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ON $\mathcal{B}(M, X)$-CC-PROJECTIVE MODULES

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Abstract: In this paper, $\mathcal{B}(M, X)$-cc-projective modules are defined as generalization of M-cc-projective modules. Let M be an X-amply supplemented module such that $\mathcal{B}(M, X)$ is closed under supplement submodules. Then M is X-lifting if and only if every module is $\mathcal{B}(M, X)$-cc-projective. Let $M = M_1 \oplus \cdots \oplus M_n$ be a finite direct sum of relatively projective modules M_i. Assume M is an amply supplemented module. Then M is $\mathcal{B}(M, X)$-cc-projective if and only if M_i is $\mathcal{B}(M_i, X)$-cc-projective, for every $i \in \{1, \cdots, n\}$.

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1. Introduction

Throughout this paper all rings are associative with unity and all modules will be unital right R-modules. Let M be a module, we write $(A \ll M)$ $A \leq M$ to indicate that A is a (small) submodule of M.

Let M be a module and $A \leq B \leq M$. If $B/A \ll M/A$, then A is called a coessential submodule of B in M. A submodule K of M is called coclosed (denoted by $K \ll_{cc} M$) if K has no proper coessential submodule in M. Let $N \leq M$, a submodule K of M is called a supplement of N in M if $M = N + K$ and $N \cap K \ll K$. A module M is called weakly supplemented if for any submodule A of M, there exists a submodule B of M such that $M = A + B$ and $A \cap B \ll M$. A module M is called amply supplemented if for any submodules

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A, B of M with $M = A + B$, there exists a supplement P of A such that $P \leq B$.

Let X and M be R-modules. Define the set

$\mathcal{B}(M, X) = \{A \leq M \mid \exists Y \leq X, \exists f \in \mathrm{Hom}(M, X/Y), \mathrm{Ker} f/A \ll M/A\}$.

Consider the property

$\mathcal{B}(M, X)$-(D_1): For all $A \in \mathcal{B}(M, X)$, there exists a direct summand A^* of M such that $A/A^* \ll M/A^*$.

Following [1], M is called X-lifting if M satisfies $\mathcal{B}(M, X)$-(D_1) and M is called X-amply supplemented if for any submodules A, B of M such that $A \in \mathcal{B}(M, X)$ and $M = A + B$, there exists a supplement P of A such that $P \leq B$.

Let M_1 and M_2 be modules. Following [2], the module M_2 is M_1-cc-projective if every homomorphism $\alpha : M_2 \to M_1/K$, where $K \leq_{cc} M_1$, can be lifted to a homomorphism $\beta : M_2 \to M_1$.

Definition 1.1. (see [1]) Let M and X be modules. A module N is called small $\mathcal{B}(M, X)$-projective if for any submodule K of M with $K \in \mathcal{B}(M, X)$, every homomorphism $\alpha : N \to M/K$ with $\mathrm{Im}\alpha \ll M/K$ can be lifted to a homomorphism $\beta : N \to M$.

Definition 1.2. Let M and X be modules. A module N is called $\mathcal{B}(M, X)$-cc-projective if every homomorphism $\alpha : N \to M/K$, where $K \leq_{cc} M$ and $K \in \mathcal{B}(M, X)$, can be lifted to a homomorphism $\beta : N \to M$.

The Prüfer p-group $\mathbb{Z}(p^\infty)$ is a lifting $\mathbb{Z}$-module and so a X-lifting $\mathbb{Z}$-module. By Proposition 2.2, $\mathbb{Z}(p^\infty)$ is $\mathcal{B}(\mathbb{Z}(p^\infty), X)$-cc-projective, but it is not self-projective.

2. $\mathcal{B}(M, X)$-cc-Projectivity

Lemma 2.1. *Let M, X be modules and K be a coclosed submodule of M with $K \in \mathcal{B}(M, X)$. If M/K is $\mathcal{B}(M, X)$-cc-projective, then K is a direct summand of M.*

Proof. By hypothesis, there exists a homomorphism $\alpha : M/K \to M$ that lifts the identity $1 : M/K \to M/K$. It is easy to see that $M = K \oplus \alpha(M/K)$. Hence K is a direct summand of M. □

Proposition 2.2. *Let M be an X-amply supplemented module M such that $\mathcal{B}(M, X)$ is closed under supplement submodules. Then the following* *statements are equivalent:*

1. *M is X-lifting.*

2. *Every module is $\mathcal{B}(M,X)$-cc-projective.*

3. *For every coclosed submodule K of M with $K \in \mathcal{B}(M,X)$, M/K is $\mathcal{B}(M,X)$-cc-projective.*

Proof. (1) $\Rightarrow$ (2) Let N be any module. Let $\alpha : N \to M/K$ be any homomorphism with $K \leq_{cc} M$ and $K \in \mathcal{B}(M,X)$. Note that every coclosed submodule K of M with $K \in \mathcal{B}(M,X)$ is a direct summand of M, now the proof is clear.

(2) $\Rightarrow$ (3) Clearly: (2) implies (3), and by Lemma 2.1, (3) implies (1).

(3) $\Rightarrow$ (1) By Lemma 2.1 and [1], Proposition 3.4. □

Lemma 2.3. *Let M_1, M_2 and X be modules. If M_1 is $\mathcal{B}(M_2,X)$-cc-projective and M_2 is weakly supplemented, then for every coclosed submodule N of M_2, M_1 is $\mathcal{B}(M_2/N,X)$-cc-projective.*

Proof. Let L/N be a coclosed submodule of M_2/N with $L/N \in \mathcal{B}(M_2/N,X)$. By [1], Lemma 2.2, $L \in \mathcal{B}(M_2,X)$ and L is coclosed in M_2 by [3], Lemma 1.4. Let $f : M_1 \to (M_2/N)/(L/N) \cong M_2/L$. Since M_1 is $\mathcal{B}(M_2,X)$-cc-projective, it is easy to see that M_1 is $\mathcal{B}(M_2/N,X)$-cc-projective. □

Lemma 2.4. *Let M, X and $\{N_i |\ i \in I\}$ be modules. Then $\oplus_{i\in I} N_i$ is $\mathcal{B}(M,X)$-cc-projective if and only if N_i is $\mathcal{B}(M,X)$-cc-projective, for every $i \in I$.*

Proof. The proof follows as for projectivity (see for example [4], Proposition 16.10). □

Corollary 2.5. *Let M_1, M_2 and X be modules and $M = M_1 \oplus M_2$ a weakly supplemented module. If M is $\mathcal{B}(M,X)$-cc-projective, then M_i is $\mathcal{B}(M_j,X)$-cc-projective, for every $i, j \in \{1,2\}$. In particular, if M is $\mathcal{B}(M,X)$-cc-projective and N is a direct summand of M, then N is $\mathcal{B}(N,X)$-cc-projective.*

Proof. By Lemmas 2.3 and 2.4. □

Lemma 2.6. *Let M_1, M_2 and X be modules such that $M = M_1 \oplus M_2$ is amply supplemented and M_1 is small $\mathcal{B}(M_2,X)$-projective. If any module N is $\mathcal{B}(M_2,X)$-cc-projective and M_1-projective, then it is $\mathcal{B}(M,X)$-cc-projective.*

Proof. Let $K \leq_{cc} M$ and $K \in \mathcal{B}(M,X)$. Then $(K+M_1)/K \in \mathcal{B}(M/K,X)$ by [1], Lemma 3.5. Consider the homomorphism $\alpha : N \to M/K$ and the natural epimorphism $\pi : M \to M/K$. Since M/K is amply supplemented, there exists a submodule H/K of M/K such that $H/K \leq (K+M_1)/K, (K+M_1)/H \ll M/H$ and $H/K \leq_{cc} M/K$. Note that $H \leq_{cc} M$. Since $K + M_1 = H + M_1$, $(H + M_1)/H \ll M/H$ and $(H + M_1)/K \in \mathcal{B}(M/K, X)$. Hence $H + M_1 \in \mathcal{B}(M, X)$ and so $H \in \mathcal{B}(M, X)$ by [1], Lemma 2.2. Since M_1 is small $\mathcal{B}(M_2, X)$-projective, there exists a submodule H' of H such that $M = H' \oplus M_2$ by [1], Proposition 2.4. Since $M_2 \cong M/H'$, N is $\mathcal{B}(M/H', X)$-cc-projective. Let β be the epimorphism from M/K to M/H defined by $\beta(m + K) = m + H$ for all $m + K \in M/K$ and π_1 the epimorphism from M/H' to $M/H \cong (M/H")/(H/H')$ defined by $\pi_1(m + H') = m + H$ for all $m + H' \in M/H'$. Since N is $\mathcal{B}(M/H', X)$-cc-projective, there exists a homomorphism $g : N \to M/H'$ such that $\pi_1 g = \beta\alpha$. Now, consider the following homomorphisms:

$$N \xrightarrow{g} M/H' \overset{f}{\cong} M_2 \xrightarrow{i_1} M_1 \xrightarrow{\pi} M/K\,,$$

where i_1 is the inclusion map and f is the isomorphism from M/H' to M_2. Then we have the homomorphism $\pi i_1 f g : N \to M/K$. Take any element n in N, and suppose $\alpha(n) = m' + K$ and $g(n) = m + H'$ with $m, m' \in M$. Therefore, $\pi_1 g(n) = \beta\alpha(n)$ implies that $m - m' \in H$. Write $m = m_1 + m_2$ where $m_1 \in M_1$ and $m_2 \in M_2$. Now, $(\pi i_1 f g - \alpha)(n) = \pi i_1 f g(n) - \alpha(n) = \pi i_1 f(m+H') - (m'+K) = \pi i_1 f(m_1+m_2+H') - (m'+K) = \pi i_1(m_2) - (m'+K) = \pi(m_2) - (m' + K) = m_2 + K - (m' + K) = m_1 + m_2 - m' - m_1 + K$ implies that $\mathrm{Im}(\pi i_1 f g - \alpha) \subseteq (H + M_1)/K = (K + M_1)/K$. Consider the inclusion map $i_2 : (K + M_1)/K \to M/K$. Since $\mathrm{Im}(\pi i_1 f g - \alpha) \subseteq \mathrm{Im}(i_2) = (K + M_1)/K$, there exists a homomorphism $\gamma : N \to (K + M_1)/K$ such that $i_2\gamma = \pi i_1 f g - \alpha$. Let $\pi_2 : M_1 \to (K + M_1)/K$ be the natural epimorphism. Since N is M_1-projective, γ can be lifted to a homomorphism $\phi : N \to M_1$. Consider, finally, the homomorphism $\theta = i_1 f g - \phi : N \to M$. Let $n \in N$, then $\pi\theta(n) = \pi(i_1 f g - \phi)(n) = \pi i_1 f g(n) - \pi\phi(n) = \pi i_1 f g(n) - \alpha(n) + \alpha(n) - \pi_2\phi(n) = (\pi i_1 f g - \alpha)(n) + \alpha(n) - i_2\gamma(n) = \alpha(n)$. Therefore, α can be lifted to the homomorphism θ and N is $\mathcal{B}(M, X)$-cc-projective. □

Theorem 2.7. *Let $M = M_1 \oplus \cdots \oplus M_n$ be a finite direct sum of relatively projective modules M_i. Assume M is amply supplemented. Then M is $\mathcal{B}(M, X)$-cc-projective if and only if M_i is $\mathcal{B}(M_i, X)$-cc-projective, for every $i \in \{1, \cdots, n\}$.*

Proof. By Lemmas 2.4 and 2.6, using induction. □

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