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USING CLUSTER MULTIFUNCTIONS FOR DECOMPOSITION THEOREMS

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Abstract: In the paper we present some properties of a cluster multifunction and the relations between original multifunction and various cluster multifunctions depending on choice of a cluster system. The main result of paper concerns using cluster multifunction for decomposition theorems.

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1. Introduction and Basic Notations

The cluster set was first time studied for functions in the last years of 19-th century. Later the notion of cluster set was enlarged for approximative cluster sets (studied by Zajicek [10]), qualitative cluster set, unilateral cluster set and so one (see Thomson [9]). Recently some unified approaches were introduced for multifunction and resultant cluster sets have been studied as multifunctions (introduced by Matejdes [6], Layek [4], Richter and Stephani [8]).

Throughout the article we use standard notation: X, Y denote topological space, respectlively, $F : X \to Y$ means a multi-valued mapping from X to Y, briefly multifunction. It can be empty-valued at some points. By Dom (F) we denote domain of F, i.e. the set of all points $x \in X$ in which $F(x)$ is non-empty. If F is single-valued mapping we will use notation f understood

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as a multifunction with the values $\{f(x)\}$. The graph of multifunction F is denoted by Gr $(F) = \{[x,y] \in X \times Y, y \in F(x)\}$. A multifunction G is a submultifunction of F ($G \subset F$) if $G(x) \subset F(x)$ for any $x \in X$. Symbols Int(A), Cl (A) denote interior and closure of a set A, $F^+(V) = \{x \in X : F(x) \subset V\}$, $F^-(V) = \{x \in X : F(x) \cap V \neq \emptyset\}$ is upper, lower inverse image of a set $V \subset Y$, respectively. By $F(A)$ we denote the set $\bigcup_{a \in A} F(a)$. We will used the next standard definitions of the continuities multifunctions.

Definition 1. A multifunction $F : X \to Y$ is upper (lower) semi-continuous at a point $x_0 \in$ Dom (F) (*usc* (*lsc*)) if for any open set $V, F(x_0) \subset V$, $(F(x_0) \cap V \neq \emptyset)$ there exists a neighbourhood U of x_0 such that $F(x) \subset V$, $(F(x) \cap V \neq \emptyset)$ for any $x \in U$. F is said to be *usc* (*lsc*) if it is *usc* (*lsc*) at any $x \in X$.

Multifunction $F : X \to Y$ is upper (lower) quasi continuous (*uqsc* (*lqsc*)) at a $x_0 \in$ Dom (F) if for any open set $V, F(x_0) \subset V(F(x_0) \cap V \neq \emptyset)$ and for any open $U, x_0 \in U$ there is a non-empty open set $G \subset U \cap$ Dom (F) such that $F(x) \subset V(F(x) \cap V \neq \emptyset)$ for any $x \in G$. F is said to be *uqsc* (*lqsc*) if it is *uqsc* (*lqsc*) at any $x \in X$.

We say that a multifunction $F : X \to Y$ is upper (lower) c-continuous (*ucc* (*lcc*)) at $x_0 \in$ Dom (F) if for any open set V such that $Y \setminus V$ is compact and $F(x_0) \subset V$, $(F(x_0) \cap V \neq \emptyset)$ there exists a neighbourhood U of x_0 such that $F(x) \subset V$, $(F(x) \cap V \neq \emptyset)$ for any $x \in U$. F is said to be *ucc* (*lcc*) if it is *ucc* (*lcc*) at any $x \in X$.

Previous definition of *ucc* can be characterized by the lower inverse image $F^-(K)$ which is closed for any compact set K. The next definition introduce notion of a cluster point of multifunction F and a mapping which assigns cluster set to a point. Let $\mathcal{E}$ be a non-empty family of non-empty subsets of X. $\mathcal{E}$ will be called cluster system.

Definition 2. A point $y \in Y$ is $\mathcal{E}$-cluster point of multifunction $F : X \to Y$ at x_0 if for any open sets U, V with $x_0 \in U, y \in V$ there is a set $E \in \mathcal{E}, E \subset U$ such that for any $x \in E$ we have $F(x) \cap V \neq \emptyset$. The set of all $\mathcal{E}$-cluster points of F at x_0 is denoted by $\mathcal{E}_F(x_0)$.

A mapping $x \mapsto \mathcal{E}_F(x) = \{\ y \in Y, y$ is $\mathcal{E}$-cluster point of F in $x\ \}$ will be called cluster multifunction of F and we shall denote it by $\mathcal{E}_F$. Multifunction $\mathcal{E}_F$ can be empty valued mapping. For example if F is function $f(x) = 0$ for $x = 0$ and $f(x) = \frac{1}{x}$ otherwise, $\mathcal{E}$ is equal to the family of non-empty open set in the reals, then $\mathcal{E}_f(0) = \emptyset$. Further we will suppose that $X_0 =$Dom$(\mathcal{E}_F)$ is non-empty.

In multifunction setting for multifunction F and its cluster multifunction with respect a system $\mathcal{E}$, the next relations are focused

$$F(x) \subset \mathcal{E}_F(x)\,, \quad F(x) \cap \mathcal{E}_F(x) \neq \emptyset\,, \quad \mathcal{E}_F(x) \subset F(x)\,.$$

For some special cluster systems we will use special notation. For example, $\mathcal{E}^0 = 2^X \setminus \{\emptyset\}$, $\mathcal{O} = \{O \subset X$, where O is non-empty open $\}$, $\mathcal{A} = \{A \subset X, A$ is not nowhere dense $\}$, $\mathcal{E}_{\mathcal{I}} = \{E \subset X, E \notin \mathcal{I}\})$, where $\mathcal{I}$ is an ideal. With respect to a cluster system $\mathcal{E}$ we can introduce the definitions of upper and lower $\mathcal{E}$-continuity.

Definition 3. A multifunction F is u-$\mathcal{E}$-continuous (l-$\mathcal{E}$)-continuous at a point $x_0 \in \mathrm{Dom}\,(F)$ if for any open sets U, V such that $F(x_0) \subset V$ $(F(x_0) \cap V \neq \emptyset)$ and $x_0 \in U$ there is a set $E \in \mathcal{E}$, $E \subset U \cap \mathrm{Dom}(F)$ such that $F(x) \subset V$ $(F(x) \cap V \neq \emptyset)$ for any $x \in E$. If F is u-$\mathcal{E}$-continuous (l-$\mathcal{E}$-continuous) at every point $x \in X_0$ then we say that F is u-$\mathcal{E}$-continuous (l-$\mathcal{E}$-continuous).

Using cluster sets we can characterize l-$\mathcal{E}$-continuity at x_0 by the inclusion between F and $\mathcal{E}_F$ as $\emptyset \neq F(x_0) \subset \mathcal{E}_F(x_0)$ and global l-$\mathcal{E}$-continuity by the inclusion $F \subset \mathcal{E}_F$. If $\mathcal{E} = \mathcal{O}$, l-$\mathcal{O}$-continuity (u-$\mathcal{O}$-continuity) coincides with the lower and upper quasi-continuity. Any multifunction F with respect to a cluster system $\mathcal{E}^0$ is l-$\mathcal{E}^0$-continuous and hence $F \subset \mathcal{E}^0_F$. The cases of some relations between various types of cluster systems and cluster multifunctions are described in the next propositions.

Proposition 4. *Let $\mathcal{E}_1$, $\mathcal{E}_2$ be cluster systems such that $\mathcal{E}_1 \subset \mathcal{E}_2$. Then $\mathcal{E}_{1F} \subset \mathcal{E}_{2F}$.*

Proof. Let U, V be neighbourhoods of x, resp. y and $y \in \mathcal{E}_{1F}(x)$. If $\mathcal{E}_1 \subset \mathcal{E}_2$ then for every set $E_1 \in \mathcal{E}_1, E_1 \subset U$ we have $E_1 \in \mathcal{E}_2$. Thus there is a set $E_2 := E_1, E_2 \in \mathcal{E}_2, E_2 \subset U$ such that $F(e) \cap V \neq \emptyset$ for every $e \in E_2$. Hence $y \in \mathcal{E}_{2F}(x)$. □

Proposition 5. *Let $\mathcal{E}_1$, $\mathcal{E}_2$ be cluster systems and $\mathcal{E}_1 \cap \mathcal{E}_2$ be cluster system too. Then:*

i) $\mathcal{E}_{1F}(x) \cup \mathcal{E}_{2F}(x) = (\mathcal{E}_1 \cup \mathcal{E}_2)_F\,(x)$.

ii) $(\mathcal{E}_1 \cap \mathcal{E}_2)_F\,(x) \subset \mathcal{E}_{1F}(x) \cap \mathcal{E}_{2F}(x)$.

Proof. i) "$\subset$" $\mathcal{E}_1 \subset (\mathcal{E}_1 \cup \mathcal{E}_2) \Rightarrow \mathcal{E}_{1F} \subset (\mathcal{E}_1 \cup \mathcal{E}_2)_F$. Analogues $\mathcal{E}_2 \subset (\mathcal{E}_1 \cup \mathcal{E}_2) \Rightarrow \mathcal{E}_{2F}$ $(\mathcal{E}_1 \cup \mathcal{E}_2)_F$. Hence $(\mathcal{E}_{1F} \cup \mathcal{E}_{2F}) \subset (\mathcal{E}_1 \cup \mathcal{E}_2)_F$.

"$\supset$" Suppose $y \in (\mathcal{E}_1 \cup \mathcal{E}_2)_F(x)$ and we will show that $y \in \mathcal{E}_{1F}(x) \cup \mathcal{E}_{2F}(x)$. If not, there are neighbourhoods U_1 of x and V_1 of y such that for every set

$E_1 \in \mathcal{E}_1,\ E_1 \subset U_1$ there is a point $e_1 \in E_1$ such that $F(e_1) \cap V_1 = \emptyset$. Similar there are neighbourhoods U_2 of x and V_2 of y such that for every set $E_2 \in \mathcal{E}_2,\ E_2 \subset U_2$ there is a point $e_2 \in E_2$ such that $F(e_2) \cap V_1 = \emptyset$. Let $U = U_1 \cap U_2$ and $V = V_1 \cap V_2$. Hence there is not a set $E \in \mathcal{E}_1 \cup \mathcal{E}_2,\ E \subset U$ such that $F(e) \cap V \neq \emptyset$ for any $e \in E$. Thus $y \notin (\mathcal{E}_1 \cup \mathcal{E}_2)_F(x)$.

ii) Since $\mathcal{E}_1 \cap \mathcal{E}_2 \subset \mathcal{E}_1$ and $\mathcal{E}_1 \cap \mathcal{E}_2 \subset \mathcal{E}_2$, $(\mathcal{E}_1 \cap \mathcal{E}_2)_F(x) \subset \mathcal{E}_{1F}(x)$ and $(\mathcal{E}_1 \cap \mathcal{E}_2)_F(x) \subset \mathcal{E}_{2F}(x)$ hence $(\mathcal{E}_1 \cap \mathcal{E}_2)_F(x) \subset \mathcal{E}_{1F}(x) \cap \mathcal{E}_{2F}(x)$, contradiction. □

2. Basic Properties of Multifunction $\mathcal{E}_F$

The following lemma (see [5]) will be used for further theorems.

Lemma 6. *If $\{x_t\}_{t \in T}$ is a net in X convergent to x_0 and $\{y_t\}_{t \in T}$ is a net in Y convergent to y_0 such that y_t is $\mathcal{E}$-cluster point of F at x_t then y_0 is $\mathcal{E}$-cluster point of F at x_0.*

Theorem 7. *Graph $\mathcal{E}_F$ is closed in $X \times Y$.*

Proof. We prove that $\mathrm{Cl}(\mathrm{Gr}\ (\mathcal{E}_F)) \subset\ \mathrm{Gr}\ (\mathcal{E}_F)$. Let $[x_0, y_0] \in \mathrm{Cl}(\mathrm{Gr}(\mathcal{E}_F))$. Then for any neighbourhoods U of x_0 and V of y_0 we have $[U \times V] \cap\ \mathrm{Gr}\ (\mathcal{E}_F) \neq \emptyset$. There is a point $[x_U, y_V] \in\ \mathrm{Gr}\ (\mathcal{E}_F)$ such that $y_V \in \mathcal{E}_F(x_U)$. Then there are the nets $\{x_U\}_{U \in \mathcal{U}(x_0)}$ converges to x_0 and $\{y_V\}_{V \in \mathcal{V}(y_0)}$ converges to y_0 such that $y_V \in \mathcal{E}_F(x_U)$, where $\mathcal{U}(x_0), \mathcal{V}(y_0)$ is the system of neighbourhoods of a point of x_0, y_0, respectlively. From Lemma 6 we have that $y_0 \in \mathcal{E}_F(x_0)$ and so $[x_0, y_0] \in\ \mathrm{Gr}\ (\mathcal{E}_F)$. □

As a corollary of Theorem 7 we have

Corollary 8. *A multifunction $\mathcal{E}_F : X_0 \to Y$ is ucc and a set $\mathcal{E}_F(x)$ is closed.*

The following properties give under which conditions a multifunction $\mathcal{E}_F$ is usc.

Lemma 9. *If Y is a compact space, then a multifunction $\mathcal{E}_F : X_0 \to Y$ is usc.*

A multifunction $F : X \to Y$ is locally bounded at x_0 if there is a compact set $K \subset Y$ and there is a neighbourhood $U(x_0)$ of x_0 such that $F(x) \subset K$ for every $x \in U(x_0)$. As a corollary of Lemma 9 we can say the next theorem.

Theorem 10. *Let Y be a Hausdorff space. If a multifunction $\mathcal{E}_F : X_0 \to Y$ is locally bounded at $x_0 \in X_0$ then $\mathcal{E}_F : X_0 \to Y$ is usc at some neighbourhood of x_0.*

Proof. Since $\mathcal{E}_F$ is locally bounded at x_0 then there is a compact set $K \subset Y$ and there is a neighbourhood U of x_0 such that $\mathcal{E}_F(U) \subset K$. Hence $\mathcal{E}_F/_U : U \to K$ is a multifunction from U to compact set K and by Lemma 9 $\mathcal{E}_F$ is *usc*. Hence $\mathcal{E}_F : X_0 \to Y$ is *usc* at neighbourhood U of x_0. □

The multifunction $F : X \to Y$ is *usco* if F be compact valued and *usc* at any $x \in \mathrm{Dom}\ (F)$.

Corollary 11. *Let Y be a regular space. Then a set of all points, where a multifunction $\mathcal{E}_F : X_0 \to Y$ is usco is open at X_0.*

The following lemma deals with the structure of a set of points where $\mathcal{E}_F$ is not *lsc*, see [3].

Lemma 12. *Let Y be a compact metric space. Then a set of all points, where a multifunction $\mathcal{E}_F : X_0 \to Y$ is not lsc is of first category in X_0.*

The some properties of a set X_0 we will introduce in the next lemmas.

Lemma 13. *Let Y be a σ-compact space. Then X_0 is F_σ in X.*

Proof. $X_0 = \mathcal{E}_F^-(Y) = \mathcal{E}_F^-\left(\bigcup_{n=1}^{\infty} K_n\right) = \bigcup_{n=1}^{\infty} \mathcal{E}_F^-(K_n)$, where $\mathcal{E}_F^-(K_n)$ is closed in X and hence a set X_0 is F_σ. □

Lemma 14. *Let Y be a σ-compact space. If a set X_0 is of second category at every $x \in X$, then $X \setminus X_0$ is nowhere dense.*

Proof. Let $X \setminus X_0$ be no nowhere dense. Then there is an open set G and a set $D \subset X \setminus X_0$ dense in G and there is also a set of second category $C \subset X_0 \cap G$. Hence there is a positive integer i such that $\mathcal{E}_F^-(K_i) \cap G$ is second category. That means $\mathrm{Int}\ (\mathcal{E}_F^-(K_i) \cap G)$ is non-empty open subset of $\mathrm{Dom}\ (F)$ what is a contradiction with the fact that D is a dense set in G. □

Corollary 15. *If a set $\mathcal{E}_F^-(K)$ is not nowhere dense, where K is compact set, then* $\mathrm{Int}\ (\mathcal{E}_F^-(K)) \neq \emptyset$.

Proof. A set $\mathcal{E}_F^-(K)$ is dense at some open set H hence $H \subset \mathrm{Cl}(\mathcal{E}_F^-(K)) = \mathcal{E}_F^-(K)$. □

We will introduce the notion of $\mathcal{E}$-small sets and its using for cluster multifunction.

Definition 16. A set $A \subset X$ is locally $\mathcal{E}$-small at a point $x \in X$ if there is a neighbourhood $U(x)$ of x such that $A \cap U(x)$ does not contain a set from $\mathcal{E}$. Let $\mathcal{D}_{\mathcal{E}}(A) = \{x \in X : A \text{ is not locally } \mathcal{E}\text{-small at } x\}$. A set $A \subset X$ is $\mathcal{E}$-small if A is locally $\mathcal{E}$-small at every $x \in X$.

Theorem 17. *Let Y be a second countable space, $F : X \to Y$ be a multifunction. Then a set $B = \{x \in X : F(x) \not\subset \mathcal{E}_F(x)\} = \bigcup_{n=1}^{\infty} B_n$, where B_n are $\mathcal{E}$-small.*

Proof. "$\subset$" We have that a set $A \setminus \mathcal{D}_{\mathcal{E}}(A)$ is $\mathcal{E}$-small for $A \subset X$. Let $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ be a basis at Y and $x_0 \in B : F(x_0) \not\subset \mathcal{E}_F(x_0)$. Since there is $y_0 \in F(x_0)$ such that $y_0 \notin \mathcal{E}_F(x_0)$. Thus exists a set $G_n \ni y_0$ and there is a neighbourhood $U(x_0)$ of x_0 such that $U(x_0) \cap F^-(G_n)$ does not contain a set from $\mathcal{E}$. Hence $x_0 \in F^-(G_n) \setminus \mathcal{D}_{\mathcal{E}}(F^-(G_n)) = B_n$.

"$\supset$" Let $x_0 \in \bigcup\limits_{n=1}^{\infty} A_n \setminus D_{\mathcal{E}}(A_n)$, where $A_n = F^-(G_n)$. Then $x_0 \in F^-(G_i) \setminus D_{\mathcal{E}}(F^-(G_i))$ for some positive integer i. Thus $x_0 \notin D_{\mathcal{E}}(F^-(G_i))$ and there is a neighbourhood $U(x_0)$ of x_0 such that $U(x_0) \cap F^-(G_i)$ does not contain a set from $\mathcal{E}$. Since exists a point $y \in F(x_0) \cap G_i$ that $y \notin \mathcal{E}_F(x_0)$ and $F(x_0) \not\subset \mathcal{E}_F(x_0)$. □

If X is a second countable space, $\mathcal{I}$ is an ideal in X and a set A is $\mathcal{E}_{\mathcal{I}}$-small, then $A \in \mathcal{I}_\sigma$. As a consequence of Theorem 17 we have the next corollary.

Corollary 18. *Let X, Y be the second countable spaces. A set $B = \{x \in X : F(x) \not\subset \mathcal{E}_{\mathcal{I}F}(x)\}$ is $\mathcal{I}_\sigma$.*

3. Main Results

In this section we present the main results concerning using cluster multifunction for decomposition theorems. Generally speaking the decomposition theorems deal with conditions under which two types of continuities are equivalent. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be any cluster systems. We will be interested in conditions concerning continuities of multifunction F under which identity $\mathcal{E}_{1F}(x) = \mathcal{E}_{2F}(x)$ holds. Then the more general l-$\mathcal{E}_1$-continuity with condition $\mathcal{E}_{1F} = \mathcal{E}_{2F}$ implies the strong l-$\mathcal{E}_2$-continuity. Let $\mathcal{E}_1 \subset \mathcal{E}_2$, then F is l-$\mathcal{E}_1$-continuous if and only if F is l-$\mathcal{E}_2$-continuous and $\mathcal{E}_{1F} = \mathcal{E}_{2F}$.

Theorem 19. *If $\mathcal{E}_{1F}(x) = \mathcal{E}_{2F}(x)$ for any $x \in B$ then F is l-$\mathcal{E}_1$-continuous on B if and only if F is l-$\mathcal{E}_2$-continuous on B.*

Proof. If a multifunction F is l-$\mathcal{E}_1$-continuous on B then $F(b) \subset \mathcal{E}_{1F}(b) = \mathcal{E}_{2F}(b)$ for any $b \in B$ and thus F is l-$\mathcal{E}_2$-continuous on B. □

Theorem 20. *If F is l-$\mathcal{E}_1$-continuous at $X \setminus A$ and l-$\mathcal{E}_2$-continuous on $X \setminus A$, $E \setminus A \neq \emptyset$ for every $E \in \mathcal{E}_1 \cup \mathcal{E}_2$, then multifunctions $\mathcal{E}_{1F}, \mathcal{E}_{2F}$ are identical on whole space X.*

Proof. We will prove inclusion $\mathcal{E}_{1F}(x) \subset \mathcal{E}_{2F}(x)$. The converse is the same. Let $x \in X$ and $y \in \mathcal{E}_{1F}(x)$, U, V are neighbourhoods of x, resp. y. Then there is a set $E_1 \in \mathcal{E}_1, E_1 \subset U$ such that $F(e_1) \cap V \neq \emptyset$ for every $e_1 \in E_1$ From $E_1 \setminus A \neq \emptyset$ exists a point $e_2 \in E_1$ where F is l-$\mathcal{E}_2$ continuous. Now there is a set $E_2 \in \mathcal{E}_2, E_2 \subset U$ such that $F(e_2) \cap V \neq \emptyset$ for every $e_2 \in E_2$. Thus $y \in \mathcal{E}_{2F}(x)$. □

As corollaries we have

Corollary 21. *If F is l-$\mathcal{E}_1$-continuous on X and l-$\mathcal{E}_2$-continuous on X then $\mathcal{E}_{1F}(x) = \mathcal{E}_{2F}(x)$ for every $x \in X$.*

Since any multifunction F is l-$\mathcal{E}^0$-continuous and Cl(Gr(F))=$\mathcal{E}^0_F$, then we have.

Corollary 22. *A multifunction F is l-$\mathcal{E}$-continuous on X if and only if $\mathcal{E}_F = Cl(Gr(F))$.*

Consequently for $\mathcal{E}_1 = \mathcal{E}_{\mathcal{I}}$, where $\mathcal{I}$ is a ideal such that $G \notin \mathcal{I}$ for any open set G and $\mathcal{E}_2 = \mathcal{O}$, $A \in \mathcal{I}$ we have Matejdes Theorem, see [5]. It is possible to read this theorem as decomposition theorem because inverse variant automatically holds.

Corollary 23. *Let F be an l-$\mathcal{E}_{\mathcal{I}}$-continuous multifunction and $X \setminus Q^-_F \in \mathcal{I}$, then F is lqsc.*

Proof. Let $A = X \setminus Q^-_F \in \mathcal{I}$. Then F is l-$\mathcal{E}_{\mathcal{I}}$-continuous on $X \setminus A$ and F is lower quasi-continuous on $X \setminus A = Q^-_F$. Hence F is lower quasi-continuous. □

Let Q^+_F, Q^-_F be a set of all points, where F is upper quasi-continuous, lower quasi-continuous, respectlively and $Q_F = Q^+_F \cap Q^-_F$. It means that if $x \in Q_F$ then for any neighbourhood U of x and for any neighbourhoods V_1, V_2 such that $F(x) \subset V_1$ and $F(x) \cap V_2 \neq \emptyset$ there are open sets G_1, G_2, $G_1 \subset U, G_2 \subset U$ such that $F(G_1) \subset V_1$ and $F(g_2) \cap V_2 \neq \emptyset$ for any $g_2 \in G_2$.

Theorem 24. *Let Y be a regular space, $F : X \to Y$ be a multifunction. If a set $Q_F = Q_F^+ \cap Q_F^-$ is dense at X then multifunctions $\mathcal{A}_F$ and $\mathcal{O}_F$ are identical on X.*

Proof. The inclusion $\mathcal{O}_F \subset \mathcal{A}_F$ is obvious because $\mathcal{O} \subset \mathcal{A}$. Now we will show that $\mathcal{A}_F(x) \subset \mathcal{O}_F(x)$ for any $x \in X$. Let $y \in \mathcal{A}_F(x)$ and U, V be neighbourhoods of x, y, respectlively. From regularity of a space Y there is an open set V_0 such that $y \in V_0 \subset \text{Cl } (V_0) \subset V$. Then there is a set $A \in \mathcal{A}, A \subset U$ and there is an open set $H \subset U$, A is dense at H such that $F(a) \cap V_0 \neq \emptyset$ for any $a \in A$. We will show that for every point $h \in H \cap Q_F$ relation $F(h) \cap$ Cl $(V_0) \neq \emptyset$ holds. We suppose that there is a point $h_0 \in H \cap Q_F$ such that $F(h_0) \cap$ Cl $(V_0) = \emptyset$. Then $F(h_0) \subset Y \setminus$ Cl (V_0). At a point h_0 is multifunction F *uqsc* then there is an open set $H_0 \subset H : F(H_0) \subset Y \setminus$ Cl (V_0) what is a contradiction with $F(a) \cap$ Cl $(V_0) \neq \emptyset$ for $a \in A$. Thus there is a point $h' \in H \cap Q_F$ such that $F(h') \cap$ Cl $(V_0) \neq \emptyset$ and hence $F(h') \cap V \neq \emptyset$. F is *lqsc* at a point h' and thus there is an open set $G \subset H$ such that $F(g) \cap V \neq \emptyset$ for any $g \in G$ hence $y \in \mathcal{O}_F(x)$. ☐

Corollary 25. *If a multifunction F is l-$\mathcal{A}$-continuous and a set Q_F is dense at X, then F is lqsc.*

As corollaries of Theorem 24 and Corollary 25 we have the Borsík-Doboš Decomposition Theorem for functions, where $\mathcal{B}$-continuity in their paper [1] and $\mathcal{A}$-continuity coincide and $u\mathcal{B}c$, $l\mathcal{B}c$ from Popa-Noiri paper [7] coincides with u-$\mathcal{A}$-continuity, l-$\mathcal{A}$-continuity, respectlively.

Corollary 26. (see [1]) *Let Y be a regular space. Then $f : X \to Y$ is quasicontinuous if and only if f is $\mathcal{B}$-continuous and a set Q_f is dense at X.*

Theorem 27. (see [7]) *Let $F : X \to Y$ be a multifunction such that $F(x)$ is α-regular α-paracompact for each $x \in X$. Then F is upper and lower quasi continuous if and only if F is $u\mathcal{B}c$ and $l\mathcal{B}c$ and Q_F is dense set at X.*

The lower variant of Theorem 27 follows from Theorem 24 and the upper variant of Theorem 27 will be proved in the next theorem.

Theorem 28. *Let Y be a regular space. If a multifunction $F : X \to Y$ with compact values is u-$\mathcal{A}$-continuous and a set $Q_F = Q_F^+ \cap Q_F^-$ is dense at X then F is uqsc.*

Proof. A multifunction F is u-$\mathcal{A}$-continuous at a point $x \in X$. Thus for any neighbourhood U of x and for any open set $V \subset Y$ such that $F(x) \subset V$ from regularity of a space Y there is an open set V_0 such that $F(x) \subset V_0 \subset$ Cl $(V_0) \subset V)$ there is a set $A \in \mathcal{A}, A \subset U$ and there is an open set $H \subset U$, A is dense in H such that $F(a) \subset V_0$ for any $a \in A$. We will show that for every point $h \in H \cap Q_F$ we have $F(h) \subset$ Cl (V_0). Let there is a point $h_0 \in H \cap Q_F$ such that $F(h_0) \not\subset$ Cl (V_0) Then $F(h_0) \cap Y\setminus$ Cl $(V_0) \neq \emptyset$. F is *lqsc* in h_0 then there is an open set $H' \subset H$ such that $F(h') \cap Y\setminus$ Cl $(V_0) \neq \emptyset$ for any $h' \in H'$ what is a contradiction with $F(a) \subset V_0$ for any $a \in A$. It means that $F(h') \not\subset$ Cl (V_0) and thus $F(h') \not\subset V_0$. Hence there is a point $h^* \in H \cap Q_F$ such that $F(h^*) \subset$ Cl (V_0) and so $F(h^*) \subset V$. Since F is *uqsc* at a point h^* hence there is an open set $G \subset H$ such that $F(g) \subset V$ for any $g \in G$ and F is *uqsc* at x. □

From Theorem 24 and Theorem 28 we have Popa-Noiri Decomposition Theorem 27. Theorem 28 holds if F is closed valued multifunction and Y is a normal space.

The next example shows that condition of density of Q_F is necessary.

Example 29. Let

$$F(x) = \begin{cases} \langle 0,1\rangle & \text{if } x \in \mathbb{I}, \text{ where } \mathbb{I} \text{ is the set of irational numbers}\,, \\ \{0\} & \text{if } x \in \mathbb{Q}, \text{ where } \mathbb{Q} \text{ is the set of rational numbers}\,. \end{cases}$$

A multifunction F is u-$\mathcal{A}$-continuous, set $Q_F^+\,(Q_F^-)$ is dense, but F is not *uqsc* in rational numbers.

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