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LIMITATIONS OF ADOMIAN DECOMPOSITION AND HOMOTOPY METHODS

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Abstract: It is usual to resort to Adomian decomposition method or to homotopy analysis method for the solution of nonlinear problem. We show by considering a simple example that, in the absence of an asymptotic analysis, these methods fail to provide any significant information beyond a finite interval.

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Key Words: Adomian decomposition method, homotopy analysis, nonlinear problem

1. Introduction

If a problem contains a small parameter, the perturbation method prove to be very useful in the solution [5]. If no such parameter exists, Adomian introduce his decomposition method [1, 2, 3] which has been applied to solve some nonlinear problems, see for example [7, 8]. Liao introduce a technique, called the homotopy analysis method which contains the Adomian decomposition method as a special case [4]. In this method there are several free parameters which can be chosen so that the solution becomes valid over a large domain. However if the initial approximation is not properly chosen both these methods yield power series which converge only for a finite interval and fail to provide any information concerning large values of the independent variable. In view of this, one may conclude that claims of these two methods are sometimes exaggerated.

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We shall consider a simple example to show that the power series produces by the Adomian decomposition method fails to show any oscillatory behavior which is the hall mark of the actual solution to the problem. Similarly no matter how we choose the arbitrary parameter h in the homotopy analysis method, the resulting function fails to oscillate. On the other hand a simple asymptotic analysis of the problem yields an approximate solution which duly exhibits all important features of the true solution.

2. Simple Problem

To highlight the deficiencies of the Adomian and the homotopy analysis method, we consider a very simple nonlinear problem

$$\frac{d^2y}{dx^2} + y^2 = x\,, \tag{1}$$

$$y(0) = 0, \qquad y'(0) = 0. \tag{2}$$

First we find an approximate analytical solution for large x. Consider a transformation

$$y = u + \sqrt{x}\,. \tag{3}$$

Then

$$y' = u' + \frac{1}{2}x^{-\frac{1}{2}}\,, \quad y'' = u'' - \frac{1}{4}x^{-\frac{3}{2}}\,, \tag{4}$$

and

$$y^2 = u^2 + x + 2u\sqrt{x}\,. \tag{5}$$

For $x \gg 1$, the second term on the right of (4) may be dropped. Also we assume $|u| \ll \sqrt{x}$, an assumption justified later. Therefore we drop u^2 in (5) and equation (1) is transformed to a linear equation

$$u'' + 2\sqrt{x}\,u = 0\,. \tag{6}$$

Now it is well-known that the equation

$$x^2\frac{d^2y}{dx^2} + (1-2s)x\frac{dy}{dx} + [(s^2 - r^2\alpha^2) + a^2r^2x^{2r}]y = 0 \tag{7}$$

has a general solution

$$y = x^s[c_1J_\alpha(ax^r) + c_2Y_\alpha(ax^r)]\,, \tag{8}$$

see [6]. Multiply (6) with x^2. We get

$$x^2u'' + 2x^{\frac{5}{2}}u = 0\,. \tag{9}$$

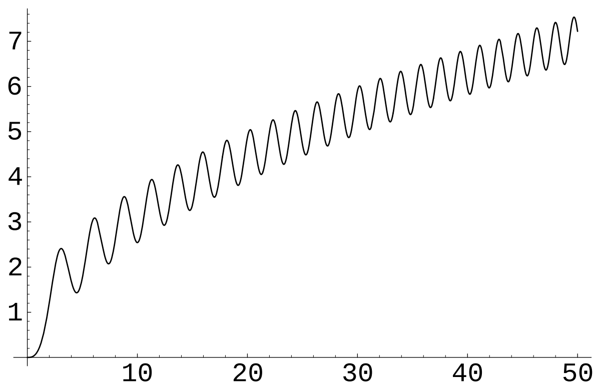

Figure 1: Numerical solution of equation (1)

Compare (7) with (9). We get

$$1-2s=0\,,\quad s^2-r^2\alpha^2=0\,,\quad 2r=\frac{5}{2}\,,\quad a^2r^2=2\,.$$

Therefore $s=\frac{1}{2}$, $r=\frac{5}{4}$, $\alpha=\frac{2}{5}$, $a=\frac{4\sqrt{2}}{5}$ and the general solution of (6) is

$$u=\sqrt{x}[c_1J_{\frac{2}{5}}(\frac{4\sqrt{2}}{5}x^{\frac{5}{4}})+c_2Y_{\frac{2}{5}}(\frac{4\sqrt{2}}{5}x^{\frac{5}{4}})]\,.$$

Since the Bessel functions are oscillatory with decreasing amplitude such that

$$J_\alpha(x)\to 0 \qquad \text{as } x\to\infty\,,$$
$$\text{and}\quad Y_\alpha(x)\to 0 \qquad \text{as } x\to\infty\,.$$

our assumption that $|u|\ll\sqrt{x}$ is justified for large x.

From (3) we see that the solution of equation (1), for large x, will be small oscillations superimposed on the parabola $y=\sqrt{x}$.

In Figure 1 we present the numerical solution of the problem which confirms our qualitative analysis. In Figure 2 we present the solution along with the curve $y=\sqrt{x}$.

3. Solution by Adomian Method

If we solve the problem by Adomian decomposition method, we get the following solution

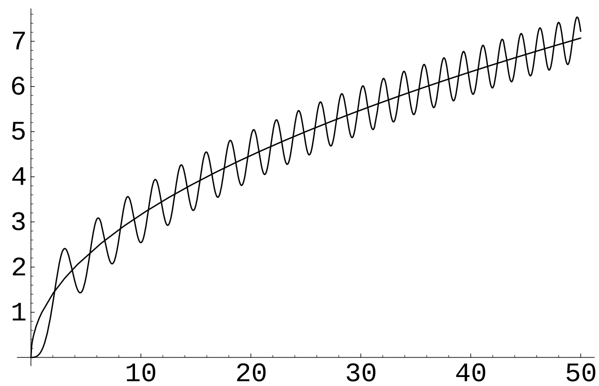

Figure 2: The solution of equation (1) together with $y = \sqrt{x}$.

$$
\begin{aligned}
y = {} & \frac{x^3}{6} - \frac{x^8}{2016} + \frac{x^{13}}{943488} - \frac{95x^{18}}{48502831104} + \frac{31x^{23}}{9203412201984} \\
& - \frac{74849x^{28}}{13507369511417413632} + \frac{189251x^{33}}{21395673306085183193088} \\
& - \frac{38700523x^{38}}{2801579230592089310166515712} + \frac{1061168785x^{43}}{50011106613036187601787024310272} \\
& - \frac{1527994261379x^{48}}{47494608245909822416547577359008530432} \\
& + \frac{986872159994333x^{53}}{20444184729624559311214550180022959197913088} \\
& - \frac{3659317869389202813517x^{58}}{50960628520373191763798652378101228449208971561009152} \\
& + \frac{63328266525722085749107x^{63}}{597156645001733061088192608566590194967830728751905243136} . \qquad (10)
\end{aligned}
$$

In Figure 3 we plot the above solution. It is clear that the series diverges when $x > 3.5$ and the oscillatory behavior depicted in Figures 1 and 2 is nowhere to be seen. If we add one more term to the solution it becomes

$$
\begin{aligned}
y = {} & 0.166667x^3 - 0.000496032x^8 + 1.0599 \times 10^{-6}x^{13} \\
& - 1.95865 \times 10^{-9}x^{18} + 3.36832 \times 10^{-12}x^{23} - 5.54135 \times 10^{-15}x^{28} \\
& + 8.84529 \times 10^{-18}x^{33} - 1.38138 \times 10^{-20}x^{38} + 2.12187 \times 10^{-23}x^{43} \\
& - 3.2172 \times 10^{-26}x^{48} + 4.82715 \times 10^{-29}x^{53} - 7.18068 \times 10^{-32}x^{58}
\end{aligned}
$$

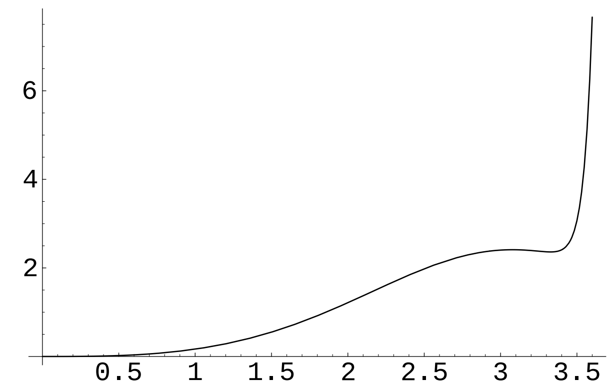

Figure 3:

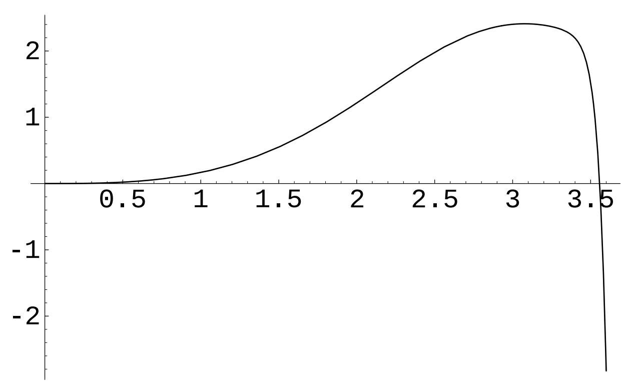

Figure 4:

$$+ 1.0605 \times 10^{-34} x^{63} - 1.55666 \times 10^{-37} x^{68} \quad (11)$$

The graph of (11) shown in Figure 4. Again the divergence of the series is clear beyond $x = 3.3$.

4. Homotopy Analysis Method

It has been shown by Liao that the Adomian solution up to m terms

$$y = \sum_{n=0}^{m} a_n x^n$$

can be improved by the homotopy analysis method to

$$y = \sum_{n=0}^{m} \mu_{m,n}(h) a_n x^n \,, \tag{12}$$

where $\mu_{m,n}(h)$ is called an approaching function and is defined as

$$\mu_{m,n}(h) = (-h)^n \sum_{k=0}^{m-n} \begin{pmatrix} n-1+k \\ k \end{pmatrix} (1+h)^k \,, \tag{13}$$

where h is arbitrary and should be chosen so that the solution (12) agrees with the exact solution over as wide an interval as possible.

If we choose $m = 7$, the homotopy solution becomes

$$\begin{aligned} y = &\frac{1}{6}(1-h)(1+h+h^2+h^3+h^4+h^5+h^6)x^3 \\ &- \frac{(1-h)^2(1+2h+3h^2+4h^3+5h^4+6h^5)}{2016}x^8 \\ &+ \frac{(1-h)^3(1+3h+6h^2+10h^3+15h^4)}{943488}x^{13} \\ &- \frac{95(1-h)^4(1+4h+10h^2+20h^3)}{48502831104}x^{18} + \frac{31(1-h)^5(1+5h+15h^2)}{9203412201984}x^{23} \\ &- \frac{74849(1-h)^5(1+5h+15h^2)}{135073695114174136 32}x^{28} + \frac{189251(1-h)^6(1+6h)}{2139567330608518319308 8}x^{33} \,. \end{aligned}$$

We plot the above solution for various values of h and observe that the homotopy solution also suffers from the convergence problem. In any case it fails to produce the oscillatory behavior of the solution shown in Figure 1 and 2 .

We note that the homotopy solution in Figures 8-10. produces a zero but the exact solution is free of zeros. This shows that a bad choice of the parameter h can lead to a solution which is totally wrong even for small values of x.

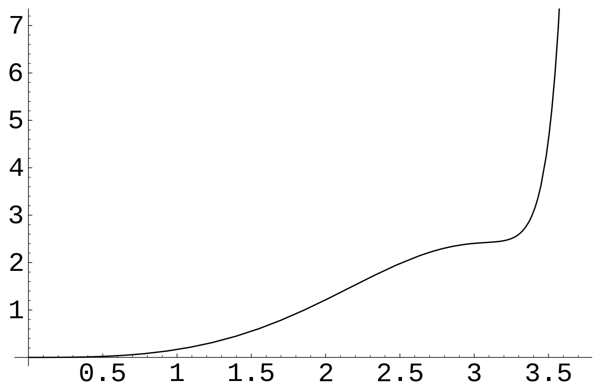

Figure 5: Homotopy solution with $h = 0.1$

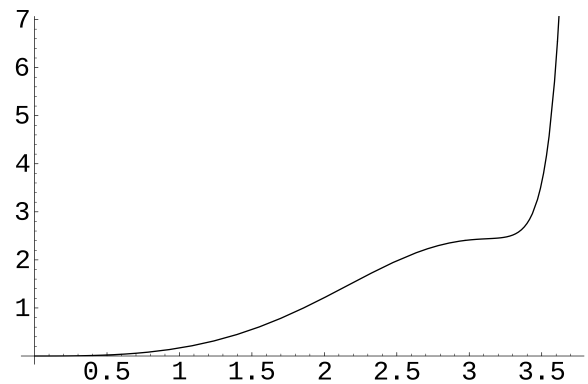

Figure 6: Homotopy solution with $h = -0.1$

5. Conclusion

By applying the Adomian and homotopy analysis methods to our simple nonlinear problem we have shown that these methods produce series solutions which converge over a relatively short interval and it is impossible to get any information for large x. In some cases these methods can yield misleading information regarding the solution of the problem as, for example, the homotopy analysis method in Figures 8-10 gives a zero of the solution whereas the true solution

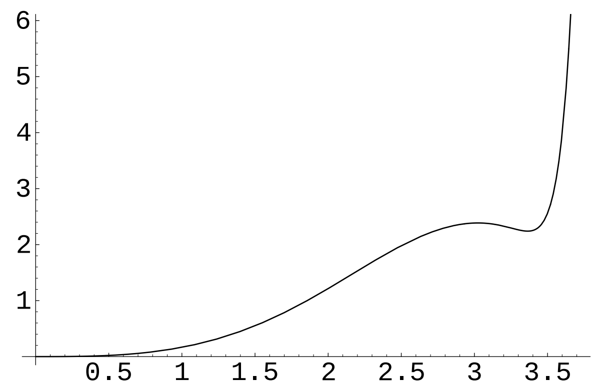

Figure 7: Homotopy solution with $h = 0.2$

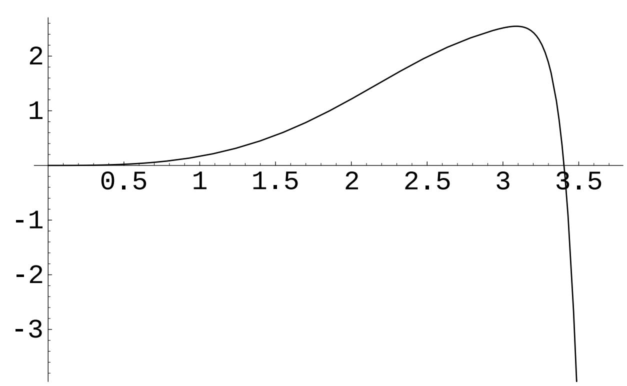

Figure 8: Homotopy solution with $h = -0.2$

possesses no zero at all. Even if we find a series solution containing a thousand terms, it does not exhibit any oscillation because the oscillatory behavior sets in at a stage which is outside the interval of convergence of the series. It appears that in order to get full information about the solution one must choose an initial function with more care or one should make an asymptotic analysis of the problem at hand.

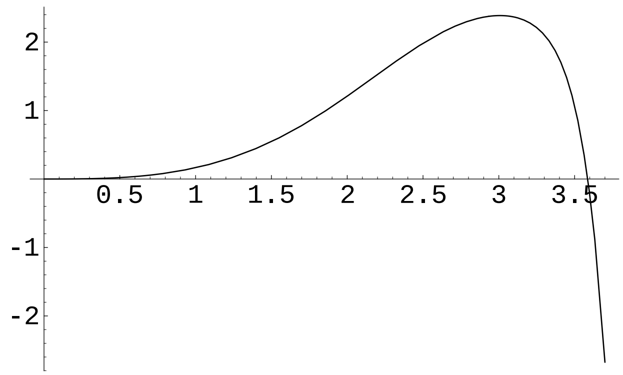

Figure 9: Homotopy solution with $h = 0.5$

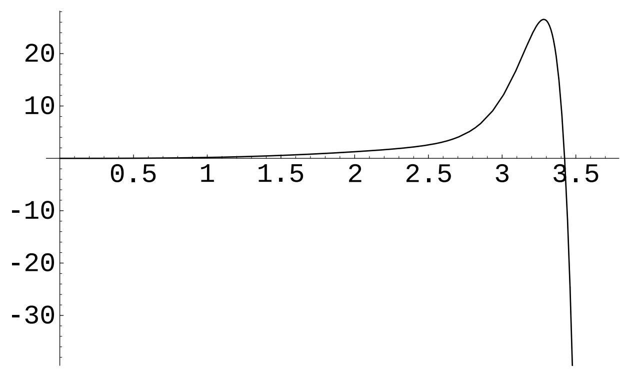

Figure 10: Homotopy solution with $h = -0.5$

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