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TWO CHARACTERIZATIONS OF THE TRIANGLE WITH THE ANGLES $\frac{\pi}{7}$, $\frac{2\pi}{7}$, $\frac{4\pi}{7}$

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Abstract: In this paper some interesting relations for the triangle with the angles $\frac{\pi}{7}$, $\frac{2\pi}{7}$, $\frac{4\pi}{7}$ are considered. Statements about lengths of its sides, Brocard angle and radius of circumscribed circle are proved.

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1. Introduction

The triangle ABC with the angles $\frac{\pi}{7}$, $\frac{2\pi}{7}$, $\frac{4\pi}{7}$ has a number of interesting properties. If a, b, c are the lengths of its sides, R the radius of its circumscribed circle, and ω its Brocard angle, then the following equalities are valid

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$$\frac{1}{a} = \frac{1}{b} + \frac{1}{c}, \tag{1}$$

$$b + c - a = R\sqrt{7}, \tag{2}$$

$$\cot \omega = \sqrt{7}. \tag{3}$$

2. Characterizations of the Triangle with Angles $\frac{\pi}{7}$, $\frac{2\pi}{7}$, $\frac{4\pi}{7}$

In this section the following theorems will be proved by mean of some lemmas.

Theorem 1. *In the class of all triangles for whose lengths of sides the equality (1) is valid, the triangle with the angles $\frac{\pi}{7}$, $\frac{2\pi}{7}$, $\frac{4\pi}{7}$ is the only (to the similarity) triangle, for which the equality (2) is valid.*

Theorem 2. *In the class of all triangles for whose lengths of sides the equality (1) is valid, the triangle with the angles $\frac{\pi}{7}$, $\frac{2\pi}{7}$, $\frac{4\pi}{7}$ is the only triangle, for which the equality (3) is valid.*

If the equality (1) is valid, then it is obvious that a is the smallest length of side. Let $a < b \leq c$ and let us put

$$b^2 + c^2 = bct. \tag{4}$$

Then from $(b-c)^2 = (t-2)bc$ it follows $t \geq 2$, and then from (1) we get

$$a = \frac{bc}{b+c}, \qquad a^2 = \frac{b^2c^2}{(b+c)^2} = \frac{bc}{t+2}.$$

Therefore

$$\begin{aligned} a^2 + b^2 + c^2 &= \left(t + \frac{1}{t+2}\right) bc = \frac{(t+1)^2}{t+2} bc, \\ (b+c)^2 - a^2 &= \left(t + 2 - \frac{1}{t+2}\right) bc = \frac{(t+1)(t+3)}{t+2} bc, \\ a^2 - (b-c)^2 &= \left[\frac{1}{t+2} - (t-2)\right] bc = \frac{5-t^2}{t+2} bc, \end{aligned}$$

$$b + c - a = b + c - \frac{bc}{b+c} = \frac{b^2 + bc + c^2}{b+c} = \frac{(t+1)bc}{b+c},$$

$$(b+c-a)^2 = \frac{(t+1)^2}{t+2} bc.$$

If Δ is the area of the triangle ABC, then from Heron's formula we get

$$16\Delta^2 = [(b+c)^2 - a^2][a^2 - (b-c)^2] = \frac{(t+1)(t+3)(5-t^2)}{(t+2)^2} b^2c^2.$$

Using the well known formulae

$$R = \frac{abc}{4\Delta}, \qquad \cot\,\omega = \frac{a^2+b^2+c^2}{4\Delta},$$

we get

$$R^2 = \frac{a^2b^2c^2}{16\Delta^2} = \frac{t+2}{(t+1)(t+3)(5-t^2)} bc,$$

$$\cot^2\omega = \frac{(a^2+b^2+c^2)^2}{16\Delta^2} = \frac{(t+1)^3}{(t+3)(5-t^2)}.$$

So we have proved the following lemma.

Lemma 3. *If for the triangle ABC the equalities (1) and (4) are valid, then the equalities*

$$\frac{(b+c-a)^2}{R^2} = \frac{(t+1)^3(t+3)(5-t^2)}{(t+2)^2}, \tag{5}$$

$$\cot^2\omega = \frac{(t+1)^3}{(t+3)(5-t^2)} \tag{6}$$

are also valid.

Let us now prove two more lemmas.

Lemma 4. *With the relationships taken from Lemma 3 each of the equalities (2) and (3) is equivalent to the equality*

$$t^3 + 3t^2 - 4t - 13 = 0 \tag{7}$$

and there is exactly one value for number t, ($t \geq 2$), for which this equation is valid.

Proof. Because of (6) the equation (3) is equivalent to the equality

$$(t+1)^3 + 7(t+3)(t^2-5) = 0,$$

which, after some arrangements and dividing by 8 gets the form (7). Because of (5) the equality (2) is equivalent to the equality

$$(t+1)^3(t+3)(t^2-5) + 7(t+2)^2 = 0,$$

which can be written in the form.

$$(t^3+3t^2-4t-13)(t^3+3t^2+2t-1) = 0.$$

It is easy to see that equation $t^3+3t^2+2t-1=0$ has no solution $t \geq 2$, and the equation (7) has just only one such a solution, and that is in the interval

(2, 3). □

Lemma 5. *If $\frac{b}{c} = x$, then in the class of the triangles, for which the equalities (1) are valid, there is exactly one value for x $(0 < x \leq 1)$ so that for this triangle the equations (2) and (3) are valid. This value satisfies the equation*

$$x^3 + 2x^2 - x - 1 = 0. \tag{8}$$

Proof. From (4) it follows $t = x + \frac{1}{x}$, which substituted in (7) after multiplication by x^3 and some arrangements gives the equation

$$x^6 + 3x^5 - x^4 - 7x^3 - x^2 + 3x + 1 = 0,$$

which can be also written in the form

$$(x^3 + 2x^2 - x - 1)(x^3 + x^2 - 2x - 1) = 0.$$

For the function $f(x) = x^3 + 2x^2 - x - 1$ we have $f(-3) = -7$, $f(-2) = 1$, $f(-1) = 1$, $f(0) = -1$, $f(1) = 1$, so its null-points t_1, t_2, t_3 are such that

$$-3 < t_1 < -2, \quad -1 < t_2 < 0, \quad 0 < t_3 < 1$$

and only last one comes in consideration. For the function $g(x) = x^3 + x^2 - 2x - 1$ we have $g(-2) = -1$, $g(-1) = 1$, $g(0) = -1$, $g(1) = -1$, $g(2) = 7$, so for its null-points t_4, t_5, t_6 it is valid

$$-2 < t_4 < -1, \quad -1 < t_5 < 0, \quad 1 < t_6 < 2$$

and none of each comes in consideration. □

Proof of Theorems 1 and 2. On the basis of lemmas, we have been dealing with so far, it is sufficient to prove that the triangle with the angles $\frac{\pi}{7}$, $\frac{2\pi}{7}$, $\frac{4\pi}{7}$ satisfies the equation (8). That triangle is the triangle ADC in a regular heptagon $ABCDEFG$, which has got the sides with the lengths a, "small" diagonals with the lengths b and "large" ones with the lengths c. Applying Ptolomey's Theorem on the quadrangles $ABCD$ and $ACDE$ we get the equality

$$b^2 = a^2 + ac \tag{9}$$

and the equality $bc = ab + ac$, which is in fact the equality (1). Because of it the equality (9), after multiplication by $(b + c)^2$, gets the form

$$b^2(b + c)^2 = b^2c^2 + bc^2(b + c),$$

i.e. by division with bc^3, because of $\frac{b}{c} = x$, the form

$$x(x + 1)^2 = x + x + 1,$$

which brings us to the equality (8). □