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RIGID SPANNED VECTOR BUNDLES ON CURVES

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Abstract: Let X be a smooth genus g curve. Here we look at the maps $f : X \to G(r,m)$ into a Grassmannian (i.e. to spanned vector bundle on X) such that the natural map $H^0(G(r,m), TG(r,m)) \to H^0(X, f^*(TG(r,m)))$ is surjective (a rigidity condition) and $f^*(TG(r,m))$ is stable, mainly when the Grassmannian is a projective space.

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Let X be a smooth and connected projective curve. Set $g := p_a(X)$. For all integers $0 < r < m$ let $G(r,m)$ denote the Grassmannian of all $(m-r)$-dimensional linear subspaces of $K^{\oplus m}$. Let $Q_{r,m}$ the rank r universal quotient line bundle of $G(r,m)$ and $S_{r,m}$ the universal rank $(m-r)$ subbundle of $G(r,m)$. Hence $TG(r,m) \cong Q_{r,m} \otimes S^*_{r,m}$ and we have the tauotological exact sequence on $G(r,m)$:

$$0 \to S_{r,m} \to \mathcal{O}^{\oplus m}_{G(r,m)} \to Q_{r,m} \to 0\,. \tag{1}$$

If $m = r+1$, then $G(r, r+1) \cong \mathbf{P}^r$ and (1) is a twist of the Euler's sequence of $T\mathbf{P}^r$. We have $h^0(G(r,m), TG(r,m)) = \dim(\mathrm{Aut}^0(G(r,m))) = m^2 - 1$. For any rank r spanned vector bundle E on X and any line subspace $V \subseteq H^0(X,E)$ such that V spans E and $\dim(V) = m$ let $u_{E,V} : X \to G(r,m)$ denote the morphism associated to the pair (E,V) by the univeral property of the Grassmannian. Hence $E \cong u^*_{E,V}(Q_{r,m})$. Set $M_{E,V} := u^*_{E,V}(S_{r,m})$. $M_{E,V}$ is the kernel of the evaluation map $V \otimes \mathcal{O}_X \to E$, which by assumption is

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surjective. Hence $\det(M_{E,v}) \cong \det(E)^*$ and $u_{E,V}^*(TG(r,m)) \cong E \otimes M_{E,V}^*$. For any smooth projective variety Z the vector space $H^0(X, f^*(TZ))$ is the deformation space of the morphism $f : X \to Z$ in which both X and Z are and $H^1(X, f^*(TZ))$ is an obstruction space for the same deformation functor. In the set-up of the map $u_{E,V}$ instead of $H^1(X, u_{E,V}^*(TG(r,m)))$ one can consider the kernel of the Petri map $V \otimes H^0(X, E^* \otimes \omega_X) \to H^0(X, End(E) \otimes \omega_X)$. We cannot have pairs (E, V) with injective Petri map outside the Brill-Noether range. We will say that the triple (X, E, V) is *nice* or the the morphism $u_{E,V}$ is *nice* if the natural map $\eta_{E,V} : H^0(G(r,m), TG(r,m)) \to H^0(X, E \otimes M_{E,V}^*)$ is surjective. If E is not trivial, then $m > r$, $M_{E,V} \neq 0$ and $u_{E,V}$ is not constant. If (X, E, V) is nice, then $V = H^0(X, E)$. Roughly speaking (i.e. forgetting the few cases for which $\eta_{E,V}$ is not injective) Riemann-Roch says that the triple (X, E, V) is nice if and only the obstruction space $H^1(X, u_{E,V}^*(TG(r,m)))$ has the smallest possible dimension for the fixed numerical data g, r, m and $\deg(\det(E))$. In the case $(r, m) = (1, 2)$, i.e. of spanned elements of $G_d^1(X)$, the niceness of (X, E, V) means that the corresponding g_d^1 of X is a reduced point of $W_d^1(X) \backslash W_d^2(X)$. We will say that $u_{E,V}$ (or (X, E, V)) is *exceptional* if $\eta_{E,V}$ is an isomorphism and $h^1(X, u_{E,V}^*(T(G(r,m))) = 0$. This is a very strong condition (see Example 5 and Remark 5).

In this short note we only list many cases such that from printed papers we get examples of nice maps. We work over an algebraically closed field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) = 0$.

Remark 1. The natural map $V \to H^0(X, M_{E,V}^*)$ is injective if and only if E has no trivial factor. Assume that this in the case and assume that (X, E, V) is nice. It is easy to check (using maps to higher dimensional Grassmannian) that $V = H^0(X, E)$ and that $V = H^0(X, M_{E,V}$, i.e. that both maps to Grassmannians $G(r,m)$ and $G(m-r, m)$ induced by (E, V) and $(M_{E,V}^*, V)$ are "linearly normal".

Remark 2. Here we assume $m = r + 1$. $\Omega_{\mathbf{P}^r}^1(2)$ is spanned. Hence there is an inclusion $j : T\mathbf{P}^r \to \mathcal{O}_{\mathbf{P}^r}(2)^{\oplus N}$ with locally free cokernel, A. Since $h^1(\mathbf{P}^r, T\mathbf{P}^r) = 0$, $h^0(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(2)^{\oplus N}) = h^0(\mathbf{P}^r, T\mathbf{P}^r) + h^0(\mathbf{P}^r, A)$. Hence $((X, E, V)$ is nice if the natural map $\alpha_{E,V} : S^2(V) \to H^0(X, \det(E)^{\otimes 2})$ is surjective and $\eta_{E,V}$ is injective, the latter conditions being always satisfied if $\alpha_{E,V}$ is bijective. $\alpha_{E,V}$ is bijective if X is a smooth complete intersection of $r-1$ hypersurfaces of degree ≥ 3.

Example 1. Fir $m > r > 0$, $m \geq 3$, and take as X a smooth complete intersection of $r(m-r)-1$ hypersurfaces $S_i \in |\mathcal{O}_{G(r,m)}(t_i)|$, $1 \leq i \leq r(m-r)-1$, of $G(r,m)$. To get that the restriction map $H^0(G(r,m), TG(r,m)) \to$

$H^0(X, TG(r,m)|X)$ is bijective it is sufficient that $t_i \geq 3$. To find examples for which $Q_{r,m}|X$, $S_{r,m}|X$ and $TG(r,m)|X$ are stable, one may use a restriction theorem (e.g. [4]), if we assume all t_i's large, but one also need to assume that X is a general complete intersection of type $(t_1, \dots, t_{r(m-r)-1})$.

To check Example 2 below we will need the following two remarks.

Remark 3. Let $D \subset \mathbf{P}^r$, $r \geq 2$, be a rational normal curve. Then $T\mathbf{P}^r|D$ is isomorphic to a direct sum of r line bundles of degree $r+1$ (see [7]) and hence it is semistable.

Remark 4. Let $B \subset \mathbf{P}^2$ be a smooth plane cubic. Hence $T\mathbf{P}^2|C$ is a rank 2 vector bundle with odd degree. Hence it is semistable if and only if it is stable. It is well-known that $T\mathbf{P}^2|C$ is stable (see [2], Remark 1.5). This may be checked using that $T\mathbf{P}^2(-1)|C$ has degree 3 and it is spanned and hence it must be indecomposable.

Example 2. Let $C \subset \mathbf{P}^2$ be an integral degree $d \geq 4$ projective curve and $u : X \to C$ the normalization map. Set $g := p_a(X)$. Let $v : X \to \mathbf{P}^2$ the composition of u with the inclusion $C \to \mathbf{P}^2$. Let $\mathcal{J} \subseteq \mathcal{O}^2_{\mathbf{P}}$ denote the conductor of the singularities of C. Thus $H^0(X, \omega_X) = H^0(\mathbf{P}^2, \mathcal{J}(d-3))$. Set $A := v^*(T\mathbf{P}^2)$. Hence A is a rank 2 vector bundle on X and $\det(A) \cong v^*(\mathcal{O}_{\mathbf{P}^2}(1)$. Since $d \geq 2$ the natural map $\eta : H^0(\mathbf{P}^2, T\mathbf{P}^2) \to H^0(X, A)$ is injective. Thus v is nice if and only if $h^0(X, A) = 8$. Since $d \geq 4$, $h^1(\mathbf{P}^2, T\mathbf{P}^2(d)) = 0$. Thus the restriction map $H^0(\mathbf{P}^2, T\mathbf{P}^2) \to H^0(C, T\mathbf{P}^2|C)$ is bijective. Riemann-Roch and Serre duality give that v is nice if and only if $h^1(\mathbf{P}^2, \mathcal{J} \otimes \Omega^1_{\mathbf{P}^2}(d-3)) = 0$. Since $\Omega^1_{\mathbf{P}^2}(2)$ is spanned (Remark 2), v is nice if $h^1(\mathbf{P}^2, \mathcal{J}(d-5)) = 0$. Remark 1 shows that if v is nice, then $h^1(\mathbf{P}^2, \mathcal{J}(d-4)) = 0$. Many examples are given for instance for nodal curve, because if C has only ordinary nodes or ordinary cusps as singularities, then $\mathcal{J}$ is the ideal sheaf of the set of all singular points (taking with the reduced structure). For every integer s such that $0 \leq s \leq (d-2)(d-1)/2$ let $V_{d,s}$ denote the variety of all degree d plane curves with exactly s nodes as their only singularities (see [8]). It is known that $V_{d,s} \neq \emptyset$, $\dim(V_{d,s}) = (d^2+2d)/2 - s$, and that the singular locus S, of a general $A \in V_{d,s}$ has the best possible postulation, i.e. $h^0(\mathbf{P}^2, \mathcal{I}_S(t)) = \max\{0, (t+2)(t+1)/2 - s\}$ and $h^1(\mathbf{P}^2, \mathcal{I}_S(t)) = \max\{0, s-(t+2)(t+1)/2\}$ (see [8]). Hence $h^1(\mathbf{P}^2, \mathcal{I}_S(d-5)) = 0$ if and only if $s \leq (d-3)(d-4)/2$. Hence we get a lot of nodal examples. Since this cohomological condition depends only on the postulation of the set $S = \mathrm{Sing}(C)$, the same is true for any nodal or cuspidal curve if either $d \geq 5$ and $\sharp(S) \leq d-5$ or $d = 4$ and C is smooth. Now we consider the stability of the rank 2 vector bundle $v^*(T\mathbf{P}^2)$, following [1]. We assume $d \geq 5$. Set $e = 1$ if d is odd and $e = 2$ if d is even. We assume

$e \le g \le (d-1)(d-2)/2$ and set $s := (d-1)(d-2)/2 - g$. Let $T \subset \mathbf{P}^2$ be a nodal union of e smooth degree 3 plane curves and $(d-3e)/2$ smooth conics. Hence T has $\lfloor (d-1)/2 \rfloor$ irreducible components and $(d-2)(d-1)/2 - e - \lfloor (d-1)/2 \rfloor$ ordinary nodes. There is a set $S \subset \mathrm{Sing}(T)$ such that $\sharp(S) = s$ and the partial normalization $u_0 : X' \to T$ in which we normalize only the nodes in $\mathrm{Sing}(T) \backslash S$ is connected. Remark 4, the case $r = 2$ of Remark 4 and [1], Lemma 1.1, give that $u^*(T\mathbf{P}^2(-1))$ is stable. Hein and Kurke [2], Theorem 2.4, proved that stability in their sense is an open condition in flat families of reduced projective curves. Hence for a sufficiently near deformation (X_λ, u_λ) with X_λ smooth for general λ the vector bundle $u_\lambda^*(T\mathbf{P}^2(-1))$ is stable. Severi theory shows that T is ia flat limit of a one-dimensional family of elements of $V_{d,s}$ in which S is the limit of the nodes of the integral elements of the family. Hence we get the existence of an integral nodal $C \subset \mathbf{P}^2$ with s ordinary nodes and such that $u^*(T\mathbf{P}^2(-1))$ is stable, where u is the composition of the normalzation map $X \to C$ and the inclusion $C \to \mathbf{P}^2$.

Example 3. Assume $(r, m) = (3, 4)$. Liaison addition gives many examples of smooth curve $X \subset \mathbf{P}^3$ such that $h^i(\mathbf{P}^3, \mathcal{I}_X(t)) = 0$ for $i = 0, 1$ and $t = 1, 2$ (see [6]). Any such embedding is nice (Remark 2). All such embeddings may be in principle constructed taking "minimal " (not necessarly reduced) examples and then using liason addition and deformations, because the so-called Lazarsfeld-Rao property is true (see [3] for more details).

Example 4. Fix a nice map $u = u_{E,V} : X \to G(r, m)$. Let $f : Y \to X$ be a finite degree $c \ge 2$ covering with Y is a smooth curve. Since we work in characteristic zero, $f_*(\mathcal{O}_Y) \cong \mathcal{O}_X \oplus A$ with A a rank $c-1$ vector bundle on X, which is rather negative. For instance, if f is a cyclic covering, then $A \cong \oplus_{i=1}^{c-1} M^{\otimes i}$ with $M \in \mathrm{Pic}(X)$ and $\deg(M) \le 0$, with $\deg(M) < 0$ if f is ramified. The morphism $u \circ f : Y \to G(r, m)$ is nice if and only if $h^0(X, u^*(TG(r, m)) \otimes A) = 0$. For fixed X, u this is true for "all sufficiently ramified coverings". Now assume that $u^*(TG(r, m))$ (resp. $u^*(Q_{r,m})$, resp. $u^*(S_{r,m})$) is semistable. Then $(u \circ f)^*(TG(r, m))$ (resp. $(u \circ f)^*(Q_{r,m})$, resp. $(u \circ f)^*(S_{r,m})$) is semistable (see [5]). If f is a double covering which is not étale and $u^*(TG(r, m))$ (resp. $u^*(Q_{r,m})$, resp. $u^*(S_{r,m})$) is stable. Then $(u \circ f)^*(TG(r, m))$ (resp. $(u \circ f)^*(Q_{r,m})$, resp. $(u \circ f)^*(S_{r,m})$) is stable.

Example 5. Assume $m = r + 1 \ge 4$. Fix the genus $g \ge 2$ and the degree d of the map. We have $\chi(u^*(T\mathbf{P}^r)) = (r+1)d + r(1-g)$. Hence $\chi(u^*(T\mathbf{P}^r)) = r^2 + r$ if and only if $\rho(g, r, d) := g - (r+1)(g + r - d) = 0$. Assume $\rho(g, r, d)$. Let X be a general smooth curve of genus g. It has a linealy normal embedding $u : X \to \mathbf{P}^r$ such that $h^1(X, u^*(T\mathbf{P}^r) = 0$ (see [7], Theorem

6.1, and the definition of the map $\mu_0(D)$ as the differential of a map to $\mathcal{M}_g$; here we use the characteristic zero assumption to pass from a result on the dimension of fibers to a result on the rank of a tangent map). Such an embedding u is exceptioptional. In almost all cases $u^*(T\mathbf{P}^r)$ is stable (see [1], Theorem 1.7).

Remark 5. Assume the existence of a degree d exceptional morphism $X \to G(r, m)$ with X of genus g. We get $\chi(u^*(TG(r, m))) = m^2 - 1$. Now we use that $\deg(u^*(T(G(r, m)) = md$ and $\mathrm{rank}(TG(r, m)) = r(m - r)$. Hence $\chi(u^*(TG(r, m))) = md + r(m - r)(1 - g)$. If $r = 2$ and $m = 4$, then $4d + 4(1 - g) \neq 15$ for any d, g.

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