International Journal of Pure and Applied Mathematics
Volume 45 No. 2 2008, 171-173

A NOTE ON PHASE RETRIEVAL OF H^2-FUNCTIONS

John N. McDonald
Department of Mathematics
Arizona State University
Tempe, AZ 85287-1804, USA
e-mail: mcdonald@math.la.asu.edu

Abstract: This note concerns the problem of determining an H^2-function g when $|g(x)|$ is known for almost all x. It is shown that the conditions $(*)$ $|g(x)| = |f(x)|$ and $(**)$ $|g(qx) - g(x)| = |f(qx) - f(x)|, 0 < q < 1$ together imply that either $g = Vf$ or $g = V\bar{f}$, where V satisfies $V(qx) = V(x)$. Under certain restrictions on f and g the function V reduces to a constant.

AMS Subject Classification: 94A12, 30D20
Key Words: phase retrieval, Hardy space of the upper half-plane

1. Introduction

The *phase retrieval* problem is to determine as much as possible about the function g when f is known and

$$|f(x)| = |g(x)|. \tag{1}$$

(to see why this problem is interesting see, e.g., [5], [6], [8], [9]). By itself (1) is usually insufficient to yield much information about g. Hence additional conditions are required. In [7] we showed that for functions f and g in H^2 the conditions (1) and

$$|f(x + b) - f(x)| = |g(x + b) - g(x)| \tag{2}$$

suffice to yield $g = Wf$ or $g = W\bar{f}$, where W has period b and reduces to a constant in certain cases.

Received: November 25, 2007

In this note we show that the methods of [7] yield somewhat stronger results when the condition (2) is replaced by

$$|f(qx) - f(x)| = |g(qx) - g(x)|, \tag{3}$$

where $q \in (0, 1)$.

2. Main Results

Let the operator γ be defined by $\gamma h(x) = h(qx) - h(x)$, where q is a constant in $(0, 1)$.

Theorem 1. *Suppose $f, g \in H^2$ and $|f(x)| = |g(x)|$ and $|\gamma f(x)| = |\gamma g(x)|$ for almost all real x. Then, either*

$$g(qx)f(x) = g(x)f(qx) \tag{4}$$

for almost all x, or

$$g(qx)\overline{f(x)} = g(x)\overline{f(qx)}. \tag{5}$$

for almost all x.

Proof. The proof is the same as the proof of Theorem 1 of [7] where the operator δ is replaced by γ and the operation $x \to x + b$ is replaced by $x \to qx$. □

Corollary 1. *Suppose that $f, g \in H^2$ satisfy (1) and (3), that f and g are continuous at 0, and that $f(0) \neq 0$. Then there is a constant c such that $g = cf$.*

Proof. Suppose that (4) holds, then for $|x|$ sufficiently small

$$\frac{g(x)}{f(x)} = \frac{g(qx)}{f(qx)} = \cdots = \frac{g(q^n x)}{f(q^n x)}$$

letting $n \to \infty$ yields $g(x) = \frac{g(0)}{f(0)} f(x)$ for x in an open interval containing 0. Hence, $g(x) = \frac{g(0)}{f(0)} f(x)$ for almost all x (see [4]).

If (5) holds, a similar argument shows $g(x) = \frac{g(0)}{\overline{f}(0)} \overline{f}(x)$ for almost all x. Hence, $\overline{f} \in H^2$. It follows that f vanishes identically. □

With methods analogous to those in [7] the following can also be proved.

Theorem 2. *Suppose $f, g \in H^2$ have singular factors satisfying condition (10) of [7]. Suppose that conditions (1), (3) and (4) are satisfied. Then there* *are Blaschke products B_1 and B_2 and a real constant β such that*

$$g(x) = e^{\imath\beta} B_1(x)\overline{B}_2(x)f(x),$$

and $B_i(qx) = B_i(x)$ for almost all x and for $i = 1, 2$.

Theorem 3. *Suppose $f, g \in H^2$ have singular factors satisfying condition (10) of [7]. Suppose that conditions (1), (3) and (5) are satisfied. Suppose that there is a Blaschke product B such that $B(qx) = B(x)$ for almost all x and $B\overline{f} \in H^2$. Then there are Blaschke products B_1 and B_2 and a real constant β such that*

$$g(x) = e^{\imath\beta} B_1(x)\overline{B}_2(x)\overline{f}(x),$$

and $B_i(qx) = B_i(x)$ for almost all x and for $i = 1, 2$.

References

[1] E.J. Akutowicz, A qualitative characterization of Blaschke products in a half plane, *American J. of Math.*, **78** (1956), 677-684

[2] E.J. Akutowicz, On the determination of the phase of Fourier integral I, *Trans. American Math. Soc.*, **83** (1956), 179-192

[3] E.J. Akutowicz, On the determination of the phase of Fourier integral II, *Proc. American Math. Soc.*, **8** (1957), 234-238

[4] P.L. Duren, *Theory of H^p Spaces*, Academic Press, New York (1970).

[5] N.E. Hurt, Phase Retrieval and zero crossings (mathematical methods in image reconstruction), *Math. and its Appl.*, Kluwer Academic Publisher, Amsterdam (1989).

[6] P. Jaming, Phase retrieval techniques for radar ambiguity problems, *J. Fourier Anal. Appl.*, **5**, No. 4 (1999), 309-329.

[7] J.N. Mc Donald, Phase retrieval of H^2 functions, *J. Math. Anal. Appl.*, **314** (2004), 152-173

[8] J. Rosenblatt, Phase retrieval, *Comm. Math. Phys.*, **95** (1984), 317-343.

[9] A. Walter, The question of phase retrieval in optics, *Opt. Acta.*, **10** (1963), 41-49.