) If $y_{-2k+1} = \cdots = y_k = -1$ or $y_{-k+1} = \cdots = y_0 = 1$, then the positive semicycle has length $3k$ and the negative has length k.

(ii) If $(1 - y_i)(1 - y_j) \neq 0$, where $i = -2k+1, \cdots, -k$, $j = -k+1, \cdots, 0$, then every semicycle has length $2k$.

Theorem 3.3. *Assume that $p < q$ holds. Then every nontrivial and oscillatory solution of equation (4) which lies in the interval $[\frac{p}{q}, 1]$, after the first semicycle, oscillates with semicycles of length at least $2k$.*

Remark 2. When $k = 1$, Theorems 3.1-3.3 were obtained in [1]. Hence the above results generalized the results in [1].

To prove the above results, we need the following results.

Lemma 3.1. (see [4]) *Assume that $f \in C[(0, \infty) \times (0, \infty), (0, \infty)]$ and $f(x, y)$ is decreasing in both arguments. Let $\overline{x}$ be a positive equilibrium of equation (11). Then every oscillatory solution of equation (11) has semicycles of length at most $2k$.*

Lemma 3.2. (see [4]) *Assume that $f \in C[(0, \infty) \times (0, \infty), (0, \infty)]$ is such that $f(x, y)$ is increasing in x for each fixed y and $f(x, y)$ is decreasing in y for each fixed x. Let $\overline{x}$ be a positive equilibrium of equation (11). Then, except possibly for the first semicycle, every oscillatory solution of equation (11) has semicycles of length at least $2k$.*

Lemma 3.3. *Let $\{y_n\}_{n=-2k+1}^{\infty}$ be a solution of equation (4). Assume that $p > q$ and $\{y_{mk+i+1}\}_{m=-2}^{\infty}$ are the subsequences of the solution $\{y_n\}_{n=-2k+1}^{\infty}$. Then the following statements are true:*

(i) If for some $M \geq 0$, $y_{(M-1)k+i+1} < \frac{p}{q}$, then $y_{(M+1)k+i+1} > 1$;

(ii) If for some $N \geq 0$, $y_{N-k}, \cdots, y_{N-2}, y_{N-1} < \frac{p}{q}$, then

$$y_{N+k},\ \cdots,\ y_{N+2k-2},\ y_{N+2k-1} > 1;$$

(iii) If for some $M \geq 0$, $y_{(M-1)k+i+1} = \frac{p}{q}$, then $y_{(M+1)k+i+1} = 1$;

(iv) If for some $N \geq 0$, $y_{N-k} = \cdots = y_{N-2} = y_{N-1} = \frac{p}{q}$, then

$$y_{N+k} = \cdots = y_{N+2k-2} = y_{N+2k-1} = 1;$$

(v) If for some $M \geq 0,\ y_{(M-1)k+i+1} > \frac{p}{q}$, *then* $y_{(M+1)k+i+1} < 1$;

(vi) If for some $N \geq 0,\ y_{N-k},\ \cdots,\ y_{N-2},\ y_{N-1} > \frac{p}{q}$, *then*

$$y_{N+k}, \cdots, y_{N+2k-2}, y_{N+2k-1} < 1;$$

(vii) If for some $M \geq 0,\ y_{(M-1)k+i+1} \geq 1$, *then* $y_{(M+1)k+i+1} < \frac{p}{q}$;

(viii) If for some $N \geq 0,\ y_{N-k},\ \cdots,\ y_{N-2},\ y_{N-1} \geq 1$, *then*

$$y_{N+k},\ \cdots,\ y_{N+2k-2},\ y_{N+2k-1} < \frac{p}{q};$$

(ix) If for some $M \geq 0,\ y_{(M-1)k+i+1} \leq 1$, *then* $y_{(M+1)k+i+1} > 1$;

(x) If for some $N \geq 0,\ y_{N-k}, \cdots,\ y_{N-2},\ y_{N-1} < 1$, *then*

$$y_{N+k},\ \cdots,\ y_{N+2k-2},\ y_{N+2k-1} > 1;$$

(xi) If for some $M \geq 0,\ y_{Mk+i+1} \geq \frac{p}{q}$, *then* $y_{(M+4)k+i+1} < y_{Mk+i+1}$;

(xii) If for some $N \geq 0,\ y_N,\ \cdots,\ y_{N+k-2},\ y_{N+k-1} \geq \frac{p}{q}$, *then*

$$y_{N+4k} < y_N,\ \cdots,\ y_{N+5k-2} < y_{N+k-2},\ y_{N+5k-1} < y_{N+k-1};$$

(xiii) If for some $M \geq 0,\ y_{Mk+i+1} \leq 1$, *then* $y_{(M+4)k+i+1} > y_{Mk+i+1}$;

(xiv) If for some $N \geq 0,\ y_{N-k},\ \cdots,\ y_{N-2},\ y_{N-1} \leq 1$, *then*

$$y_{N+4k} > y_N,\ \cdots,\ y_{N+5k-2} > y_{N+k-2},\ y_{N+5k-1} > y_{N+k-1};$$

(xv) If for some $M \geq 0,\ 1 \leq y_{(M-1)k+i+1} \leq \frac{p}{q}$, *then* $1 \leq y_{(M+1)k+i+1} \leq \frac{p}{q}$;

(xvi) If for some $N \geq 0,\ 1 \leq y_{N-k},\ \cdots,\ y_{N-2},\ y_{N-1} \leq \frac{p}{q}$, *then*

$$1 \leq y_{N+k},\ \cdots,\ y_{N+2k-2},\ y_{N+2k-1} \leq \frac{p}{q};$$

(xvii) If for some $M \geq 0,\ 1 \leq y_{(M-1)k+i+1},\ y_{Mk+i+1} \leq \frac{p}{q}$, *then* $y_{mk+i+1} \in \left[1, \frac{p}{q}\right]$, *for* $m > M$;

(xviii) If for some $N \geq 0,\ 1 \leq y_{N-k},\ \ldots,\ y_{N+k-1} \leq \frac{p}{q}$, *then*

$$y_n \in \left[1, \frac{p}{q}\right], \quad \text{for } n > N,$$

that is, $\left[1, \frac{p}{q}\right]$ *is an invariant interval for equation (4);*

(xviv) $1 < \overline{y} < \frac{p}{q}$.

Proof. Let $\{y_n\}_{n=-2k+1}^{\infty}$ be a solution of equation (4). Then a straightforward calculation gives the following identities:

$$y_{n+1} - 1 = q\frac{\frac{p}{q} - y_{n-2k+1}}{y_{n-k+1} + qy_{n-2k+1}}, \tag{12}$$

$$y_{n+1} - \frac{p}{q} = \frac{(q-p)y_{n-k+1} + pq(1 - y_{n-2k+1})}{q(y_{n-k+1} + qy_{n-2k+1})}, \tag{13}$$

and

$$y_{n-k+1} - y_{n+3k+1} = \frac{qy_{n+1}(y_{n-k+1} - \frac{p}{q})}{q(p + y_{n+1}) + y_{n+2k+1}(y_{n+1} + qy_{n-k+1})} + \frac{(y_{n-k+1} - 1)(qy_{n-k+1}y_{n+2k+1} + y_{n+1}y_{n+2k+1})}{q(p + y_{n+1}) + y_{n+2k+1}(y_{n+1} + qy_{n-k+1})}. \tag{14}$$

Assume that $\{y_{mk+i+1}\}_{m=-2}^{\infty}$ are the subsequences of $\{y_n\}_{n=-2k+1}^{\infty}$, for all $i \in \{0, 1, \ldots, k-1\}$. Using (12)-(14) for subsequences $\{y_{mk+i+1}\}_{m=-2}^{\infty}$, we can write

$$y_{(m+1)k+i+1} - 1 = q\frac{\frac{p}{q} - y_{(m-1)k+i+1}}{y_{mk+i+1} + qy_{(m-1)k+i+1}}, \tag{15}$$

$$y_{(m+1)k+i+1} - \frac{p}{q} = \frac{(q-p)y_{mk+i+1} + pq(1 - y_{(m-1)k+i+1})}{q(y_{mk+i+1} + qy_{(m-1)k+i+1})}, \tag{16}$$

and

$$\begin{aligned} &y_{mk+i+1} - y_{(m+4)k+i+1} \\ &= \frac{qy_{(m+1)k+i+1}(y_{mk+i+1} - \frac{p}{q})}{q(p + y_{(m+1)k+i+1}) + y_{(m+3)k+i+1}(y_{(m+1)k+i+1} + qy_{mk+i+1})} \\ &+ \frac{(y_{mk+i+1} - 1)(qy_{mk+i+1}y_{(m+3)k+i+1} + y_{(m+1)k+i+1}y_{(m+3)k+i+1})}{q(p + y_{(m+1)k+i+1}) + y_{(m+3)k+i+1}(y_{(m+1)k+i+1} + qy_{mk+i+1})}. \end{aligned} \tag{17}$$

By (10), we get the same identities for all $m \geq 0$ as in (15)-(17)

$$y_{m+1}^i - 1 = q\frac{\frac{p}{q} - y_{m-1}^i}{y_m^i + qy_{m-1}^i}, \tag{18}$$

$$y_{m+1}^i - \frac{p}{q} = \frac{(q-p)y_m^i + pq(1 - y_{m-1}^i)}{q(y_m^i + qy_{m-1}^i)}, \tag{19}$$

and

$$y_m^i - y_{m+4}^i = \frac{qy_{m+1}^i(y_m^i - \frac{p}{q}) + (y_m^i - 1)[qy_m^i y_{m+3}^i + y_{m+1}^i y_{m+3}^i]}{q(p + y_{m+1}^i) + y_{m+3}^i(y_{m+1}^i + qy_m^i)}. \tag{20}$$

Since (18)-(20) can be identities as second order rational difference equations, the result follows by similar methods of the proof of Lemma 6.4.2 of [6] or [1]. The proof of the lemma is finished. □

Proof of Theorem 3.1. The result follows from Lemma 3.1 and Lemma 3.3. The proof of the theorem is finished. □

Lemma 3.4. *Let* $\{y_n\}_{n=-2k+1}^{\infty}$ *be a solution of equation (4). Suppose* *that* $p = q$ *and* $\{y_{mk+i+1}\}_{m=-2}^{\infty}$ *are the subsequence of* $\{y_n\}_{n=-2k+1}^{\infty}$. *Then the following statements are true:*

(i) If for some $M \geq 0$, $y_{(M-1)k+i+1} < 1$, *then* $y_{(M+1)k+i+1} > 1$;

(ii) If for some $N \geq 0$, $y_{N-k}, \ \cdots, \ y_{N-2}, \ y_{N-1} < 1$, *then*

$$y_{N+k}, \ \cdots, \ y_{N+2k-2}, \ y_{N+2k-1} > 1;$$

(iii) If for some $M \geq 0$, $y_{(M-1)k+i+1} = 1$, *then* $y_{(M+1)k+i+1} = 1$;

(iv) If for some $N \geq 0$, $y_{N-k} = \cdots = y_{N-2} = y_{N-1} = 1$, *then*

$$y_{N+k} = \cdots = y_{N+2k-2} = y_{N+2k-1} = 1;$$

(v) If for some $M \geq 0$, $y_{(M-1)k+i+1} > 1$, *then* $y_{(M+1)k+i+1} < 1$;

(vi) If for some $N \geq 0$, $y_{N-k}, \ \cdots, \ y_{N-2}, \ y_{N-1} > 1$, *then*

$$y_{N+k}, \ \cdots, \ y_{N+2k-2}, \ y_{N+2k-1} < 1;$$

(vii) If for some $M \geq 0$, $y_{Mk+i+1} < 1$, *then* $y_{Mk+i+1} < y_{(M+4)k+i+1} < 1$;

(viii) If for some $N \geq 0$, $y_N, \ \cdots, \ y_{N+k-2}, \ y_{N+k-1} < 1$, *then*

$$y_N < y_{N+4k} < 1, \ \cdots, \ y_{N+k-2} < y_{N+5k-2} < 1, \ y_{N+k-1} < y_{N+5k-1} < 1;$$

(ix) If for some $M \geq 0$, $y_{Mk+i+1} > 1$, *then* $y_{Mk+i+1} > y_{(M+4)k+i+1} > 1$;

(x) If for some $N \geq 0$, $y_N, \ \cdots, \ y_{N+k-2}, \ y_{N+k-1} > 1$, *then*

$$y_N > y_{N+4k} > 1, \ \cdots, \ y_{N+k-2} > y_{N+5k-2} > 1, \ y_{N+k-1} > y_{N+5k-1} > 1.$$

Proof. In this case equation (4) reduces to

$$y_{n+1} = \frac{y_{n-k+1} + p}{y_{n-k+1} + p y_{n-2k+1}}, \tag{21}$$

with the unique equilibrium point $\overline{y} = 1$. Also (12)-(14) reduce to

$$y_{n+1} - 1 = p \frac{1 - y_{n-2k+1}}{y_{n-k+1} + p y_{n-2k+1}} \quad \text{for all} \quad n \geq 0 \tag{22}$$

and

$$y_{n-k+1} - y_{n+3k+1} = (y_{n-k+1} - 1) \frac{p y_{n+1} + p y_{n-k+1} y_{n+2k+1} + y_{n+1} y_{n+2k+1}}{p(p + y_{n+1}) + y_{n+2k+1}(y_{n+1} + p y_{n-k+1})} \quad \text{for all} \quad n \geq 0. \tag{23}$$

Therefore, using equations (21) and (10), we get

$$y^i_{m+1} = \frac{y^i_m + p}{y^i_m + p y^i_{m-1}}, \quad m = 0, 1, 2, \ldots \tag{24}$$

From equation (24), we see that identities (18)-(20) can be reduced to

$$y^i_{m+1} - 1 = p\frac{1 - y^i_{m-1}}{y^i_m + py^i_{m-1}} \quad \text{for all} \quad m \geq 0, \tag{25}$$

and

$$y^i_m - y^i_{m+4} = (y^i_m - 1)\frac{py^i_{m+1} + py^i_m y^i_{m+3} + y^i_{m+1}y^i_{m+3}}{p(p + y^i_{m+1}) + y^i_{m+3}(y^i_{m+1} + py^i_m)}. \tag{26}$$

Then the results follow immediately from [6] or [1]. The proof of the lemma is finished. □

Proof of Theorem 3.2. The result immediately follows from Lemma 3.4. The proof of the theorem is finished. □

Lemma 3.5. *Let $\{y_n\}_{n=-2k+1}^{\infty}$ be a solution of equation (4). Assume that $p < q$ and $\{y_{mk+i+1}\}_{m=-2}^{\infty}$ are the subsequences of $\{y_n\}_{n=-2k+1}^{\infty}$. Then the following statements are true:*

(i) If for some $M \geq 0$, $y_{(M-1)k+i+1} < \frac{p}{q}$, then $y_{(M+1)k+i+1} > 1$;

(ii) If for some $N \geq 0$, $y_{N-k}, \cdots, y_{N-2}, y_{N-1} < \frac{p}{q}$, then

$$y_{N+k}, \ \cdots, \ y_{N+2k-2}, \ y_{N+2k-1} > 1;$$

(iii) If for some $M \geq 0$, $y_{(M-1)k+i+1} = \frac{p}{q}$, then $y_{(M+1)k+i+1} = 1$;

(iv) If for some $N \geq 0$, $y_{N-k} = \cdots = y_{N-2} = y_{N-1} = \frac{p}{q}$, then

$$y_{N+k} = \cdots = y_{N+2k-2} = y_{N+2k-1} = 1;$$

(v) If for some $M \geq 0$, $y_{(M-1)k+i+1} > \frac{p}{q}$, then $y_{(M+1)k+i+1} < 1$;

(vi) If for some $N \geq 0$, $y_{N-k}, \ \cdots, \ y_{N-2}, \ y_{N-1} > \frac{p}{q}$, then

$$y_{N+k}, \ \cdots, \ y_{N+2k-2}, \ y_{N+2k-1} < 1;$$

(vii) If for some $M \geq 0$, $y_{(M-1)k+i+1} \leq 1$, then $y_{(M+1)k+i+1} > \frac{p}{q}$;

(viii) If for some $N \geq 0$, $y_{N-k}, \ \cdots, \ y_{N-2}, \ y_{N-1} \leq 1$, then

$$y_{N+k}, \ \cdots, \ y_{N+2k-2}, \ y_{N+2k-1} > \frac{p}{q};$$

(ix) If for some $M \geq 0$, $y_{(M-1)k+i+1} \leq \frac{p}{q}$, then $y_{(M+1)k+i+1} > \frac{p}{q}$;

(x) If for some $N \geq 0$, $y_{N-k}, \ \cdots, \ y_{N-2}, \ y_{N-1} \leq \frac{p}{q}$, then

$$y_{N+k}, \ \cdots, \ y_{N+2k-2}, \ y_{N+2k-1} > \frac{p}{q};$$

(xi) If for some $M \geq 0$, $y_{Mk+i+1} \geq 1$, then $y_{(M+4)k+i+1} < y_{Mk+i+1}$;

(xii) If for some $N \geq 0$, $y_N, \ \cdots, \ y_{N+k-2}, \ y_{N+k-1} \geq 1$, then

$$y_{N+4k} < y_N, \ \cdots, \ y_{N+5k-2} < y_{N+k-2}, \ y_{N+5k-1} < y_{N+k-1};$$

(xiii) If for some $M \geq 0$, $y_{Mk+i+1} \leq \frac{p}{q}$, *then* $y_{(M+4)k+i+1} > y_{Mk+i+1}$;

(xiv) If for some $N \geq 0$, $y_N, \ \dots, \ y_{N+k-2}, \ y_{N+k-1} \leq \frac{p}{q}$, *then*

$$y_{N+4k} > y_N, \ \cdots, \ y_{N+5k-2} > y_{N+k-2}, \ y_{N+5k-1} > y_{N+k-1};$$

(xv) If for some $M \geq 0$, $\frac{p}{q} \leq y_{(M-1)k+i+1} \leq 1$, *then* $\frac{p}{q} \leq y_{(M+1)k+i+1} \leq 1$;

(xvi) If for some $N \geq 0$, $\frac{p}{q} \leq y_{N-k}, \ \dots, \ y_{N-2}, \ y_{N-1} \leq 1$, *then*

$$\frac{p}{q} \leq y_{N+k}, \ \cdots, \ y_{N+2k-1} \leq 1;$$

(xvii) If for some $M \geq 0$, $\frac{p}{q} \leq y_{(M-1)k+i+1}, y_{Mk+i+1} \leq 1$, *then* $y_{mk+i+1} \in \left[\frac{p}{q}, 1\right]$, *for* $m > M$;

(xviii) If for some $N \geq 0$, $\frac{p}{q} \leq y_{N-k}, \ \cdots, \ y_{N+k-1} \leq 1$, *then*

$$y_n \in \left[\frac{p}{q}, 1\right], \quad \text{for} \ \ n > N.$$

In other word, $\left[\frac{p}{q}, 1\right]$ *is an invariant interval for equation (4);*

(xviv) $\frac{p}{q} < \overline{y} < 1$.

Proof. The proof is similar to [1] or Lemma 6.4.4 of [6], since the equations (18)-(20) can be assumed as second order recursive sequences. The proof of the lemma is finished. □

Proof of Theorem 3.3. The result follows from Lemma 3.2 and Lemma 3.5. The proof of the theorem is finished. □

4. Global Asymptotic Stability

In this section, we discuss the global asymptotic stability behavior of solutions of equation (4). We state the main result in this section as follows.

Theorem 4.1. *The equilibrium* $\bar{y}$ *of equation (4) is globally asymptotically stable when* $q \leq 4p + 1$.

Remark 3. In [1], the author proved that the equilibrium $\bar{y}$ of equation (5) is globally asymptotically stable when $q \leq 4p + 1$. Hence our result generalized the result in [1].

To prove the above result, we need the following results.

Lemma 4.1. (see [6]) *Let* $[a, b]$ *be an interval of real numbers and assume that*

$$f : [a, b] \times [a, b] \to [a, b]$$

is a continuous function satisfying the following properties:

(i) $f(x, y)$ *is non-increasing in each of its arguments.*

(ii) If $(m, M) \in [a, b] \times [a, b]$ *is a solution of the system*

$$f(m, m) = M \quad \text{and} \quad f(M, M) = m,$$

then $m = M$.

Then equation (11) has a unique equilibrium $\overline{x} \in [a, b]$ *and every solution of it converges to* $\overline{x}$.

Lemma 4.2. (see [6]) *Let* $[a, b]$ *be an interval of real numbers and assume that*

$$f : [a, b] \times [a, b] \to [a, b]$$

is a continuous function satisfying the following properties:

(i) $f(x, y)$ *is nondecreasing in* $x \in [a, b]$ *for each* $y \in [a, b]$*, and* $f(x, y)$ *is nonincreasing in* $y \in [a, b]$ *for each* $x \in [a, b]$.

(ii) If $(m, M) \in [a, b] \times [a, b]$ *is a solution of the system*

$$f(m, M) = m \quad \text{and} \quad f(M, m) = M,$$

then $m = M$.

Then equation (11) has a unique equilibrium $\overline{x} \in [a, b]$ *and every solution of it converges to* $\overline{x}$.

Proof of Theorem 4.1. By Theorem 2.1, we have known that the equilibrium point $\bar{y}$ is locally asymptotically stable. Hence it remains to show that $\bar{y}$ is a global attractor of every solution of equation (4). The proof essentially similar to the proof of Theorem 6.4.4 of [6]. But for the reader's convenient, we give only the proof for the case of $p < q$.

From Lemma 3.5 we see that every solution of equation (4) eventually lie in the interval $[p/q, 1]$. In this interval, the function

$$g(x, y) = \frac{x + p}{x + qy}$$

is increasing in x and decreasing in y. In addition, from

$$g(m, M) = \frac{m + p}{m + qM} = m \quad \text{and} \quad g(M, m) = \frac{M + p}{M + qm} = M,$$

we see that $m = M$. By Lemma 4.2, we get the desired result. The proof of the theorem is finished. $\square$

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