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ORTHOGONAL CLASSES AND THEIR MORE OR LESS CONCEALED SUBCLASSES

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Abstract: We investigate some rather small orthogonal classes with respect to functorial "approximations" of tilting or cotilting modules.

Dedicated to the memory of
Professor A.L.S. Corner.

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1. Introduction

Many tilting or cotilting–type objects, more or less concrete, for instance modules, bimodules and complexes, admit two types of definitions, either very short or reasonably long. In addition to this, short and long definitions have "dual" properties. On the one hand, exactly one "continuous" property, concerning the relationship between two classes of modules, shows up in some short definitions of tilting or cotilting modules. On the other hand, either two or three "discrete" properties, concerning two special modules (plus direct summands of their direct sums or products) show up in some long definitions of the same modules, in some of their generalizations, and in the classical definition of tilting modules given by Brenner and Butler in [6].

In this paper we investigate some functorial "approximations" of tilting or

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cotilting modules, that is the so called "large partial tilting or cotilting modules". We may sum up as follows the omnipresence of these modules. Sometimes (under a "global" hypothesis on the "minimal size" of certain orthogonal classes [13, Theorems 3 and 4]), the following facts happen in the big world of direct summands of tilting or cotilting modules.

— Many direct summands of tilting or cotilting modules keep a very strong functorial property, involving the whole category of modules, namely the property of being "large partial tilting or cotilting".

— Some obvious direct summands of tilting or cotilting modules (that is some indecomposable projective – injective modules with a "big" complement) play an inessential functorial role. More precisely, we can cancel them, without losing the property of being "large partial tilting or cotilting".

The strategy used in [13] leads to the construction of modules of small "size", defined over algebras of finite Representation Type. In this situation, the big gap between our "approximations" and the tilting or cotilting modules is very easy to recognize, if use the "discrete" and long definitions of tilting or cotilting modules. As we shall see, the examples constructed in this note (where we almost always use "continuous" and short definitions of tilting or cotilting modules), lead to the conclusion that the above gap may be bigger than expected, and less easy to check. To justify the previous assertion, we point out two "practical" reasons, based on the visualization of more or less concealed classes of modules in the best possible situation, that is inside finite Auslander–Reiten quivers.

— An orthogonal class and a proper subclass, formed by modules generated or cogenerated in a special way, may have the same indecomposable modules except one (Proposition 4).

— Projective or injective indecomposable modules are not enough to compare orthogonal classes and their subclasses involved in short definitions of partial tilting or cotilting modules (Example D(d); Proposition 4(iii)).

This paper is organized as follows. In Section 2, we recall some definitions and we fix the notation used in the sequel. In Section 3, we construct some examples and we compare orthogonal classes and some of their subclasses.

2. Preliminaries

Let R be a ring. We denote by $R - Mod$ the category of all left R–modules. If $M \in R - Mod$, then we write $\operatorname{Add} M$ (resp. $\operatorname{Prod} M$) for the class of all modules isomorphic to direct summands of direct sums (resp. direct products) of copies of M. Next, for every cardinal λ, we denote by $M^{(\lambda)}$ (resp. M^{λ}) the direct sum (resp. direct product) of λ copies of M. Moreover, for any $n \geq 1$, we denote by $\mathrm{Gen}_n(M)$ (resp. $\mathrm{Cogen}_n(M)$) the class of all modules X such that there is an exact sequence of the form

$$M^{(\alpha_n)} \to \cdots \to M^{(\alpha_2)} \to M^{(\alpha_1)} \to X \to 0$$

$$(\text{resp.} \qquad 0 \to X \to M^{\alpha_1} \to M^{\alpha_2} \to \cdots \to M^{\alpha_n})$$

for some cardinals $\alpha_1, \ldots, \alpha_n$. On the other hand, we denote by $\mathrm{Gen}_\infty(M)$ (resp. $\mathrm{Cogen}_\infty(M)$) the following module:

$$\mathrm{Gen}_\infty(M) = \cap_{i\geq 1} \mathrm{Gen}_i(M) \quad (\text{resp. } \mathrm{Cogen}_\infty(M) = \cap_{i\geq 1} \mathrm{Cogen}_i(M)).$$

Finally, we write $M^{\perp\infty}$ (resp. ${}^{\perp\infty}M$) for the following orthogonal class:

$$M^{\perp\infty} = \{X \in R-Mod |\, \mathrm{Ext}^i_R(M, X) = 0 \text{ for every } i \geq 1\}$$

$$(\text{resp. } {}^{\perp\infty}M = \{X \in R-Mod |\, \mathrm{Ext}^i_R(X, M) = 0 \text{ for every } i \geq 1\}).$$

We shall say that an R–module T is a *partial n–tilting module* if the following conditions hold:

— The projective dimension of T is at most n;

— $\mathrm{Ext}^i_R(T, T^{(\lambda)}) = 0$ for every $i \geq 1$ and every cardinal λ.

Given a partial n–tilting module T, we shall say that T is an *n–tilting module* if there is a long exact sequence of the form

$$0 \to R \to T_0 \to T_1 \to \cdots \to T_n \to 0,$$

where $T_i \in \mathrm{Add}\, T$ for every $i = 0, \ldots, n$. From now on, we shall say, for brevity, that a partial n–tilting module is a *large partial n–tilting module* if $\mathrm{Ker}\,\mathrm{Hom}(T, -) \cap T^{\perp\infty} = 0$.

Dually, we say that an R–module C is a *partial n-cotilting module* if the following conditions hold:

— The injective dimension of C is at most n;

— $\mathrm{Ext}^i_R(C^{\lambda}, C) = 0$ for every $i \geq 1$ and every cardinal λ.

Given a partial n–cotilting module C, we shall say that C is an *n–cotilting module* if there is a long exact sequence of the form

$$0 \to C_n \to \cdots \to C_1 \to C_0 \to E \to 0,$$

where E is an injective cogenerator of $R-Mod$ and $C_i \in \mathrm{Prod}\, C$ for every $i = 0, \ldots, n$. In the sequel, we shall say, for brevity, that a partial n–cotilting module is a *large partial n–cotilting module* if $\mathrm{Ker}\,\mathrm{Hom}(-, C) \cap {}^{\perp\infty}C = 0$.

Maintaining the notation and terminology introduced above, we recall the compact and elegant characterizations of Bazzoni [5], namely the short definitions of some of the modules already defined by means of longer definitions.

— (see [5, Proposition 3.6]) If T is an n–tilting module, then $\mathrm{Gen}_n(T) = \mathrm{Gen}_\infty = T^{\perp_\infty}$. Dually, if C is an n–cotilting module, then $\mathrm{Cogen}_n(C) = \mathrm{Cogen}_\infty(C) = {}^{\perp_\infty}C$.

— (see [5, Lemma 3.12]) Let T (resp. C) be a partial n–tilting (resp. n–cotilting) module. Then $\mathrm{Gen}_n(T) \subseteq T^{\perp_\infty}$ (resp. $\mathrm{Cogen}_n(C) \subseteq {}^{\perp_\infty}C$), and T (resp. C) is an n–tilting (resp. n–cotilting) module iff $T^{\perp_\infty}$ is generated by T (resp. ${}^{\perp_\infty}C$ is cogenerated by C).

We list in the sequel some results on large partial tilting or cotilting modules.

— For every $n \geq 1$, every n–tilting module T (resp. n–cotilting module C) is a large partial n–tilting (resp. n–cotilting) module [5, p. 371].

— A finitely generated module T is a 1–tilting module if and only if T is a large partial 1–tilting module [7, Theorem 1] (also see [8, Theorem 2.3.1 and Section 3.1]).

— A module C is a 1–cotilting module if and only if C is a large partial 1–cotilting module (see [3, Proposition 2.3], [9, Theorem 1.7], [10], and [8, Section 2.5]).

— Every $n \geq 2$ is the projective (resp. injective) dimension of a non faithful large partial n–tilting (resp. n–cotilting) module M, such that the orthogonal class $M^{\perp_\infty}$ (resp. ${}^{\perp_\infty}M$) is as small as possible [13, Example 5].

We also recall that faithful large partial 2–tilting (resp. large partial 2–cotilting) modules are not necessarily 2–tilting (resp. 2–cotilting) modules [11, Corollary 6]. On the other hand, for every large partial n-tilting module T, we clearly have

— $\mathrm{Hom}(T, I) \neq 0$ for every non–zero injective module I.

Dually, for every large partial n–cotilting module C, we obviously have

— $\mathrm{Hom}(P, C) \neq 0$ for every non–zero projective module P.

Consequently, if there are only finitely many simple modules, then every large partial tilting (resp. cotilting) module M of finite length is *sincere* [4], that is every simple module appears as a composition factor of M.

It actually occurs that every simple module S has multiplicity one as a composition factor of a large partial m–tilting or m–cotilting Λ–module M, annihilated by I, with one of the following properties:

— M is not faithful, Λ has finite representation type, and M is not an n-tilting or n–cotilting over Λ/I for any n [13, Example 7].

— Λ and Λ/I do not have the same representation type, and the Λ/I–module M is a projective –injective generator and cogenerator [13, Proposition 8].

It is easy to construct proper large partial n–tilting (resp. n–cotilting) modules M such that the indecomposable modules belonging to $M^{\perp_\infty}$ (resp. ${}^{\perp_\infty}M$) are either injective (resp. projective) or summands of M [13, Examples 5, 6 and 7]. As we shall see, it is also easy to find large partial n–tilting (resp. n–cotilting) modules M such that, non obvious modules belong or do not belong to $\mathrm{Gen}_n(M)$ and $M^{\perp_\infty} \setminus \mathrm{Gen}_n(M)$ (resp. $\mathrm{Cogen}_n(M)$ and ${}^{\perp_\infty}M \setminus \mathrm{Cogen}_n(M)$). Indeed, if T (resp. C) is a large partial n–tilting (resp. n–cotilting) module, then the following facts may occur:

— $\mathrm{Gen}_n(T)$ (resp. $\mathrm{Cogen}_n(C)$) is the class of all injective (resp. projective) modules (see conditions (a) and (b) of Examples A and D).

— $\mathrm{Gen}_i(T)$ and $\mathrm{Cogen}_i(C)$ are not closed under direct summands for every $i = 2, \ldots, n$ (see condition (d) of Examples B and C).

In the sequel, K always denotes an algebraically closed field, and we always identify modules and their isomorphism classes. Moreover, if Λ is a K–algebra given by a quiver and relations, according to [18], then we often replace indecomposable finite dimensional modules by some obvious pictures, describing their composition factors. In particular, over a representation–finite algebra given by a quiver, we often denote by x the simple module corresponding to the vertex x.

For unexplained terminology, we refer to [1] and [4]. We refer to [8, 14, 15, 17, 19, 20, 21, 23] for other, more or less combinatorial aspects of tilting–type objects. Finally, we refer to the Handbook of Tilting Theory [2] for the interplay between Tilting Theory and other parts of mathematics.

3. Examples and Proofs

We shall often use the following preliminary lemma.

Lemma 1. *Let A be a representation–finite K–algebra of finite global dimension m. The following facts hold:*

(i) The regular module ${}_AA$ is an n–cotilting module for some $n \leq m$.

(ii) The injective cogenerator ${}_AD = \mathrm{Hom}_K(A_A, K)$ is an n–tilting module

for some $n \leq m$.

Proof. See [11, Lemmas 1 and 2]. □

We begin with an example of global dimension two.

Example A. *There exist a representation–finite algebra Λ of global dimension two, and two Λ–modules T and C with the following properties:*

(a) T (resp. C) is a faithful large partial 2–tilting (resp. 2–cotilting) module.

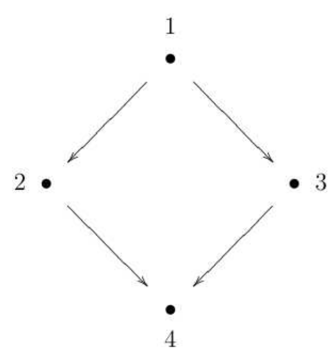
(b) $\mathrm{Gen}_2(T)$ (resp. $\mathrm{Cogen}_2(C)$) is the class of all injective (resp. projective) modules.

(c) $\mathrm{Gen}_\infty(T) = \mathrm{Add}(T) = \mathrm{Gen}_3(T) \neq \mathrm{Gen}_2(T)$ *and*
$$\mathrm{Cogen}_\infty(C) = \mathrm{Add}(C) = \mathrm{Cogen}_3(C) \neq \mathrm{Cogen}_2(C).$$

(d) $T^{\perp_\infty} \setminus \mathrm{Gen}_2(T)$ (resp. ${}^{\perp_\infty}C \setminus \mathrm{Cogen}_2(C)$) contains exactly one indecomposable module M, and M is projective (resp. injective).

(e) $\mathrm{Gen}_2(T) \setminus \mathrm{Gen}_\infty(T)$ (resp. $\mathrm{Cogen}_2(C) \setminus \mathrm{Cogen}_\infty(C)$) contains exactly one indecomposable module N, and N is simple.

Proof. (Construction) Let A denote the K–algebra given by the quiver



with 4 arrows $\alpha_1, \ldots, \alpha_4$ and relations $\alpha_i \alpha_j = 0$ for every i and j. Next, let T and C denote the following modules:
$$T = \begin{smallmatrix} 2\,3 \\ 4 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 3 \end{smallmatrix} \qquad \text{and} \qquad C = \begin{smallmatrix} 1 \\ 2\,3 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 4 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \end{smallmatrix}.$$
Thus (a) holds [11, Corollary 6 and Corollary 7 (a)].

Moreover, we obviously have $1 \in \mathrm{Gen}_2(T)$; 2 , $3 \notin \mathrm{Gen}_2(T)$; $4 \in \mathrm{Cogen}_2(C)$ and 2 , $3 \notin \mathrm{Cogen}_2(C)$. Hence also (b) holds.

On the other hand, $\begin{smallmatrix} 1 \\ 2\,3 \end{smallmatrix}$ (resp. $\begin{smallmatrix} 2\,3 \\ 4 \end{smallmatrix}$) is the unique indecomposable non injective (resp. non projective) module belonging to $T^{\perp_\infty}$ (resp. ${}^{\perp_\infty}C$). Consequently (d) holds. We finally observe that the indecomposable modules generated by T (resp. cogenerated by C) are $\begin{matrix}2\ 3\\ 4\end{matrix}$, $\begin{matrix}1\\ 2\end{matrix}$, $\begin{matrix}1\\ 3\end{matrix}$, 1 , 2 , 3 (resp. $\begin{matrix}1\\ 2\ 3\end{matrix}$, $\begin{matrix}2\\ 4\end{matrix}$, $\begin{matrix}3\\ 4\end{matrix}$, 2 , 3 , 4). Since $\mathrm{Gen}_2(T)$ and $\mathrm{Cogen}_2(C)$ do not contain 2 and 3, we conclude that $1 \notin \mathrm{Gen}_3(T) = \mathrm{Add}(T) = \mathrm{Gen}_\infty(T)$ and $4 \notin \mathrm{Cogen}_3(C) = \mathrm{Add}(C) = \mathrm{Cogen}_\infty(C)$. Therefore (c) holds, and the simple module $N = 1$ (resp. $N = 4$) satisfies (e). □

As we shall see in the next examples, the classes $\mathrm{Gen}_n(X)$ and $\mathrm{Cogen}_n(X)$ fail to be closed under direct summands even for very special modules X.

Example B. *For every $n \geq 3$ there is a module T with the following properties:*

(a) T is faithful large partial n–tilting module of projective dimension n.

(b) $\mathrm{Gen}_2(T)$ contains any injective module.

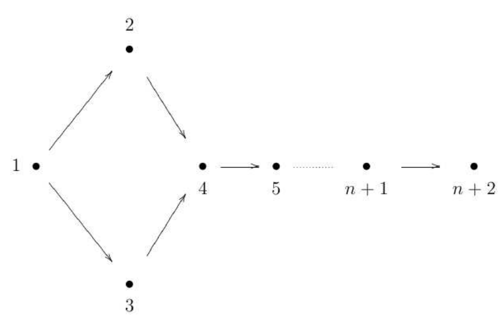
(c) $\mathrm{Gen}_\infty(T) = \mathrm{Add}(T) = \mathrm{Gen}_{n+1}(T) \neq \mathrm{Gen}_n(T)$.

(d) $\mathrm{Gen}_i(T)$ is not closed under direct summands for every $i = 2, \dots, n$.

(e) $T^{\perp_\infty} \setminus \mathrm{Gen}_2(T)$ (resp. $T^{\perp_\infty} \setminus \mathrm{Gen}_n(T)$) contains exactly one (resp. two) indecomposable module M (resp. modules M and L), and M is projective.

(f) $\mathrm{Gen}_\infty(T)$ and $\mathrm{Gen}_n(T)$ have the same indecomposable modules.

Proof. (Construction) Let B denote the algebra, obtained from the algebra A in Example A after $n-2$ one–point coextensions, given by the quiver



with $n+2$ arrows $\alpha_1, \dots, \alpha_{n+2}$ satisfying the relations $\alpha_i \alpha_j = 0$ for every i and j. Let T denote the following injective module of projective dimension n:

$$T = \begin{matrix}1\\ 2\end{matrix} \oplus \begin{matrix}1\\ 3\end{matrix} \oplus \begin{matrix}2\ 3\\ 4\end{matrix} \oplus \begin{matrix}4\\ 5\end{matrix} \oplus \cdots \oplus \begin{matrix}n+1\\ n+2\end{matrix}.$$

Then the simple projective module $n+2$ is the unique indecomposable module belonging to $\operatorname{Ker}\operatorname{Hom}_B(T,-)$. On the other hand, by dimension shifting, we have $\operatorname{Ext}^1_B(n+1,-) \simeq \operatorname{Ext}^{n-1}_B\left(\begin{smallmatrix}2\ 3\\4\end{smallmatrix},-\right)$. Consequently, we obtain $\operatorname{Ker}\operatorname{Hom}(T,-)\cap T^{\perp_\infty}=0$. Hence (a) follows from Lemma 1.

Since the sequence $\begin{smallmatrix}2\ 3\\4\end{smallmatrix} \longrightarrow \begin{smallmatrix}1\\2\end{smallmatrix} \longrightarrow 1 \longrightarrow 0$ is exact, we clearly have $1\in \operatorname{Gen}_2(T)$. Thus also (b) holds. Next, let $0\to X\to T'\to 1\to 0$ and $0\to Y\to T''\to i\to 0$ be exact sequences with $T',T''\in \operatorname{Add}(T)$ and $i=2,3$. Then X (resp. Y) has a direct summand isomorphic to $2,3$ or $\begin{smallmatrix}1\\2\ 3\end{smallmatrix}$ (resp. $\begin{smallmatrix}2\\4\end{smallmatrix}$ or $\begin{smallmatrix}3\\4\end{smallmatrix}$). Since $\begin{smallmatrix}1\\2\ 3\end{smallmatrix}, \begin{smallmatrix}2\\4\end{smallmatrix}, \begin{smallmatrix}3\\4\end{smallmatrix} \notin \operatorname{Gen}_1(T)$ and $2,3\notin \operatorname{Gen}_2(T)$, we obtain

Claim 1. *$1\notin \operatorname{Gen}_3(T)$ and $2,3\notin\operatorname{Gen}_2(T)$.* *(1)*

We also note that

Claim 2. *$n+2\notin\operatorname{Gen}_1(T)$ and $i\in\operatorname{Gen}_{n+2-i}(T)\setminus\operatorname{Gen}_{n+3-i}(T)$ for every $i=4,\dots,n+1$.* *(2)*

Finally, the following sequences are exact:

Claim 3. $\begin{smallmatrix}n+1\\n+2\end{smallmatrix} \to\cdots\to \begin{smallmatrix}4\\5\end{smallmatrix} \to \begin{smallmatrix}2\ 3\\4\end{smallmatrix} \to \begin{smallmatrix}1\\2\end{smallmatrix}\oplus\begin{smallmatrix}1\\3\end{smallmatrix} \to 1\oplus 1 \to 0\,.$
(3)

Claim 4. $\begin{smallmatrix}n+1\\n+2\end{smallmatrix} \longrightarrow\dots\longrightarrow \begin{smallmatrix}4\\5\end{smallmatrix} \longrightarrow \begin{smallmatrix}2\ 3\\4\end{smallmatrix} \longrightarrow 2\oplus 3 \longrightarrow 0\,.$ *(4)*

Consequently, we have

Claim 5. *$1\oplus 1\in\operatorname{Gen}_n(T)$ and $2\oplus 3\in\operatorname{Gen}_{n-1}(T)$.* *(5)*

Putting (1) and (5) together, we conclude that (c), (d) and (f) hold. Moreover, as in Example A, it its easy to see that $\begin{smallmatrix}1\\2\ 3\end{smallmatrix}$ is the unique indecomposable non injective module belonging to $T^{\perp_\infty}$.

By (1) and (b), this remark implies that the modules $M=\begin{smallmatrix}1\\2\ 3\end{smallmatrix}$ and $L=1$ satisfy condition (e). □

Example C. *For every $n\geq 3$ there is a module C with the following properties:*

(a) C *is a faithful large partial* n*–cotilting module of injective dimension* n*.*

(b) $\mathrm{Cogen}_2(C)$ *contains any projective module.*

(c) $\mathrm{Cogen}_\infty(C) = \mathrm{Add}(C) = \mathrm{Cogen}_{n+1}(C) \neq \mathrm{Cogen}_n(C)$*.*

(d) $\mathrm{Cogen}_i(C)$ *is not closed under direct summands for every* $i = 2, \dots, n$*.*

(e) ${}^{\perp\infty}C \setminus \mathrm{Cogen}_2(C)$ *(resp.* ${}^{\perp\infty}C \setminus \mathrm{Cogen}_n(C)$*) contains exactly one (resp. two) indecomposable module* M *(resp. modules* M *and* L*), and* M *is injective.*

(f) $\mathrm{Cogen}_\infty(C)$ *and* $\mathrm{Cogen}_n(C)$ *have the same indecomposable modules.*

Proof. (Construction) Let B denote the algebra considered in Example B, and let R denote the algebra B^{op}. Finally, with obvious notation, let C be the following projective R–module:

$$C = \begin{matrix} 2 \\ 1 \end{matrix} \oplus \begin{matrix} 3 \\ 1 \end{matrix} \oplus \begin{matrix} 4 \\ 2\,3 \end{matrix} \oplus \begin{matrix} 5 \\ 4 \end{matrix} \oplus \begin{matrix} n+2 \\ n+1 \end{matrix} .$$

Now, (a), (b), (c), (d) and (f) follow, by duality, from the properties of the module T constructed in Example B. Moreover, the modules $M = \begin{matrix} 2\,3 \\ 1 \end{matrix}$ and $L = \ 1$ satisfy condition (e). □

Some "minimal" examples of proper large partial 2–tilting (resp. 2–cotilting) modules M (for instance, Example A and [13, Examples 5 and 6 with $n = 2$]) have the following property:

Claim 6. *Every indecomposable module* $X \in M^{\perp\infty} \setminus \mathrm{Gen}_2(M)$ *(resp.* $X \in {}^{\perp\infty}M \setminus \mathrm{Cogen}_2(M)$*) is either projective or injective.* *(+)*

To see that (+) is a rather special property, it suffices to consider the following example.

Example D. *There exists a representation–finite algebra* Λ *of global dimension two and two* Λ*–modules* T *and* C *with the following properties:*

(a) T *(resp.* C*) is a faithful large partial 2–tilting (resp. 2–cotilting) module.*

(b) $\mathrm{Gen}_2(T)$ *(resp.* $\mathrm{Cogen}_2(C)$*) is the class of all injective (resp. projective) modules.*

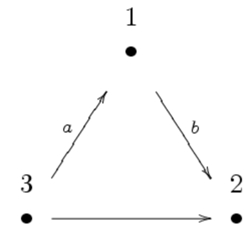
(c) $\mathrm{Gen}_\infty(T) = \mathrm{Add}(T) = \mathrm{Gen}_3(T) \neq \mathrm{Gen}_2(T)$ *and*
$\mathrm{Cogen}_\infty(C) = \mathrm{Add}(C) = \mathrm{Cogen}_3(C) \neq \mathrm{Cogen}_2(C)$*.*

(d) $T^{\perp\infty} \setminus \mathrm{Gen}_2(T)$ *(resp.* ${}^{\perp\infty}C \setminus \mathrm{Cogen}_2(C)$*) contains exactly one indecomposable module* M*, and* M *is neither projective nor injective.*

(e) $\mathrm{Gen}_2(T) \setminus \mathrm{Gen}_\infty(T)$ *(resp.* $\mathrm{Cogen}_2(C) \setminus \mathrm{Cogen}_\infty(C)$*) contains exactly*

one indecomposable module N, and N is simple.

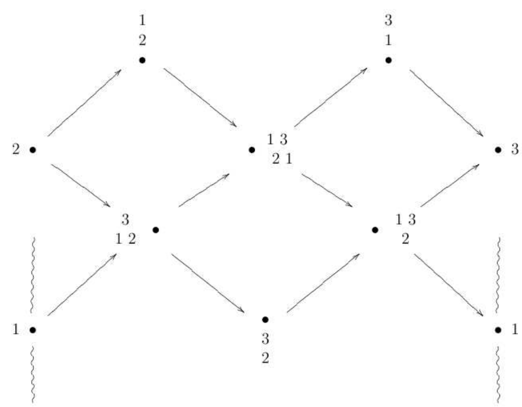
Proof. (Construction) We will use the algebra defined in Ringel's paper [22, page 97], that is the K–algebra Λ given by the quiver:



with relation $ba = 0$. Next, let T and C denote the following modules:

$$T = \begin{matrix} 1\,3 \\ 2 \end{matrix} \oplus \begin{matrix} 3 \\ 1 \end{matrix} \qquad \text{and} \qquad C = \begin{matrix} 3 \\ 1\,2 \end{matrix} \oplus \begin{matrix} 1 \\ 2 \end{matrix}.$$

Then the global dimension of Λ is equal to 2, and coincides with the projective (resp. injective) dimension of T (resp. C). Moreover, it is easy to check (and it follows from Lemma 1) that T (resp. C) is a partial 2–tilting (resp. 2–cotilting) module. We also note that the Auslander–Reiten quiver of Λ is of the form



Hence the following facts hold:

(1) The indecomposable modules belonging to $\operatorname{Ker}\operatorname{Hom}(T,-)$ (resp. $\operatorname{Ker}\operatorname{Hom}(-,C)$) are $\begin{matrix} 1 \\ 2 \end{matrix}$, $\begin{matrix} 3 \\ 2 \end{matrix}$ and 2 (resp. $\begin{matrix} 3 \\ 1 \end{matrix}$, $\begin{matrix} 3 \\ 2 \end{matrix}$ and 3).

(2) $\operatorname{Ext}^1_\Lambda\left(\begin{matrix} 3 \\ 1 \end{matrix}, \begin{matrix} 1 \\ 2 \end{matrix} \right) \neq 0$.

(3) $\mathrm{Ext}^2_\Lambda\left(\begin{matrix}1\,3\\2\end{matrix}, X\right) \simeq \mathrm{Ext}^1_\Lambda\left(1 \oplus 2, X\right) \neq 0$ if $X = \begin{matrix}3\\2\end{matrix},\ 2$.

(4) $\mathrm{Ext}^2_\Lambda\left(X, \begin{matrix}3\\1\,2\end{matrix}\right) \simeq \mathrm{Ext}^1_\Lambda\left(X, 1 \oplus 3\right) \neq 0$ if $X = \begin{matrix}3\\2\end{matrix},\ 3$.

Putting (1), (2), (3) and (4) together, we conclude that T (resp. C) is a large partial 2–tilting (resp. 2–cotilting) module. Hence (a) holds.

On the other hand, the following sequences are exact:

Claim 7. $\begin{matrix}1\,3\\2\end{matrix} \longrightarrow \begin{matrix}3\\1\end{matrix} \longrightarrow 3 \longrightarrow 0$;

$0 \longrightarrow 2 \longrightarrow \begin{matrix}1\\2\end{matrix} \longrightarrow \begin{matrix}3\\1\,2\end{matrix}$.

Since $1 \notin \mathrm{Gen}_2(T)$ and $1 \notin \mathrm{Cogen}_2(C)$, we have $3 \notin \mathrm{Gen}_3(T)$, $2 \notin \mathrm{Cogen}_3(C)$ and $\begin{matrix}1\,3\\2\end{matrix}, \begin{matrix}3\\1\end{matrix}$ and 3 (resp. $\begin{matrix}3\\1\,2\end{matrix}, \begin{matrix}1\\2\end{matrix}$ and 2) are the indecomposable modules belonging to $\mathrm{Gen}_2(T)$ (resp. $\mathrm{Cogen}_2(C)$). Therefore also (b), (c) and (e) hold. Moreover, we clearly have

Claim 8. $\mathrm{Ext}^1_\Lambda(T, 1) \neq 0$, $\mathrm{Ext}^1_\Lambda(1, C) \neq 0$ *and* $\mathrm{Ext}^1_\Lambda(T, X) \neq 0$ *(resp.* $\mathrm{Ext}^1_\Lambda(X, C) \neq 0$*) for every indecomposable projective (resp. injective) module* X*.*

Now let M denote the module $\begin{matrix}1\,3\\2\,1\end{matrix}$. Then we deduce from the Auslander–Reiten formula [4, Proposition 4.6] that $\mathrm{Ext}^1_\Lambda(T, M) = 0$ and $\mathrm{Ext}^1_\Lambda(M, C) = 0$.

Since the projective and injective dimensions of $\begin{matrix}1\,3\\2\,1\end{matrix}$ are equal to one, we have

Claim 9. $\begin{matrix}1\,3\\2\,1\end{matrix} \in T^{\perp_\infty}$ *and* $\begin{matrix}1\,3\\2\,1\end{matrix} \in {}^{\perp_\infty}C$.

Consequently, by (2), (3), (4), (6) and (7), $M = \begin{matrix}1\,3\\2\,1\end{matrix}$ is the unique indecomposable non injective (resp. non projective) module belonging to $T^{\perp_\infty}$ (resp. ${}^{\perp_\infty}C$). Hence we deduce from (b) that $\begin{matrix}1\,3\\2\,1\end{matrix}$ is the unique indecomposable module belonging to $T^{\perp_\infty} \setminus \mathrm{Gen}_2(T)$ and to ${}^{\perp_\infty}C \setminus \mathrm{Cogen}_2(C)$. Thus also (d) holds. □

As we shall see, there are quite different large partial n–tilting (resp. n–

cotilting) modules X such that $\mathrm{Gen}_\infty(X) = \mathrm{Gen}_n(X)$ and $X^{\perp_\infty} \setminus \mathrm{Gen}_\infty(X)$ (resp. $\mathrm{Cogen}_\infty(X) = \mathrm{Cogen}_n(X)$ and ${}^{\perp_\infty}X \setminus \mathrm{Cogen}_\infty(X)$) contains exactly one indecomposable module.

Example E. *There are non faithful modules T, C, V and W with the following properties:*

(a) T (resp. C) is a non injective (resp. non projective) large partial 2–tilting (resp. 2–cotilting) module.

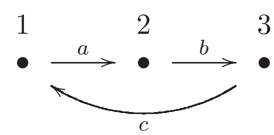
(b) $\mathrm{Gen}_\infty(T) = \mathrm{Gen}_2(T) \neq \mathrm{Add}(T)$ and $\mathrm{Cogen}_\infty(C) = \mathrm{Cogen}_2(C) \neq \mathrm{Add}(C)$.

(c) V (resp. W) is an injective (resp. a projective) large partial 3–tilting (resp. 3–cotilting) module.

(d) $\mathrm{Gen}_\infty(V) = \mathrm{Gen}_3(V) = \mathrm{Add}(V)$ and $\mathrm{Cogen}_\infty(W) = \mathrm{Cogen}_3(W) = \mathrm{Add}(W)$.

(e) $T^{\perp_\infty} \setminus \mathrm{Gen}_\infty(T)$ and $V^{\perp_\infty} \setminus \mathrm{Gen}_\infty(V)$ (resp. ${}^{\perp_\infty}C \setminus \mathrm{Cogen}_\infty(C)$ and ${}^{\perp_\infty}W \setminus \mathrm{Cogen}_\infty(W)$) contain exactly one indecomposable module M, and M is projective–injective.

Proof. (Construction) Let Λ be the K–algebra given by a quiver of the form



with relations $ac = 0$ and $cb = 0$. Next, let T, C, V and W denote the following modules:

$$T = \begin{matrix}1\\2\\3\end{matrix} \oplus 3\ , \quad C = \begin{matrix}1\\2\\3\end{matrix} \oplus 1\ , \quad V = \begin{matrix}1\\2\\3\end{matrix} \oplus \begin{matrix}1\\2\end{matrix}\ , \quad W = \begin{matrix}1\\2\\3\end{matrix} \oplus \begin{matrix}2\\3\end{matrix}\ .$$

Then (a) holds [13, Example 7], and

Claim 10. $\begin{matrix}1\\2\\3\end{matrix}$, 3 *and* $\begin{matrix}1\\2\end{matrix}$ *(resp.* $\begin{matrix}1\\2\\3\end{matrix}$, 1 *and* $\begin{matrix}2\\3\end{matrix}$*) are the indecomposable modules in $\mathrm{Gen}_2(T)$ (resp. $\mathrm{Cogen}_2(C)$).*

Consequently, also (b) holds. Moreover, the module $L = V \oplus \begin{matrix}3\\1\end{matrix}$ (resp. $L = W \oplus \begin{matrix}3\\1\end{matrix}$) is an injective 3-tilting (resp. a projective 3–cotilting) module such that $L^{\perp_\infty}$ (resp. ${}^{\perp_\infty}L$) is the class of all injective (resp. projective) modules. This remark and [13, Theorems 3 and 4] guarantee that (c) holds. This means that

Claim 11. $\begin{smallmatrix}1\\2\\3\end{smallmatrix}$, $\begin{smallmatrix}1\\2\end{smallmatrix}$ *and* $\begin{smallmatrix}3\\1\end{smallmatrix}$ *(resp.* $\begin{smallmatrix}1\\2\\3\end{smallmatrix}$, $\begin{smallmatrix}2\\3\end{smallmatrix}$ *and* $\begin{smallmatrix}3\\1\end{smallmatrix}$*) are the indecomposable modules belonging to* $V^{\perp_\infty}$ *(resp.* ${}^{\perp_\infty}W$*).*

We also observe that

Claim 12. *3 (resp. 1) is the unique indecomposable module belonging to* $T^{\perp_\infty} \setminus V^{\perp_\infty}$ *(resp.* ${}^{\perp_\infty}C \setminus {}^{\perp_\infty}W$*).*

Since $1 \notin \mathrm{Gen}_2(V)$ and $3 \notin \mathrm{Cogen}_2(W)$, we have

Claim 13. $\mathrm{Gen}_2(V) = \mathrm{Add}(V)$ *and* $\mathrm{Cogen}_2(W) = \mathrm{Add}(W)$.

Hence also (d) holds. Putting (1), (2), (3) and (4) together, we deduce from (b) and (d) that the module $M = \begin{smallmatrix}3\\1\end{smallmatrix}$ satisfies condition (e). □

We can now prove

Proposition 2. *Let T (resp. C) be a large partial m–tilting (resp. m–cotilting) module. Assume* $\mathrm{Gen}_\infty(T) = \mathrm{Gen}_r(T) \neq \mathrm{Gen}_{r-1}(T)$ *(resp.* $\mathrm{Cogen}_\infty(C) = \mathrm{Cogen}_r(C) \neq \mathrm{Cogen}_{r-1}(C)$*) for some $r \geq 2$. Then we may have $r = m$, $r > m$ and $r < m$.*

Proof. Let T and C be the module constructed in Example E. Then conditions (a) and (c) tell us that we have $r = m = 2$. Next, let T and C be the modules constructed in Examples A, B and C. Then, by conditions (a) and (b), we have $r = m + 1$ and $m \geq 2$. Finally, for any $n > 2$, two uniserial modules of the form

$$T = \begin{matrix}2\\ \vdots\\ n\\ 1\end{matrix} \qquad \text{and} \qquad C = \begin{matrix}1\\ 2\\ \vdots\\ n\end{matrix}$$

(defined over a Nakayama algebra considered in [16, Example 3.2]) satisfy the hypotheses of Proposition 2 with $r = 2$ and $m = 2n - 2$ (see [12, Example 7] and [13, Example 6]). □

In the next statement we compare other subclasses of left or right orthogonal classes.

Proposition 3. *Let L (resp. M) be a large partial n–tilting (resp. n–cotilting) module such that L (resp. M) is not an n–tilting (resp. n–cotilting) module. The following cases may occur:*

(i) $\mathrm{Gen}_\infty(L) = \mathrm{Gen}_n(L) = \mathrm{Add}(L)$ *and* $\mathrm{Cogen}_\infty(M) = \mathrm{Cogen}_n(M) = \mathrm{Add}(M)$.

(ii) $\mathrm{Gen}_\infty(L) = \mathrm{Gen}_n(L) \neq \mathrm{Add}(L)$ *and* $\mathrm{Cogen}_\infty(M) = \mathrm{Cogen}_n(M) \neq \mathrm{Add}(M)$.

(iii) $\mathrm{Gen}_\infty(L) \neq \mathrm{Gen}_n(L)$, $\mathrm{Cogen}_\infty(M) \neq \mathrm{Cogen}_n(M)$, *but every module in* $\mathrm{Gen}_n(L) \setminus \mathrm{Gen}_\infty(L)$ *and in* $\mathrm{Cogen}_n(M) \setminus \mathrm{Cogen}_\infty(M)$ *is decomposable.*

(iv) $\mathrm{Gen}_n(L) \setminus \mathrm{Gen}_\infty(L)$ *and* $\mathrm{Cogen}_n(M) \setminus \mathrm{Cogen}_\infty(M)$ *contain exactly one indecomposable module.*

Proof. (i) Let L and M be the modules V and W constructed in Example E. Then the conclusion that (i) holds follows from conditions (c) and (d) in Example E.

(ii) Let L and M denote the modules T and C constructed in Example E. Now the conclusion that L and M satisfy (ii) follows from conditions (a) and (b) in Example E.

(iii) Let L and M be the modules T and C constructed in Examples B and C, respectively. Then conditions (a), (c) and (f) of these examples guarantee that (iii) holds.

(iv) Let L and M denote the modules T and C constructed in Example A. Then (iv) follows from conditions (a) and (e) in Example A. □

The next result shows that quite different indecomposable modules occur as the unique indecomposable modules of rather special classes of the form $M^{\perp_\infty} \setminus \mathrm{Gen}_n(M)$ or ${}^{\perp_\infty}M \setminus \mathrm{Cogen}_n(M)$.

Proposition 4. *Let T (resp. C) be a large partial n–tilting (resp. n–cotilting) module. Assume* $T^{\perp_\infty} \setminus \mathrm{Gen}_n(T)$ *(resp.* ${}^{\perp_\infty}C \setminus \mathrm{Cogen}_n(C)$*) contains exactly one indecomposable module X (resp. Y). Then the following facts may occur:*

(i) $X = Y$ is projective–injective.

(ii) X is projective but not injective, and Y is injective but not projective.

(iii) $X = Y$ is neither projective nor injective.

Proof. (i) This follows from conditions (a) and (e) of Example E, where $X = Y$ is the module $\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}$.

(ii) This follows from conditions (a) and (d) of Example A, where we have $X = \begin{smallmatrix} 1 \\ 2\ 3 \end{smallmatrix}$ and $Y = \begin{smallmatrix} 2\ 3 \\ 4 \end{smallmatrix}$.

(iii) This follows from conditions (a) and (d) of Example D, where $X = Y$ is the module $\begin{smallmatrix} 1\ 3 \\ 2\ 1 \end{smallmatrix}$. □

We end with a result on semisimple and homogeneous modules.

Proposition 5. *Let T (resp. C) be a large partial n–tilting (resp. n–cotilting) module. Let S be a simple module such that $S \oplus S \in \mathrm{Gen}_n(T)$ (resp. $S \oplus S \in \mathrm{Cogen}_n(C)$). Then S does not necessarily belong to $\mathrm{Gen}_n(T)$ (resp. $\mathrm{Cogen}_n(C)$).*

Proof. Let T and C be the modules constructed in Examples B and C respectively. Then our proof of condition (d) shows that the simple injective (resp. projective) module $S = 1$ has all the desired properties. □

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