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MODELS AND RISK ANALYSIS OF UNCERTAIN COMPLEX SYSTEMS

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Abstract: A modeling methodology is developed for complex systems, systems of several interacting components, with observable component performance normal fields that are also separable over the space of uncertainties. The algebra of operator representations of system components is completed by using separable equivalents in place of sums which are in general nonseparable. The end product is a computation of the mean and variance of the performance of the modeled system at each point in the space of system uncertainties.

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1. Wiener Fields and Spaces of Hellinger Integrable Functions

An emerging class of decision problems under conditions of uncertainty and risk requires a flexible modeling methodology for representing multiple component systems [17, 18]. Providing a framework or setting for these models in the mathematical literature motivates this paper. Certain tools used in the methodology have a long history: reproducing kernel Hilbert spaces [1], Hellinger integrals [7, 22], Wiener fields [4], representations of operators on spaces of functions of sev-

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eral variables [12], and Central Limit Theorem [5]. However, the requirements of a modeling methodology exposes new problems to be addressed in this setting. In particular, the representation of separable random fields [28], the class of fields most like continuous random processes, and separable approximations for sums of separable fields, which in general are nonseparable, are addressed.

Our approach to random fields is from a modeling viewpoint, as opposed to a data driven approach, using theoretical means and covariances of random fields to construct more complex models out of simpler component models. The models are not dynamic, in fact time is usually not a system variable, but rather linear operators on appropriate inner product spaces of functions. The end product is a computation of the mean and variance of the modeled field at each point in the domain of the random field (the space of uncertainties).

1.1. Representation of Discrete Surfaces

Suppose each of S and T is a positive number and F is a field defined on $[0, S] \times [0, T]$. If $0 = s_1 < s_2 < \cdots < s_m = S$ and $0 = t_1 < t_2 < \cdots < t_n = T$ then let F_{st} denote the discrete surface defined by

$$F_{st}(i, p) = F(s_i, t_p)\,.$$

F_{st} can also be thought of as an $m \times n$ matrix. Let K_{ss} and K_{tt} denote the discrete covariance kernels of the standard Wiener process W on $[0, S]$ and $[0, T]$, respectively. That is, $K_{ss}(i, j) = EW(s_i)W(s_j)$ and $K_{tt}(p, q) = EW(t_p)W(t_q)$. For the rest of the paper we fix S and T as positive numbers, m and n as positive integers, and $s = \{s_p\}_{p=1}^m$ and $t = \{t_q\}_{q=1}^n$ as partitions of $[0, S]$ and $[0, T]$, respectively.

1.2. Definition of a Random Wiener Field

Random Wiener fields play a fundamental role in the representation of more general random fields. We can use Wiener fields to define the basic inner product spaces in which more general random fields with mean zero can at first be identified with positive definite operators.

The standard Wiener field [4, 15], also denoted by W, on $[0, S] \times [0, T]$ has the following defining properties:

— sample fields are continuous,

— $W(a, 0) = W(0, b) = 0$,

— $E[W(a,b)] = 0$,

— $E[(W(b,d) - W(a,d) - W(b,c) + W(a,c))^2] = E[(dW(a,b,c,d))^2] = (b-a)(d-c)$,

— if $[a,b] \times [c,d] \cap [z,w] \times [x,y]$ is empty or has no interior then

$$E[dW(a,b,c,d)dW(z,w,x,y)] = 0,$$

where $E[\cdot]$ is the expectation operator.

Notice that for $0 < a, b$ the process $\frac{1}{\sqrt{a}} W(a, \ )$ is the standard Wiener process on $[0, T]$ and $\frac{1}{\sqrt{b}} W(\ , b)$ is the standard Wiener process on $[0, S]$.

1.3. The Wiener Kernel for Fields

Suppose F is a field defined on $[0, S] \times [0, T] \times [0, S] \times [0, T]$, such as the covariance kernel of the Wiener field. Let $F_{(st)^2}$ denote the discrete function defined by

$$F_{(st)^2}(i, p, j, q) = F(s_i, t_p, s_j, t_q)\,.$$

Let K_{ss}^C and K_{tt}^C be upper triangular matrices with nonnegative diagonals such that $(K_{ss}^C)^T K_{ss}^C = K_{ss}$ and $(K_{tt}^C)^T K_{tt}^C = K_{tt}$. In an extension of classic usage we refer to K_{ss}^C and K_{tt}^C as *upper Cholesky factors* of K_{ss} and K_{tt}, respectively. We can obtain simulations of W_{st} in terms of K_{ss}^C and K_{tt}^C as follows:

$$W_{st} = (K_{ss}^C)^T Z_{st} K_{tt}^C\,,$$

where Z_{st} is a $(0, 1)$-normal $m \times n$ matrix of independent random variables.

Theorem 1. *W_{st} is a discretization of the standard Wiener field.*

Lemma 1. *If U_{ss} is a nonnegative $m \times m$ matrix with upper Cholesky factor U_{ss}^C in the above sense and V_{tt} is a nonnegative $n \times n$ matrix with upper Cholesky factor V_{tt}^C then*

$$E([(U_{ss}^C)^T Z_{st} V_{tt}^C](i, p)(U_{ss}^C)^T Z_{st} V_{tt}^C](j, q) = U_{ss}(i, j) V_{tt}(p, q)$$

for $1 \le i, j \le m$ and $1 \le p, q \le n$.

Proof of Lemma 1.

$$E([(U_{ss}^C)^T Z_{st} V_{tt}^C](i, p)(U_{ss}^C)^T Z_{st} V_{tt}^C](j, q)$$

$$= E\left(\sum_{\ell_1=1}^{m} \sum_{\ell_2=1}^{n} (U_{ss}^C)^T(i, \ell_1) Z_{st}(\ell_1, \ell_2) V_{tt}^C(\ell_2, p) \right)$$

$$\times \left(\sum_{\bar{\ell}_1=1}^{m} \sum_{\bar{\ell}_2=1}^{n} (U_{ss}^C)^T(j,\, \bar{\ell}_1) Z_{st}(\bar{\ell}_1,\, \bar{\ell}_2) V_{tt}^C(\bar{\ell}_2,\, q) \right)$$

$$= \sum_{\ell_1=1}^{m} \sum_{\ell_2=1}^{n} (U_{ss}^C)^T(i,\, \ell_1)(U_{ss}^C)^T(j,\, \ell_1) V_{tt}^C(\ell_2,\, p) V_{tt}^C(\ell_2,\, q)$$

$$= \sum_{\ell_1=1}^{m} (U_{ss}^C)^T(i,\, \ell_1) U_{ss}^C(\ell_1,\, j) \sum_{\ell_2=1}^{n} (V_{tt}^C)^T(p,\, \ell_2) V_{tt}^C(\ell_2,\, q) = U_{ss}(i,\, j) V_{tt}(p,\, q)\,.$$

Proof of Theorem 1. First, $EW_{st} = 0$, the $m \times n$ zero matrix, and $W_{st}(1,\ ) = W_{st}(\ ,\, 1) = 0$. Also from the lemma with $U_{ss} = K_{ss}$ and $V_{tt} = K_{tt}$

$$EW_{st}(i,\, p) W_{st}(j,\, q) = E[(K_{ss}^C)^T Z_{st} K_{tt}^C](i,\, p)[(K_{ss}^C)^T Z_{st} K_{tt}^C](j,\, q)$$
$$= K_{ss}(i,\, j) K_{tt}(p,\, q)\,.$$

Therefore $K_{(st)^2}(i,\, p,\, j,\, q) = K_{ss}(i,\, j) K_{tt}(p,\, q)$. Furthermore, for $i \le j$ and $p \le q$,

$$\begin{aligned} & E(W_{st}(j,\, q) - W_{st}(i,\, q) - W_{st}(j,\, p) + W_{st}(i,\, p))^2 \\ &= E(W_{st}(j,\, q)^2 + EW_{st}(i,\, q)^2 + EW_{st}(j,\, p)^2 + EW_{st}(i,\, p)^2 \\ &\quad - 2EW_{st}(j,\, q) W_{st}(i,\, q) - 2EW_{st}(j,\, q) W_{st}(j,\, p) \\ &\quad + 2EW_{st}(j,\, q) W_{st}(i,\, p) + 2EW_{st}(i,\, q) W_{st}(j,\, p) \\ &\quad - 2EW_{st}(i,\, q) W_{st}(i,\, p) - 2EW_{st}(j,\, p) W_{st}(i,\, p) \\ &= K_{ss}(j,\, j) K_{tt}(q,\, q) + K_{ss}(i,\, i) K_{tt}(p,\, p) \\ &\quad - K_{ss}(i,\, i) K_{tt}(q,\, q) - K_{ss}(j,\, j) K_{tt}(p,\, p) \\ &= (K_{ss}(j,\, j) - K_{ss}(i,\, i))(K_{tt}(q,\, q) - K_{tt}(p,\, p)) \\ &= (s_j - s_i)(t_q - t_p)\,. \end{aligned}$$

Finally, suppose $[i,\, j] \times [p,\, q] \cap [\bar{i},\, \bar{j}] \times [\bar{p},\, \bar{q}] = \phi$. Assume to be definite that $i < j < \bar{i} < \bar{j}$. The other cases work the same way. Then

$$\begin{aligned} & E(W_{st}(j,\, q) - W_{st}(i,\, q) - W_{st}(j,\, p) + W_{st}(i,\, p)) \\ &\qquad \times (W_{st}(\bar{j},\, \bar{q}) - W_{st}(\bar{i},\, \bar{q}) - W_{st}(\bar{j},\, \bar{p}) + W_{st}(\bar{i},\, \bar{p})) \\ &\qquad = (K_{ss}(j,\, \bar{j}) - K_{ss}(j,\, \bar{i}) - K_{ss}(i,\, \bar{j}) + K_{ss}(i,\, \bar{i})) \\ &\qquad\quad \times (K_{tt}(q,\, \bar{q}) - K_{tt}(q,\, \bar{p}) - K_{tt}(p,\, \bar{q}) + K_{tt}(p,\, \bar{p})) = 0\,. \quad \square \end{aligned}$$

The discrete representations and the method of proof in this first theorem are the result of our modeling viewpoint. Implementation of the theory to be developed, because of the matrix emphasis, becomes almost immediate in MATLAB, our computational environment. Although most of the material

in Section 2 is familiar, we will present everything in terms of finite matrix manipulations.

In addition to the role of the standard Wiener field W in the structure of the theory taken up in the next subsection, the Wiener field is the prototypical of the random fields of interest. The Wiener kernel can be factored, i.e., $K_{(st)^2}(i, p, j, q) = K_{ss}(i, j)K_{tt}(p, q)$, for $0 \leq i, j \leq m$ and $0 \leq p, q \leq n$, and K_{ss} and K_{tt} can in turn be factored and the factors used to simulate the discrete Wiener field W_{st}. See the note in Section 2.3 for more general Wiener fields [4].

1.4. Separable Random Fields

For scalar fields defined on two-dimensional rectangles, sufficient conditions on the system covariance kernel are given for the development of a system linearization based on a factorization of the system covariance kernel. The mathematical question of limitations imposed by the covariance condition can be discussed in terms of properties of the resulting linearizations. A set of reasonable conditions on linear systems can be shown to be equivalent to the covariance condition. Thus the methods apply to a rich class of random fields.

The central idea in this subsection is to produce a condition on the covariance kernel of a random field (function of two variables) which for fields F generated by linear systems, i.e., of the form $F = AW$, is sufficient to produce a representation of A (here W is the standard Wiener field, discussed earlier). If F is generated by a nonlinear system $\hat{A}$ then the method produces a linearization of $\hat{A}$, namely A. In this case the utility of the linearization depends upon the particular application and how nearly the assumption of the condition (perhaps uncheckable) is to holding.

The condition is the following: for each pair of points (a, b) and (c, d) in $[0, S] \times [0, T]$

$$\begin{aligned} &\operatorname{cov}(F(a,b),\, F(c,d)) \\ &\quad = \operatorname{cov}(F(a,T),\, F(c,T)) \cdot \operatorname{cov}(F(S,b),\, F(S,d))/var(F(S,\, T))\,, \end{aligned} \tag{1}$$

where $\operatorname{cov}(\cdot)$ is the covariance operator (see [28], p. 82). Random fields satisfying the condition are said to be *separable*. Note that the condition is on the observation field F rather than on the underlying unmodeled system. Also note that the standard Wiener field is separable.

In general, the independence expressed in the condition might not hold. However, such independence is implicit in the common engineering practice of

exploring a given physical system by allowing only one quantity to vary at a time.

Vanmarcke [28], with an extensive bibliography, provides a foundation reference for separable fields. Separability in Vanmarcke is given in terms of correlation functions rather than covariance kernels. See Yaglom [27], also using correlations rather than covariances and with a large bibliography, for alternative representations for random fields. Models incorporating time are different because time is different from a space variable, i.e., for time there is a past, a present, and a future. Space variables are not linearly ordered. While related to our random fields, random space-time functions have a different flavor [8]. In fact our random fields would correspond to the limited case where spatial behavior is the same at all time instants.

Wiener fields were introduced by Chentsov [4]. A Wiener field is *not* Lévy's Brownian motion process of two parameters [10, 11, 2]. Our use of "separable" is not to be confused with the use for abstract metric spaces or infinite dimensional spaces, i.e., the existence of a countable dense subset. In this use separable processes are processes whose sample paths are separable subsets of a function space [14, 13].

1.5. Reproducing Kernel Hilbert (RKH) Spaces

Covariance kernels for random fields have the nonnegative function property that makes them reproducing kernels of some complete inner product space of functions of two variables [1]. Of interest are those kernels which can be associated with positive linear operators on the space determined by the Wiener kernel. A scalar function R on $[0,\, S] \times [0,\, T] \times [0,\, S] \times [0,\, T]$ is said to be nonnegative definite provided for each sequence $\{\mathbf{u}^p\}_{p=1}^{\ell}$, where $\mathbf{u^p}$ is an ordered pair in $[0,\, S] \times [0,\, T]$, and sequence $\{a_p\}_{p=1}^{n}$ of nonzero real numbers

$$\sum_{p=1}^{n}\sum_{q=1}^{n} R(\mathbf{u}^p,\, \mathbf{u}^q)a_p a_q \geq 0\,.$$

The space of Hellinger integrable functions of two variables provide an explicit representation of the RKH space determined by the Wiener kernel [12]. Functions f defined on $[0,\, S]\times[0,\, T]$ are said to be Hellinger integrable provided the set of approximating sums

$$\left\{\sum_s \sum_t \frac{(df(u,\, v))^2}{du\, dv}\right.$$

$$= \sum_{i=2}^{m} \sum_{p=2}^{n} \frac{(f(s_i, t_p) - f(s_{i-1}, t_p) - f(s_i, t_{p-1}) + f(s_{i-1}, t_{p-1}))^2}{(s_i - s_{i-1})(t_p - t_{p-1})} \Bigg\}$$

is bounded. The least upper bound is denoted by $\int\int_{[0,\,S]\times[0,\,T]} \frac{(df(u,\,v))^2}{du\,dv}$.

The covariance kernel for the standard Wiener surface W is

$$E(W(u,\,v)W(x,\,y)) = \min(u,\,v)\min(x,\,y) = K(u,\,v)K(x,\,y)\,.$$

The complete inner product space of functions $\{G,\,Q\}$ determined by this kernel is the space of Hellinger integrable functions f on $[0,\,S]\times[0,\,T]$ such that $f(0,\,\cdot) = 0$ and $f(\cdot\,,\,0) = 0$. Further, the inner product is defined by $Q(f,\,g) = \int\int_{[0,\,S]\times[0,\,T]} \frac{df(u,\,v)\,dg(u,\,v)}{du\,dv}$.

1.6. Approximating Sums and Discrete Representations

We can consider approximating sums as the results of matrix products.

Theorem 2.

$$\sum_t \sum_s \frac{df(s,\,t)\,dg(s,\,t)}{ds\,dt} = \mathrm{trace}((K_{tt}^C)^{-T} f_{st}^T K_{ss}^{-1} g_{st} (K_{tt}^C)^{-1})\,.$$

Proof of Theorem 2. Recall that for $2 \le i,\,p$

$$[(K_{ss}^C)^{-T} f_{st} (K_{tt}^C)^{-1}](i,\,p) = \frac{\{f(s_i,\,t_p) - f(s_{i-1},\,t_p) - f(s_i,\,t_{p-1}) + f(s_{i-1},\,t_{p-1})\}}{(\sqrt{s_i - s_{i-1}}\sqrt{t_p - t_{p-1}})}$$

Let $A = (K_{ss}^C)^{-T} f_{st} (K_{tt}^C)^{-1}$ and $B = (K_{ss}^C)^{-T} g_{st} (K_{tt}^C)^{-1}$. Then

$$\begin{aligned}\mathrm{trace}((K_{tt}^C)^{-T} f_{st}^T K_{ss}^{-1} g_{st} (K_{tt}^C)^{-1}) &= \mathrm{trace}(((K_{ss}^C)^{-T} f_{st} (K_{tt}^C)^{-1})^T ((K_{ss}^C)^{-T} g_{st} (K_{tt}^C)^{-1})) \\ &= \mathrm{trace}(A^T B) = \sum_{p=1}^{n} [A^T B](p,\,p) = \sum_{p=1}^{n} \sum_{i=1}^{m} A(i,\,p) B(i,\,p) \\ &= \sum_t \sum_s \frac{df(s,\,t)\,dg(s,\,t)}{ds\,dt}\,. \quad \square\end{aligned}$$

From the general theory of RKH spaces obtained using Hellinger integrals [12], we know that $Q(f,\,K(\ ,\,u)K(\ ,\,v)) = f(u,\,v)$, i.e., the Wiener kernel is the reproducing kernel for $\{G,\,Q\}$, but it is useful to use our representation of Q to establish this fact in our context.

Theorem 3. *For f in $\{G, Q\}$ and (u, v) in $[0, S] \times [0, T]$*

$$Q(f, K(\ , u)K(\ , v)) = f(u, v)\,.$$

Proof of Theorem 3. Suppose $s_i = u$ and $t_p = v$. Then

$$\begin{aligned} Q(f, K(\ , u)K(\ , v)) &\sim \mathrm{trace}((K^C_{tt})^{-T} f^T_{st} K^{-1}_{ss} K_{ss}(\ , i) K_{tt}(\ , p)^T (K^C_{tt})^{-1}) \\ &= \mathrm{trace}((K^C_{tt})^{-T} f^T_{uv} I_{uu}(\ , i) K^C_{vv}(\ , p)^T) \\ &= \mathrm{trace}((K^C_{tt})^{-T} f_{st}(i,\ )^T K^C_{tt}(\ , p)^T) = f_{st}(i,\ )(K^C_{tt})^{-1} K^C_{tt}(\ , p) \\ &= f_{st}(i,\ ) I_{tt}(\ , p) = f_{st}(i, p) = f(u, v)\,. \quad \square \end{aligned}$$

Again, from the general theory we can characterize all continuous linear operators on $\{G, Q\}$ and their matrix representations [12]. Our interest is in a special class of operators which the following theorem enables us to characterize in terms of their matrix representations.

Theorem 4. *Suppose L is the matrix representation of a continuous linear transformation A on $\{G, Q\}$. These are equivalent:*

1. There is an $m \times m$ matrix A_L and an $n \times n$ matrix A_R such that $[Af]_{st}(j, q) = ((K^C_{ss})^{-1} A_L(\ , j))^T f_{st} (K^C_{tt})^{-1} A_R(\ , q)$.

2. There is an $m \times m$ matrix L^1_{ss} and an $n \times n$ matrix L^2_{tt} such that $L_{(st)^2}(i, j, p, q) = [L^1_{ss}(\ , j)(L^2_{tt}(\ , q))^T](i, p)$.

Proof of Theorem 4. Suppose $(Af)_{st} = ((K^C_{ss})^{-1} A_L)^T f_{st} (K^C_{tt})^{-1} A_R$, where A_L is an $m \times m$ matrix, f_{st} is an $m \times n$ matrix, and A_R is an $n \times n$ matrix. Then

$$\begin{aligned} (AK(\ , u)K(\ , v))_{st} &= ((K^C_{ss})^{-1} A_L)^T K_{ss}(\ , i) K_{tt}(\ , p)^T (K^C_{tt})^{-1} A_R \\ &= A^T_L (K^C_{ss})^{-T} (K^C_{ss})^T K^C_{ss}(\ , i) ((K^C_{tt})^T (K^C_{tt})(\ , p))^T (K^C_{tt})^{-1} A_R \\ &= A^T_L K^C_{ss}(\ , i) K^C_{tt}(\ , p)^T A_R\,. \end{aligned}$$

Therefore

$$\begin{aligned} L_{(st)^2}(i, j, p, q) &= Q(AK(\ , s_i)K(\ , t_p),\, K(\ , s_j)K(\ , t_q)) \\ &= \mathrm{trace}((K^C_{tt})^{-T} A^T_R K^C_{tt}(\ , p) K^C_{ss}(\ , i)^T A_L K^{-1}_{ss} K_{ss}(\ , j) K_{tt}(\ , q)^T (K^C_{tt})^{-1}) \\ &= \mathrm{trace}((K^C_{tt})^{-T} A^T_R K^C_{tt}(\ , p) K^C_{ss}(\ , i)^T A_L(\ , j) K_{tt}(\ , q)^T (K^C_{tt})^{-1}) \\ &= (A_L(\ , j))^T K^C_{ss}(\ , i) K^C_{tt}(\ , p)^T A_R (K^C_{tt})^{-1} K^C_{tt}(\ , q) \\ &= (A_L(\ , j))^T K^C_{ss}(\ , i) K^C_{tt}(\ , p)^T A_R(\ , q) = [L^1_{ss}(\ , j)(L^2_{tt}(\ , q))^T](i, p)\,. \end{aligned}$$

On the other hand, suppose that

$$L_{(st)^2}(i, j, p, q) = [L^1_{ss}(\ , j)(L^2_{tt}(\ , q))^T](i, p)$$

and

$$[(Af)_{st}](j,\, q) = \mathrm{trace}((K_{tt}^C)^{-T} f_{st}^T K_{ss}^{-1} L_{(st)^2}(\ ,\, j,\ \ ,\, q)(K_{tt}^C)^{-1})\,.$$

Let $A_L = (K_{ss}^C)^{-T} L_{ss}^1$ and $A_R = (K_{tt}^C)^{-T} L_{tt}^2$. Then

$$\begin{aligned}[(Af)_{st}](j,\, q) &= \mathrm{trace}((K_{tt}^C)^{-T}(f_{st})^T K_{ss}^{-1} L_{ss}^1(\ ,\, j)(L_{tt}^2(\ ,\, q))^T (K_{tt}^C)^{-1}) \\ &= \mathrm{trace}((K_{tt}^C)^{-T}(f_{st})^T (K_{ss}^C)^{-1} A_L(\ ,\, j) A_R(\ ,\, q)^T) \\ &= ((K_{ss}^C)^{-1} A_L(\ ,\, j))^T f_{st} (K_{tt}^C)^{-1} A_R(\ ,\, q)\,.\end{aligned}$$

See [22] for a survey of Hellinger's contributions to integration and operator theory. The collection of benchmark papers edited by Weinert [26] contains the foundation papers for applications of reproducing kernel Hilbert space methods to signal analysis. Note the small change in notation. We use RKH space representations meaning operator representations on the reproducing kernel Hilbert space. For the Parzen/Kailath meaning ($RKHS$ representations) the random process is represented by the reproducing kernel Hilbert space itself.

2. Linear Operators Associated with Separable Random Fields

The method of stochastic linearization [3] is considered in the context of separable random surfaces. The factorization of the surface kernels following from this assumption enables us to present a representation of linear operators generating the random surfaces.

2.1. Producing Linearizations of Separable Random Fields

We provide a recipe for representations of linear systems generating separable random fields. The recipe works for a large class of systems. Finite dimensional approximations for a linear operator A require two matrices. For instance,

$$AW_{st} = ((K_{ss}^C)^{-1} A_L)^T W_{st} (K_{tt}^C)^{-1} A_R,$$

where A_L is an $m \times m$ matrix, W_{st} is an $m \times n$ matrix, the discretization of the standard Wiener field, and A_R is an $n \times n$ matrix. The representation problem for a linear transformation A in the general case reduces to a search for appropriate matrices A_L and A_R.

Theorem 5. *If F is a separable random field defined on $[0,\, S] \times [0,\, T]$ then there are appropriate matrices A_L and A_R such that*

$$AW_{st} = ((K_{ss}^C)^{-1} A_L)^T W_{st} (K_{tt}^C)^{-1} A_R$$

has the mean and covariance of F_{st}.

Proof of Theorem 5. Let U_{ss} and V_{tt} be matrices defined by

$$U_{ss}(i,\, j) = \mathrm{cov}(F_{st}(i,\, n),\, F_{st}(j,\, n))/std(F(S,\, T))$$

and

$$V_{tt}(p,\, q) = \mathrm{cov}(F_{st}(m,\, p),\, F_{st}(m,\, q))/std(F(S,\, T))$$

Then $U_{ss} = (U_{ss}^C)^T U_{ss}^C$ and $V_{tt} = (V_{tt}^C)^T V_{tt}^C$, where each of U_{ss}^C and V_{tt}^C is an upper triangular matrix with nonnegative entries on the main diagonal. For $A_L = U_{ss}^C$ and $A_R = V_{tt}^C$ we have, using Lemma 1,

$$\begin{aligned}\mathrm{cov}([AW_{st}](i,\, p),\, [AW_{st}](j,\, q)) &= U_{ss}(i,\, j)V_{tt}(p,\, q)\\ &= \mathrm{cov}([AW_{st}](i,\, n),\, [AW_{st}](j,\, n))/std(F(S,\, T))\\ &\quad \times \mathrm{cov}([AW_{st}](m,\, p),\, [AW_{st}](m,\, q))/std(F(S,\, T))\,. \quad \square\end{aligned}$$

Hence, AW_{st} has the mean and covariance of F_{st} and so A is the stochastic linearization of the system generating F_{st}.

Corollary 1. *For $1 \le i \le m$ and $1 \le p \le n$*

$$cov([AW_{st}](i,\, p),\, [AW_{st}](i,\, p)) = U_{ss}(i,\, i)V_{tt}(p,\, p)\,.$$

That is, the variance of AW_{st} at each point can be readily computed from the diagonals of U_{ss} and V_{tt}.

Corollary 2. *The risk surface for F is specified by two increasing functions k_1 and k_2 with $k_1(0) = k_2(0) = 0$.*

2.2. Basic Results for Linearizations

Our goal is risk analysis for complex systems constructed of simpler components. Combining components requires algebraic rules for combining linearizations. From the basic representation of a separable field H, we say an operator A is separable provided

$$H_{st} = AF_{st} = (A_L)^T (K_{ss}^C)^{-T} F_{st} (K_{tt}^C)^{-1} A_R$$

for each field F in the domain of A. Recall from Theorem 4 that the definition could have been given equivalently in terms of factorization of the matrix representation. Further, the definition results in the random field $H_{st} = AW_{st}$, using an extension of A to the continuous functions, being separable. Composition (or products) and inverses of the separable linear transformations yield separable linear transformations. A similar result fails for addition. In general,

$(A+B)_L \neq A_L + B_L$ and $(A+B)_R \neq A_R + B_R$. See discussion below for approximation results.

Theorem 6. *If each of A and B is a separable linear transformation, then*

$$(AB)_L = B_L(K_{ss}^C)^{-1}A_L \text{ and } (AB)_R = B_R(K_{tt}^C)^{-1}A_R\,.$$

Furthermore,

$$(A^{-1})_L = K_{ss}^C(A_L)^{-1}K_{ss}^C \text{ and } (A^{-1})_R = K_{tt}^C(A_R)^{-1}K_{tt}^C\,.$$

Proof of Theorem 6.

$$E\{[ABW_{st}](i,\,p)[ABW_{st}](i,\,p)\} = [A_L^T(K_{ss}^C)^{-T}B_L^TB_L(K_{ss}^C)^{-1}A_L](i,\,i) \times [A_R^T(K_{tt}^C)^{-T}B_R^TB_R(K_{tt}^C)^{-1}A_R](p,\,p)\,,$$

i.e. $U_{ss} = A_L^T(K_{ss}^C)^{-T}B_L^TB_L(K_{ss}^C)^{-1}A_L$ or $U_{ss}^C = (AB)_L = B_L(K_{ss}^C)^{-1}A_L$. In the same way, $(AB)_R = B_R(K_{tt}^C)^{-1}A_R$. □

2.3. Special Results

Formulas for K_{ss}, U_{ss}, and V_{ss}. The nonnegative definite matrices K_{ss}, U_{ss}, and V_{ss} have a special form and combine in nice ways. For instance

$$U_{ss} = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ 0 & k(s_2) & k(s_2) & k(s_2) & \dots \\ 0 & k(s_2) & k(s_3) & k(s_3) & \dots \\ 0 & k(s_2) & k(s_3) & k(s_4) & \dots \\ \cdot & \cdot & \cdot & \cdot & \dots \\ \cdot & \cdot & \cdot & \cdot & \dots \\ \cdot & \cdot & \cdot & \cdot & \dots \end{bmatrix},$$

$$U_{ss}^C = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ 0 & \sqrt{k(s_2)} & \sqrt{k(s_2)} & \sqrt{k(s_2)} & \dots \\ 0 & 0 & \sqrt{k(s_3)-k(s_2)} & \sqrt{k(s_3)-k(s_2)} & \dots \\ 0 & 0 & 0 & \sqrt{k(s_4)-k(s_3)} & \dots \\ \cdot & \cdot & \cdot & \cdot & \dots \\ \cdot & \cdot & \cdot & \cdot & \dots \\ \cdot & \cdot & \cdot & \cdot & \dots \end{bmatrix}.$$

Starting with U_{ss} we get U_{ss}^C and conclude, since the principal subdeterminants of U_{ss}^C are nonnegative, that U_{ss} is nonnegative definite. Assuming equal

increments for the partition $\{s_p\}_{p=0}^n$, we have

$$(K_{ss}^C)^{-1} = \frac{1}{\sqrt{s_1}} \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & -1 & 0 & \dots \\ 0 & 0 & 1 & -1 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ . & . & . & . & \dots \\ . & . & . & . & \dots \\ . & . & . & . & \dots \end{bmatrix},$$

$$U_{ss}^C(K_{ss}^C)^{-1} = \frac{1}{\sqrt{s_1}} \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ 0 & \sqrt{k(s_2)} & 0 & 0 & \dots \\ 0 & 0 & \sqrt{k(s_3)-k(s_2)} & 0 & \dots \\ 0 & 0 & 0 & \sqrt{k(s_4)-k(s_3)} & \dots \\ . & . & . & . & \dots \\ . & . & . & . & \dots \\ . & . & . & . & \dots \end{bmatrix},$$

$$[U_{ss}^C(K_{ss}^C)^{-1}V_{ss}^C](i,\, j) = \begin{cases} 0 & \text{if } i = 1 \text{ or } j < i\,, \\ \frac{1}{\sqrt{s_1}}\sqrt{k_1(s_i)-k(s_{i-1})}\sqrt{k_2(s_i)-k(s_{i-1})} & \text{otherwise}\,, \end{cases}$$

$$[(U_{ss}^C(K_{ss}^C)^{-1}V_{ss}^C)'(U_{ss}^C(K_{ss}^C)^{-1}V_{ss}^C)](i,\, j) = \begin{cases} 0 & \text{if } i = 1 \text{ or } j < i\,, \\ \frac{1}{s_1}\sum_{\ell=2}^{i}(k_1(s_\ell)-k(s_{\ell-1}))(k_2(s_\ell)-k(s_{\ell-1})) & \text{otherwise}\,. \end{cases}$$

Note for $i \le j$ that

$$\begin{aligned}[(U_{ss}^C(K_{ss}^C)^{-1}V_{ss}^C)'(U_{ss}^C(K_{ss}^C)^{-1}V_{ss}^C)](i,\, j) &= [(U_{ss}^C(K_{ss}^C)^{-1}V_{ss}^C)'(U_{ss}^C(K_{ss}^C)^{-1}V_{ss}^C)](j,\, i) \\ &= [(U_{ss}^C(K_{ss}^C)^{-1}V_{ss}^C)'(U_{ss}^C(K_{ss}^C)^{-1}V_{ss}^C)](i,\, i)\,. \end{aligned}$$

2.4. A Note on Kernels for Fields with More Than Two Independent Variables

Suppose d is a positive integer and $\{k_i\}_1^d$ is a sequence of increasing functions on $[0,\, 1]$ with $k_i(0) = 0$, for $i = 1,\, \dots,\, d$. Let

$$R(\mathbf{u},\, \mathbf{v}) = \Pi_{i=1}^d k_i(u_i \wedge v_i)$$

for d-sequences $\mathbf{u} = \{u_i\}_{i=1}^d$ and $\mathbf{v} = \{v_i\}_{i=1}^d$ with values in [0, 1], where $a \wedge b = \min(a, b)$.

Theorem 7. *R is positive definite, i.e., if $\{\mathbf{u}^p\}_{p=1}^n$ is a sequence of d-sequences $\mathbf{u}^p$ with values in (0, 1] and $\{a_p\}_{p=1}^n$ is a nonzero sequence of real numbers then*

$$\sum_{p=1}^{n}\sum_{q=1}^{n} R(\mathbf{u}^p, \mathbf{u}^q)a_p a_q > 0\,.$$

Example. The Wiener field [4, 15] on $[0, 1]^d$ has covariance kernel

$$K(\mathbf{u}, \mathbf{v}) = \Pi_{i=1}^d (u_i \wedge v_i)\,.$$

Lemma 2. *If M is an $n \times n$ symmetric matrix such that:*

1. *$M(1, 1) > 0$,*
2. *$M(p, q) = M(q, p) = M(p, p)$, for $1 \le p \le q \le n$, and*
3. *$\{M(p, p)\}_{p=1}^n$ is an increasing sequence.*

Then M is positive definite.

Proof of Lemma 2. Note that

$$M^C = \begin{bmatrix} \sqrt{M(1,1)} & \sqrt{M(1,1)} & \sqrt{M(1,1)} & \sqrt{M(1,1)} & \dots \\ 0 & \sqrt{M(2,2)-M(1,1)} & \sqrt{M(2,2)-M(1,1)} & \sqrt{M(2,2)-M(1,1)} & \dots \\ 0 & 0 & \sqrt{M(3,3)-M(2,2)} & \sqrt{M(3,3)-M(2,2)} & \dots \\ 0 & 0 & 0 & \sqrt{M(4,4)-M(3,3)} & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \end{bmatrix}$$

is the upper Cholesky factor of M. Therefore the principal subdeterminants of M are positive and so M is positive definite. □

Proof for Theorem 7. Suppose $d = 2$. Let $U(p, q) = k_1(u_p \wedge u_q)$ and $V(p, q) = k_2(v_p \wedge v_q)$ for $1 \le p, q \le n$. Let $\mathbf{u}^p = (u_p, v_p)$ for $p = 1, \dots, n$ and

$$M(p, q) = R(\mathbf{u}^p, \mathbf{u}^q) = k_1(u_p \wedge u_q)k_2(v_p \wedge v_q) = U(p, q)V(p, q)$$

for $1 \le p, q \le n$. By Lemma 2, M is positive definite and the theorem follows for $d = 2$. The extension to $d > 2$ is immediate. Interest in building more complicated models from simpler components also arises in data driven studies. This pursuit is well represented by [6] which goes over some of the same ground covered in this paper. The viewpoint is quite different, however. Rather than considering the object random field as the result of a "visible" interaction of components, the modeling viewpoint, the field is analyzed in terms of available observables of the field itself. The goal of the research thrust is the most general representation of a random field that can be unraveled in terms of the

observables of the field. □

Our starting point is the class of separable random fields, the most tractable case. Since the envisioned application is to decision problems with uncertainty and risk, we are able to restrict our attention to an even smaller class of fields. The principal difficulty is developing an algebra for combining the simpler models. Dealing with this difficulty is a principal focus of this paper.

2.5. Covergence Results

In following subsections, we present some of the convergence results.

2.5.1. Central Limit Theorem

Assume that $\{u_p\}$ is a sequence of positive integers with limit ∞ and $\{\hat{X}_{jp}\}$ is a double sequence such that, for each positive integer p, $1 \le j \le u(p)$. Let $S_p = \sum_{j=1}^{u(p)} \hat{X}_{jp}$.

Theorem 8. (see Chung, 2001) *If $E\hat{X}_{jp} = 0$ and $E(|\hat{X}_{jp}|^3)$ is finite, for $p \ge 1$ and $1 \le j \le u(p)$, $\sum_{j=1}^{u(p)} E(\hat{X}_{jp})^2 = 1$, for $p \ge 1$, and $\sum_{j=1}^{u(p)} E(|\hat{X}_{jp}|^3)$ has limit 0 as $p \to \infty$ then $\{S_p\}$ converges in distribution to a (0, 1)-normal random variable.*

Suppose that $\{s_p\}$ is a sequence of partitions of [0, 1], $0 = s_0(0) < s_0(1) < \cdots < s_0(m) = 1$, s_{p+1} refines s_p, and $mesh(s_p) \to 0$ as $p \to \infty$. Assume that $\{u_i\}_{i=0}^m$ is a sequence of nondecreasing integer valued sequences such that: $s_p(u_i(p)) = s_0(i)$, for $0 \le p$ and $0 \le i \le m$. Note that $s_p(u_{(.)}(p)) = s_0$. Also, for $0 \le i \le m$, $u_i(p) \to \infty$ as $p \to \infty$. Further, suppose that $\{t_q\}$ is a sequence of partitions of [0, 1], $0 = t_0(0) < t_0(1) < \cdots < t_0(n) = 1$, t_{q+1} refines t_q, and $mesh(t_q) \to 0$ as $q \to \infty$. Assume that $\{v_j\}_{j=0}^n$ is a sequence of nondecreasing integer valued sequences such that $t_q(v_j(q)) = t_0(j)$, for $0 \le q$ and $0 \le j \le n$. Suppose that:

1. Z is a random function on [0, 1] × [0, 1] such that:

(a) $Z(s, t)$ is (0, 1)-normal.

(b) $Z(s, t)$ and $Z(u, v)$ are independent for $(s, t) \ne (u, v)$.

2. Each of k_L and k_R is a continuous increasing function on [0, 1], $k_L(0) = k_R(0) = 0$, and $k_L(1) = k_R(1) = 1$.

3. R_L and R_R are nonnegative definite functions on [0, 1] × [0, 1] defined

(a) $R_L(s, t) = k_L(\min(s, t))$.

(b) $R_R(s, t) = k_R(\min(s, t))$.

Theorem 9. *If i and j are fixed, $s = s_0(i)$, and $t = t_0(j)$, then*

$$S_{pq} = \sum_{\alpha=1}^{u_i(p)} \sum_{\beta=1}^{v_j(q)} \sqrt{k_L(s_p(\alpha)) - k_L(s_p(\alpha-1))} \times \sqrt{k_R(t_q(\beta)) - k_R(t_q(\beta-1))}\, Z_{s_p t_q}(\alpha, \beta)$$

has limit in distribution the $(0, (k_L(s)k_R(t))^{1/2})$-normal random variable $\int_0^t \int_0^s Z(\alpha, \beta)\,(dk_L(\alpha))^{1/2}(dk_R(\beta))^{1/2}$ as $p, q \to \infty$.

Proof of Theorem 9. If q is fixed and $1 \le \beta \le v_j(q)$ then the Central Limit Theorem allows us to conclude that

$$\sum_{\alpha=1}^{u_i(p)} \sqrt{k_L(s_p(\alpha)) - k_L(s_p(\alpha-1))} Z_{s_p t_q}(\alpha, \beta)$$

has limit in distribution the $(0, k_L(s)^{1/2})$-normal random variable

$$\int_0^s Z(\alpha, t_q(\beta))\,(dk_L(\alpha))^{1/2}$$

as $p \to \infty$. Further, we can conclude that

$$\sum_{\beta=1}^{v_j(q)} \sqrt{k_R(t_q(\beta)) - k_R(t_q(\beta-1))} \int_0^s Z(\alpha, t_q(\beta))\,(dk_L(\alpha))^{1/2}$$

has limit in distribution the $(0, (k_L(s)k_R(t))^{1/2})$-normal random variable $\int_0^t \int_0^s Z(\alpha, \beta)\,(dk_L(\alpha))^{1/2}(dk_R(\beta))^{1/2}$ as $q \to \infty$. Hence the result. □

Let $U = R_L$ and $V = R_R$. Then, for $0 \le i \le m$ and $0 \le j \le n$, $S_{pq} = [(U^C_{s_p s_p})^T Z_{s_p t_q} V^C_{t_q t_q}](u_i(p), v_j(q))$ has limit in distribution

$$\int_0^{t_0(j)} \int_0^{s_0(i)} Z(\alpha, \beta)\,(dk_L(\alpha))^{1/2}(dk_R(\beta))^{1/2}$$

as $p, q \to \infty$.

Note that a more general result is available, i.e., for more general R_L and R_R, but the limit will lack the nice integral representation. Assuming the setup above, i.e., $\{\{s_p\}, \{u_i\}, \{t_q\}, \{v_j\}, Z\}$, $U = R_L$, and $V = R_R$, then, for i and j fixed, $s = s_0(i)$, and $t = t_0(j)$, $S_{pq} = [(U^C_{s_p s_p})^T Z_{s_p t_q} V^C_{t_q t_q}](u_i(p), v_j(q))$ has limit in distribution a $(0, (R_L(s, s)R_R(t, t))^{1/2})$-normal random variable.

We can extend a continuous linear operator A on $\{G, Q\}$ with associated discretized covariances $U_{s_p s_p}$ and $V_{t_q t_q}$ to a domain including random functions

as follows:

$$[\hat{A}F](s,\,t) = [AE(F)](s,\,t) + \lim_{p}\lim_{q}[(U^C_{s_p s_p})^T (K^C_{s_p s_p})^{-T}(F - E(F))_{s_p t_q} \\ (K^C_{t_q t_q})^{-1} V^C_{t_q t_q}](u_i(p),\, v_j(q))\,.$$

We have to impose conditions on $F - E(F)$ in order satisfy the hypothesis of the Central Limit Theorem but we do not have to assume normality. On the other hand, for simulations we can substitute a zero mean normal field for $F - E(F)$ simplifying the numerics and arrive at the same limit. Because these extensions are always available, we concentrate on continuous linear operators defined on $\{G,\, Q\}$ for decision models developed in the next section.

3. Risk Analysis of an Uncertain Complex System

A system with at least two interacting components is a complex system. Broadly defined risk is the probability of an undesirable decision outcome. In our setup the distribution of outcomes depends on the uncertainties, our independent variables. The goal of decision making for complex systems is a subjective balance of expected payoff and risk. Risk is only half of the equation.

In a recent INFORMS Tutorial, Rockafellar [21] surveyed a variety of approaches to risk and the axioms of risk. The goal of the axioms is a coherent risk assessment, which we interpret as a consistent assessment leading to a consistent decision methodology. More on consistency later.

Rockafellar assumes uncertainties can be quantified, i.e., the randomness he chooses to categorize as uncertainties has a distribution. Our investigation of Ellsburg's paradox [20] leads us in a different direction. Uncertainties should be modeled as independent variables (no distributions) and decision problems with uncertainties should be resolved using multi-criteria decision methods.

The goal of balance can be achieved by either maximizing $\mu - \alpha\,\sigma$ or minimizing $\mu + \alpha\,\sigma$, where μ is the mean of a component performance, σ^2 is the variance of the performance, and α is a positive parameter set by the decision maker reflecting his/her risk tolerance (see [25]). Note that the probability that performance is less than $\mu - \sigma$, an undesirable event, is approximately 0.1586.

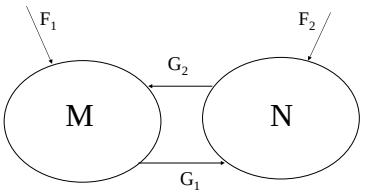

Figure 1: A two component model representing the interaction of two random fields F_1 and F_2

3.1. Network Representations of Complex Systems

The following example, Figure 1, has all of the essential features of more complex systems and will serve to illustrate general results. We want to study the interaction of random fields, in particular, estimates of $\mu_i - \alpha\,\sigma_i$, α a real parameter, computed from G_i, $i = 1, 2$ (see Figure 1). Our general approach is through the construction of more complex models by combining simpler components. Notice that the nodes (M and N) represent linear operators and the arcs (F_i and G_i) represent random fields. The model could be used for a simplified investment decision [19] or some other multicriteria decision problem [17, 18].

In these, and other decision problems, the domain of the random fields is interpreted as the space of uncertainties and the decision variable $\mu - \alpha\,\sigma$ as an acceptable balance of expected payoff and risk. Formally, taking M and N as linear operators on an appropriate space of fields, we have

$$G_1 = M(F_1 + G_2)\,, \quad G_2 = N(F_2 + G_1)\,,$$
$$(I - MN)G_1 = M(F_1 + NF_2)\,, \quad G_1 = (I - MN)^{-1}M(F_1 + NF_2)\,.$$

Thus we need to be able to deal with sums of fields and products and inverses of sums of operators. In general, the sum of two separable fields is not separable. At one level, this is not a problem since we can still express the covariance of the sum in terms of the covariances of each of the summands. However, for model building we need a single linear operator representation of the sum and this representation should be separable so we can combine the result with other operator representations of fields. We have to be content with separable approximations to the sum rather than separable representations.

3.2. Separable Approximations of Sums of Fields

There are two cases, i.e., we need separable approximations for a discrete random field H_{st} when either

1. $H_{st} = (A_L)^T Z_{st} A_R + (B_L)^T \bar{Z}_{st} B_R$, where Z_{st} and $\bar{Z}_{st}$ are independent (0, 1)-normal, and

2. $H_{st} = (A_L)^T Z_{st} A_R + (B_L)^T Z_{st} B_R$, where Z_{st} is (0, 1)-normal.

Referencing the example the first case arises, for instance in G_1, when $A_L = F_L^1$, $A_R = F_R^1$, $B_L = F_L^2 (K_{ss}^C)^{-1} N_L$ and $B_R = F_L^2 (K_{tt}^C)^{-1} N_R$. The second case arises, again in G_1, when $A_L = A_R = K_{ss}^C$, $B_L = N_L (K_{ss}^C)^{-1} M_L$, and $B_R = N_R (K_{tt}^C)^{-1} M_R$.

For the first case,

$$\begin{aligned}\operatorname{cov}([H_{st}](i,\,p),\,[H_{st}](j,\,q)) = {} & [(A_L)^T A_L](i,\,j)[(A_R)^T A_R](p,\,q) \\ & + [(B_L)^T B_L](i,\,j)[(B_R)^T B_R](p,\,q)\,.\end{aligned}$$

Note that for $i \le j$ and $p \le q$ we have

$$\operatorname{cov}([H_{st}](i,\,p),\,[H_{st}](j,\,q)) = \operatorname{cov}([H_{st}](i,\,p),\,[H_{st}](i,\,p))\,.$$

Therefore we seek increasing functions k_1 and k_2 on $[0,\,1]$ such that $k_1(0) = k_2(0) = 0$ and $\max_{i,\,p}(|\operatorname{cov}([H_{st}](i,\,p),\,[H_{st}](i,\,p)) - k_1(s_i)k_2(t_p)|)$ is as small as possible.

Suppose each of F and G is a random field on $[0,\,S] \times [0,\,T]$ with risk fields R_F and R_G, respectively. Our modeling assumption is that we only "observe" $F(\,\cdot\,,\,T)$ and $F(S,\,\cdot\,)$, i.e., we only "know" $R_F(\,\cdot\,,\,T)$ and $R_F(S,\,\cdot\,)$. Something similar holds for G. We say that F and G are *equivalent* if $R_F(\,\cdot\,,\,T) = R_G(\,\cdot\,,\,T)$ and $R_F(S,\,\cdot\,) = R_G(S,\,\cdot\,)$. Further, we say G is the *separable equivalent* of F if $R_G(s,\,t) = R_F(s,\,T)R_F(S,\,t)/R_F(S,\,T)$, for $(s,\,t)$ in $[0,\,S] \times [0,\,T]$. Note that the separable equivalent is always available.

Turning to the second case, suppose
$H_{st} = (A_L)^T (K_{ss}^C)^{-T} W_{st} (K_{tt}^C)^{-1} A_R + (B_L)^T (K_{ss}^C)^{-T} W_{st} (K_{tt}^C)^{-1} B_R$.

We begin by computing the covariance for H_{st}.

Theorem 10.

$$\begin{aligned}cov([H_{st}](i,\,p),\,[H_{st}](j,\,q)) = {} & [(A_L)^T A_L](i,\,j)[(A_R)^T A_R](p,\,q) \\ & + [(B_L)^T A_L](i,\,j)[(B_R)^T A_R](p,q) + [(A_L)^T B_L](i,\,j)[(A_R)^T B_R](p,q) \\ & + [(B_L)^T B_L](i,\,j)[(B_R)^T B_R](p,\,q)\,.\end{aligned}$$

Proof of Theorem 10. Using Lemma 1,

$$\begin{aligned}
&\operatorname{cov}([H_{st}](i,\,p),\,[H_{st}](j,\,q))\\
&= E([(A_L)^T Z_{st} A_R + (B_L)^T Z_{st} B_R](i,\,p)[(A_L)^T Z_{st} A_R + (B_L)^T Z_{st} B_R](j,\,q))\\
&\qquad = E([(A_L)^T Z_{st} A_R](i,\,p)[(A_L)^T Z_{st} A_R](j,\,q))\\
&\qquad + E([(A_L)^T Z_{st} A_R](i,\,p)[(B_L)^T Z_{st} B_R](j,\,q))\\
&\qquad + E([(B_L)^T Z_{st} B_R](i,\,p)[(A_L)^T Z_{st} A_R](j,\,q))\\
&\qquad + E([(B_L)^T Z_{st} B_R](i,\,p)[(B_L)^T Z_{st} B_R](j,\,q))\\
&\quad = [(A_L)^T A_L](i,\,j)[(A_R)^T A_R](p,\,q) + [(B_L)^T A_L](i,\,j)[(B_R)^T A_R](p,\,q)\\
&\quad + [(A_L)^T B_L](i,\,j)[(A_R)^T B_R](p,\,q) + [(B_L)^T B_L](i,\,j)[(B_R)^T B_R](p,\,q)\,.
\end{aligned}$$

3.3. Separable Approximations of Inverses of Sums of Operators

In the modeling context we require a separable approximation of $(I - A)^{-1}$, where A is separable. If we use the separable approximation B to $I - A$ then one of the basic results gives a separable representation of B^{-1}.

3.3.1. Separable Representation of the Inverse of a Sum: The Dependent Case

Suppose

$$H_{st} = W_{st} - (A_L)^T (K_{ss}^C)^{-T} W_{st} (K_{tt}^C)^{-1} A_R\,,$$

then

$$\begin{aligned}
&\operatorname{cov}([H_{st}](i,\,p),\,[H_{st}](i,\,p))\\
&\quad = [K_{ss}(i,\,i)[K_{tt}(p,\,p) - 2[(A_L)^T K_{ss}^C](i,\,i)[(A_R)^T K_{tt}^C](p,p)\\
&+ \quad [(A_L)^T A_L](i,\,i)[(A_R)^T A_R](p,\,p)\\
&\quad = [diag(K_{ss}) diag(K_{tt})^T](i,\,p)\\
&- \quad 2[diag((A_L)^T K_{ss}^C) diag((A_R)^T K_{tt}^C)^T](i,\,p)\\
&+ \quad [diag((A_L)^T A_L) diag((A_R)^T A_R)^T](i,\,p)\,.
\end{aligned}$$

From the basic result

$$((I - A)^{-1})_L \sim (B_L)^{-1} = K_{ss}^C (U_{ss}^C)^{-1} K_{ss}^C$$

and

$$((I - A)^{-1})_R \sim (B_R)^{-1} = K_{tt}^C (V_{tt}^C)^{-1} K_{tt}^C\,.$$

Thus given F_1 and F_2 we are able to simulate the discrete version of

$$G_1 = (I - MN)^{-1}M(F_1 + NF_2)$$

and hence can estimate $\mu_1 = EG_1$ and $\sigma_1^2 = var(G_1)$. In the same way we can estimate μ_2 and σ_2. These are simple tasks in the *MatLab* environment. We are ready to take up the decision process.

3.3.2. The Decision Process

The discussion of the example of a complex decision problem can be completed by briefly indicating how the modeling methodology which we have developed can be employed to resolve a hypothetical decision problem, choosing from a finite set of alternatives.

1. An outline for comparing decision alternatives for the example

(a) We assume the criteria interact with the interactions modeled by Figure 1.

(b) Associated with the j-th alternative will be a pair of continuous linear operators M^j and N^j.

(c) We assume the random external influences F_1 and F_2 are the same for all alternatives.

(d) The evaluation of the j-th alternative results in two random fields G_1^j and G_2^j, different for each alternative.

(e) We compute the mean and standard deviation fields μ_i^j and σ_i^j for G_i^j, $i = 1, 2$.

2. Two steps to problem resolution

(a) Multi-criteria optimization. Since the goal is to balance μ and σ, we would either consider maximizing $\mu - \sigma$ or minimizing $\mu + \sigma$ based on the example's objective, where μ is the mean of a component performance and σ^2 is the variance of the performance. Let us assume that we are interested in maximizing $\mu - \sigma$ for our decision problem represented by Figure 1. Therefore for each of the finite set of alternative, we compute $R_i^j = (\mu_i^j - \sigma_i^j)$, $i = 1, 2$.

Given two alternatives, in multi-criteria decision making for a minimization problem, Alternative 1 is said to *dominate* Alternative 2 if for all uncertainties $R_i^1 \leq R_i^2$ and for at least one discretized uncertainty $R_i^1 < R_i^2$. Otherwise, both alternatives are said to be *nondominated*.

The first step of problem resolution involves in finding all nondominated alternatives. If the first step provides an unique nondominated alternative, this

is the preferred alternative. On the other hand, if at the end of first step we have multiple nondominated alternatives, we go to the second step of the problem resolution.

(b) Identifying the preferred choice. If more than one alternative is nondominated, we do not have a clear preferred choice. To choose a preferred alternative we first define an *ideal alternative*. An ideal alternative is any infeasible alternative which clearly dominates all the nondominated alternatives from step one. Then, we use L_2 norm to pick a preferred alternative from the set of nondominated alternatives which is closest to the ideal alternative.

(3) Questions of consistency for our approach are resolved [20] by establishing a mapping between Savage's informal probability structure [24] and our preference rules. Only the first four of Savage's postulates hold but those suffice to show the methodology is rational.

The models open a new field of application to decision problems. Requirements for decision problems dictate our approach at various points: the discretization/matrix representations ease the computational burden, two dimensional domains represent a significant enlargement of the class of possible applications, the emphasis on risk comes from decision problems. Decisions in an environment of uncertainty and risk are ubiquitous. The modeling tools discussed in this paper enable us to consider larger, more complex problems. Problems for developing a decision making methodology are not discussed but would include efficient multi-criteria optimization algorithms and implementation of decision preference rules for multi-criteria decisions.

3.4. An Engineering Approach to Risk Analysis of Complex Systems

Engineering systems, such as submarines or power generation plants, can be composed of many parts. A method for estimating risk of failure for such complex systems proceeds as follows [9]. Estimate expected time to failure for each part, perhaps based on experiment. The possible failures are partitioned by significance, for example as major or minor. Risk for each category is the probability of failure during an operating period, for example the probability of a major failure during the next year. The probabilities increase over time limiting the useful "safe" lifetime of the system. Part of the engineering art is deciding when a series of minor failures can cascade into a major failure, for example the pattern of failures in the Three Mile Island incident. Risk estimates in this methodology are not functions of "uncertainties" although the estimates of future risk can be updated using operating histories. Many

aircraft operate long after the end of the original designed "lifespan" because of hardware updates and refurbishment. Whether our "soft" approach has anything to offer to "hard" engineering problems remains to be seen.

4. Conclusions

The paper is a mixture of hard results and plausible methods based on numerical experiments. The results are firmly based in RKH space theory but theory which is restricted to special cases. The numerical experiments need to be followed up with a more systematic study.

Separable approximations enable us to build complex models from simpler components. We have a complete algebra of separable components, at least in terms of approximations. Separable approximations to nonseparable fields (operators) obtained through addition seem unavoidable. The approximations are not heavily penalized, at least in numerical experiments, and so seem to be useful for modeling.

The approximations fit within an established framework, i.e., RKH spaces. Discretizations of various objects, for instance fields and covariance kernels, are not approximations but exact. The random fields of interest are not elements of the underlying RKH space. However, the covariance kernel of a random field provides the matrix representation of a linear transformation of the RKH space which can be extended to a larger space including the Wiener field enabling the representation of the random field. The representations provide a complete characterization of the zero mean random fields. The representations are more tractable, at least for our purposes, than distribution function representations, etc..

Convergence is a natural question for continuous parameter models. From a data viewpoint, the usual collection methodology starts with discrete random functions but some representations (transform methods) immediately progress to continuous parameter models. Denied the possibility of infinite sampling, the convergence question is mute.

Some optimization problems, at first glance, seem to be approachable as large mathematical programming problems after a suitable discretization. Such an approach leaves open the problem of convergence of the optimal solutions of the subproblems. A difficulty which is often overlooked. Our multi-criteria optimization problems on the discrete models might leave us open to similar criticisms. However, our discretizations leave us in the original RKH space

setting, i.e., we can claim that the discrete problem is the "real" problem.

Are we losing something by only modeling risk? Of course we are. Many different system models, models based on different covariances, can share the same risk profiles. However, if the decision is to turn on analysis of risk then we might as well choose the simpler models.

References

[1] N. Aronszajn, Theory of reproducing kernels, *Amer. Math. Soc. Trans.*, **68** (1950), 337-404.

[2] S.M. Berman, A version of the Levy-Baxter Theorem for the increments of Brownian motion of several parameters, *Proc. Amer. Math. Soc.*, **18** (1967), 1051-1055.

[3] B.K. Bhartia, E.H. Vanmarcke, Associate linear-system approach to non-linear random vibration, *J. Eng. Mech. – ASCE*, **117** (1991), 2407-2428.

[4] N.N. Chentsov, Wiener random fields depending on several parameters, *Dokl. Akad. Nauk SSSR(N.S.)*, **106** (1956), 607-609.

[5] K.L. Chung, *A Course in Probability Theory Revised*, Academic Press (2001).

[6] S. De Iaco, D.E. Myers, D. Posa, Space-time analysis using a general product-sum model, *Stat. and Prob. Letters*, **52** (2001), 21-28.

[7] E. Hellinger, *Die Orthogonalinvarianten Quadratischer Formen von Unendlichvielen Variablen*, Inaugural - Dissertation, Georg - August - Universität Göttingen (1907).

[8] P.C. Kyriakidis, A.G. Journel, Geostatistical space-time models: A review, *Mathematical Geology*, **31** (1999), 651-684.

[9] R.D. Leitch, *Reliability Analysis for Engineers: An Introduction*, Oxford University Press, Oxford (1995).

[10] P. Lévy, Le movement Brownien plan, *Amer. J. Math.*, **62** (1940), 487-550.

[11] P. Lévy, Sur le Brownian dépendent de plusieurs parametres, *C.R. Acad. Sci. Paris*, **220** (1945), 420-422.

[12] J.S. Mac Nerney, Finitely additive set functions, *Houston J. Math., Supplement* (1980).

[13] D. Monrad, On the p-variation of Gaussian random fields with separable increments, In: *Seminar on Stochastic Processes (1982)*, Birkhäuser, Boston (1983), 203-218.

[14] C.J. Naresh, D. Monrad, Gaussian measures in B_p, *Ann. Prob.*, **11** (1983), 46-57.

[15] M. Ossiander, E.C. Waymire, Certain positive-definite kernels, *Proc. Amer. Math. Soc.*, **107** (1989), 487-492.

[16] J.A. Reneke, Stochastic linearization of nonlinear point dissipative systems, *Internat. J. of Math. and Math. Sci.*, **65** (2004), 3541-3563.

[17] J.A. Reneke, M.J. Saltzman, M.M. Wiecek, Affordable upgrades of complex systems: A multilevel, performance-based approach, *Modeling Uncertainty: An Examination of Stochastic Theory, Methods, and Applications* (Ed-s: M. Dror, P. L'Ecuyer, F. Szidarovszky), Kluwer, Boston (2002), 301-331.

[18] J.A. Reneke, M.M. Wiecek, Performance-based multicriteria decision making for complex systems: An example, In: *Proceedings of the 6-th World Multiconference on Systemics, Cybernetics and Informatics SCI*, Volume XVIII, Information Systems Development III (2002), 292-298.

[19] J.A. Reneke, M.M. Wiecek,Research support decisions under conditions of uncertainty and risk, In: *Proceedings of the World Congress of Nonlinear Analysis* (2004).

[20] S. Samson, J.A. Reneke, Multicriteria approach resolving Ellsberg's paradox, submitted (2008).

[21] R.T. Rockafellar, Coherent approaches to risk in optimization under uncertainty, *Tutorials in Operation Research, INFORMS 2007* (2007).

[22] J. Rovnyak, Ernst David Hellinger 1883-1950: Göttingen, Frankfurt idyll, and the New World, *Topics in Operator Theory 48, Ernst D. Hellinger Memorial Volume* (Ed-s: L. de Branges, I. Gohberg, J. Rovnyak), Birkhäuser, Boston (1990), 1-41.

[23] P. Sampson, D. Damian, P. Guttorp, Advances in modeling and inference for environmental processes with nonstationary spatial covariance, In: *GeoENV III–Geostastistics for Environmental Applications* (Ed-s: P. Monestiez, D. Allard, R. Froideveaux), Kluwer, Dordrecht (2001), 17-32.

[24] L.J. Savage, *The Foundations of Statistics*, J. Wiley and Sons, New York and London (1954).

[25] S. Samson, J.A. Reneke, M.M. Wiecek, A survey and a new modeling paradigm for uncetainty and risk, *Reliability Engineering and System Safety*, submitted (2007).

[26] *Reproducing Kernel Hilbert Spaces: Applications in Statistical signal Processing* (Ed. H. Weinert), Benchmark Papers in Electrical Engineering and Computer Science, Volume 25, Hutchinson Ross Publishing Company, Stroudsburg, PA (1982).

[27] A.M. Yaglom, *Correlation Theory of Stationary and Related Random Functions I*, Springer-Verlag, New York (1986).

[28] E. Vanmarcke, *Random Fields: Analysis and Synthesis*, The MIT Press, Cambridge, MA (1983).