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ITERATED MULTIPLICATION MAPS FOR STABLE VECTOR BUNDLES ON CURVES

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Abstract: Let C be a smooth curve of genus $g \ge 2$. Here we study the surjectivity of the iterated multiplication map $\sigma_{t,E} : S^t(H^0(C,E)) \otimes H^0(C,\omega_C) \to H^0(C, S^t(E) \otimes \omega_C)$, $t \ge 1$ when E is a stable vector bundle with low $h^1(C,E)$.

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1. Introduction

Let C be a smooth and connected projective curve of genus $g \ge 2$ defined over an algebraically closed field $\mathbb{K}$. For all integer r, d such that $r > 0$ let $U(r,d)$ denote the moduli space of all stable vector bundles on C with rank r and degree d. $U(r,d)$ is a non-empty integral variety of dimension $r^2(g-1)+1$. If $r \ge 2$ and $d \ge r(2g-2)$, then $h^1(X,E) = 0$ for every $E \in U(r,d)$, because the definition of stability implies the non-existence of a non-zero morphism $E \to \omega_C$ and then we may apply Serre duality. For all vector bundles E, F and all linear subspaces $V \subseteq H^0(C,E)$, $W \subseteq H^0(C,F)$ let $\mu_{E,V;F,W} V \otimes W \to H^0(C, E \otimes F)$ denote the multiplication map. Omit V (resp. W) if $V =$

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$H^0(C,E)$ (resp. $W = H^0(C,F)$. Let $\mu_E : H^0(C,E) \otimes H^0(C,\omega_C) \to H^0(C, E \otimes \omega_C)$ denote the multiplication map, i.e. set $\mu_E := \mu_{E,\omega_C}$. For every integer $t \geq 1$ let $\mu_{t,E} : H^0(C,E)^{\otimes t} \otimes H^0(C,\omega_C) \to H^0(C, E^{\otimes t} \otimes \omega_C)$ (resp. $\sigma_{t,E} : S^t(H^0(C,E)) \otimes H^0(C,\omega_C) \to H^0(C, S^t(E) \otimes \omega_C))$ denote the multiplication map (resp. the symmetric multiplication map). For the geometric significance of the surjectivity of $\mu_{1,E}$, see [4]. If E is general in $U(r,d)$ and $d \geq r(g+1)$, then $h^1(C,E) = 0$ and $\mu_{1,E}$ is surjective (see [4], [3]). Here we ask for the non-general case. For non-special stable vector bundles there is no exceptional cases, while for special vector bundles we will see that hyperelliptic curves have different properties. In Section 2 we will prove the following result.

Theorem 1. *Assume either* $\mathrm{char}(\mathbb{K}) = 0$ *or* $\mathrm{char}(\mathbb{K}) \geq 2g-1$. *Fix integers* $r \geq 2$, *and* $d \geq r(g+3)$. *Fix a general* $E \in U(r,d)$. *Then* $\mu_{t,E}$ *for all integers* $t \geq 1$.

Let E be a vector bundle on C with rank $r \geq 2$ and degree d. Is there any geometry involved in the cohomology group $H^1(C,E)$? For the cohonology group $H^0(C,E)$ the answer is affermative. One first distinguishes if E is spanned or not. If E is spanned, then $H^0(C,E)$ induces a morphism h_M from C into a Grassmannian and one can study the morphism h_E (is it an embedding?). If E is not spanned, then one take the subsheaf F of E spanned by $H^0(C,E)$ and get two integers $\mathrm{rank}(F)$ and $\deg(F)$ and a morphism h_F. One can also study the stability of F. Serre duality gives $H^1(C,E)^* \cong H^0(C, E^* \otimes \omega_C)$ and hence we may look at the previous questions for the vector bundle $E^* \otimes \omega_C$. We assume $h^1(C,E) > 0$. Call $\rho(E)$ the rank of the image of the evaluation map $H^0(C, E^* \otimes \omega_C) \otimes \omega_C \to E^* \otimes \omega_C$. Serre duality shows that there is a natural map $u_{E,1} : E \to H^1(C,E) \otimes \omega_C)$ and $\rho(E) = \mathrm{rank}(\mathrm{Im}(u_{E,1})$. Obviously, $\rho(E) \leq \min\{\mathrm{rank}(E), h^1(C,E)\}$ and $\rho(E) > 0$ if $h^1(C,E) > 0$. In Section 2 we will prove the following results.

Theorem 2. *Assume either* $\mathrm{char}(\mathbb{K}) = 0$ *or* $\mathrm{char}(\mathbb{K}) \geq 2g-3$. *Assume that* C *is not hyperelliptic. Fix integers* $t \geq 1$, $r \geq s > 1$ *and* d *such that* $r(g+2) \leq d \leq r(2g-2) - s - 1$. *Then there exists* $E \in U(r,d)$ *such that* $h^1(C,E) = s$, $\mu_{t,E}$ *is surjective, and* $\rho(E) = s$.

Proposition 1. *Assume that* C *is hyperelliptic. Fix integers* r, d, t *such that* $r \geq 2$. *Let* E *be a semistable vector bundle with rank* r *and degree* d *such that* $h^1(C,E) > 0$.

(a) Assume $d \geq 2g-1$ *if* $r = 2$ *and* $d \geq r(g+1)$ *if* $r \geq 3$. *If* E *is not stable and* $r \geq 3$, *then assume* $d > r(g+1)$. *Then* $\mu_{1,E}$ *is not surjective.*

(b) Assume $t \geq 2$. *Then* $\mu_{t,E}$ *is not surjective.*

2. Proofs and Related Results

Remark 1. Fix an integer $t \geq 2$. If either $\mathrm{char}(\mathbb{K}) = 0$ or $\mathrm{char}(\mathbb{K}) > t$, then $S^t(E)$ is a direct factor of $E^{\otimes t}$. Hence if either $\mathrm{char}(\mathbb{K}) = 0$ or $\mathrm{char}(\mathbb{K}) > t$, then the surjectivity of $\mu_{t,E}$ implies the surjectivity of $\sigma_{t,E}$.

Remark 2. Fix integers $r \geq 2$ and d. Let E be a vector bundle of rank r and degree d. Any non-zero element of $H^1(C,E)$ induces a non-zero morphism $E \to \omega_C$. Hence $h^1(C,E) = 0$ if either $d \geq r(2g-2)$ and d is stable or $d \geq 1+r(2g-2)$ and E is semistable. Notice that $\deg(E \otimes \omega_C) = d + r(2g-2)$. Hence $h^1(C, E \otimes \omega_C(-P)) = 0$ for all $P \in C$ if either $d \geq r$ and E is stable or $d \geq r+1$ and E is semistable. Hence $E \otimes \omega_C$ is spanned if either $d \geq r$ and E is stable or $d \geq r+1$ and E is semistable.

Remark 3. Let A be any vector bundle such that $A \otimes \omega_C$ is spanned. If $\mu_{1,A}$ is surjective, then A is spanned.

Let E be a rank r vector bundle on C and $V \subseteq H^0(C,E)$ such that V spans E and $\dim(V) = r+1$. For any vector bundle F on C we have an exact sequence

$$0 \to F \otimes \det(E)^* \to V \otimes F \to F \to 0\,. \tag{1}$$

Hence from (1) we have the following rank r extension of the classical base point free pencil trick.

Lemma 1. *Let E be a rank r vector bundle on C and $V \subseteq H^0(C,E)$ such that V spans E and $\dim(V) = r+1$. Then*

$$\begin{aligned}\max\{h^1(C, F \otimes \det(E)^*) - (r+1)h^1(C,F), 0\} &\leq \dim(\mathrm{Coker}(\mu_{E,V;F})) \\ &\leq h^1(C, F \otimes \det(E)^*)\end{aligned}$$

for any vector bundle F on C. If $h^1(C, E \otimes F) = 0$, then

$$\dim(\mathrm{Coker}(\mu_{E,V;F}) = h^1(C, F \otimes \det(E)^*) - (r+1)h^1(C,F)\,.$$

If $h^0(C, F \otimes \det(E)^*) = 0$, then $\dim(\mathrm{Im}(\mu_{E,V;F})) = (r+1) \cdot h^0(C,F)$.

Lemma 2. *Fix integers $r \geq 2$, $t \geq 1$, and r line bundles L_i, $1 \leq i \leq r$, on C. Set $E := \oplus_{i=1}^r L_i$. $\mu_{t,E}$ is surjective if and only if for all choices $a_1, \ldots, a_t$ of t integers such that $1 \leq a_i \leq r$ for all i the multiplication map $\otimes_{i=1}^t H^0(C, L_{a_i}) \otimes H^0(C, \omega_C) \to H^0(C, (\otimes_{i=1}^t L_{a_i}) \otimes \omega_C)$ is surjective.*

Proof. Set $\Delta := \{1, \ldots, r\}^t$. We have $E^{\otimes t} \cong \oplus_{(a_1,\ldots,a_t)\in\Delta} \otimes_{i=1}^t L_{a_i}$ and this decomposition commutes with the global section functor and the multiplication by ω_C. □

Remark 4. Fix integers $r \geq 2$, $t \geq 1$, and r line bundles L_i, $1 \leq i \leq r$, on C. Set $E := \oplus_{i=1}^{r} L_i$. Hence $S^t(E)$ is a direct factor of $E^{\otimes t}$ in arbitrary characteristic. Thus Lemma 2 shows that $\sigma_{t,E}$ is surjective if for all choices $a_1, \dots, a_t$ of t integers such that $1 \leq a_i \leq r$ for all i the multiplication map $\otimes_{i=1}^{t} H^0(C, L_{a_i}) \otimes H^0(C, \omega_C) \to H^0(C, (\otimes_{i=1}^{t} L_{a_i}) \otimes \omega_C)$ is surjective.

Lemma 3. *Fix a spanned $L \in \mathrm{Pic}(C)$ such that $h^0(C, L) = 2$ and L is spanned. Then μ_L is surjective and $h^1(C, L) = \dim(\mathrm{Ker}(\mu_L))$.*

Proof. Set $e := h^1(C, L)$. Riemann-Roch gives $\deg(L) = g + 1 - e$. Apply Lemma 1 with $E := L$ and $V := H^0(C, L)$. Since $h^0(C, \omega_C \otimes L) = h^1(C, L)$, we get $h^1(C, L) = \dim(\mathrm{Ker}(\mu_L)$. Thus μ_L is surjective if and only if $h^0(C, L \otimes \omega_C) = 2g - e$. Since $h^0(C, L) \geq 2$, $H^1(C, L \otimes \omega_C) = 0$. Hence $h^0(C, L \otimes \omega_C) = \deg(L \otimes \omega_C) + 1 - g = 2g - 2 + g + 1 - e + 1 - g = 2g - e$, concluding the proof. □

We recall the following theorem of Castelnuovo (see [1], p. 151). For the positive characteristic case use the positive characteristic case of uniform position principle (see [9], Theorem 2.5) in part K-2 of [1], p. 152.

Lemma 4. *Fix $L \in \mathrm{Pic}(C)$ such that $h^0(C, L) \geq 4$, L is spanned and the morphism $h_L : C \to \mathbf{P}^n$, $n := h^0(C, L) - 1$, associated to the complete linear system $|L|$ is biirational onto its image. Assume either* $\mathrm{char}(\mathbb{K}) = 0$ *or* $\mathrm{char}(\mathbb{K}) \geq d - n + 1$. *Then $\sigma_{t,L}$ is surjective for all $t \geq 1$.*

Lemma 5. *Fix an integer $t \geq 1$, and t line bundles L_i, $1 \leq i \leq t$, (not necessarly distinct) on C. Assume $h^0(C, L_i) \geq 4$ for all i, that each L_i is spanned, and that all morphisms $h_{L_i} : C \to \mathbf{P}^{n_i}$, $n_i := h^0(C, L_i)$ are birational onto their images. Then the multiplication map $\alpha_t : (\otimes_{i=1}^{t} H^0(C, L_i)) \otimes H^0(C, (\otimes_{i=1}^{r} L_i) \otimes \omega_C)$ is surjective.*

Proof. If $t = 1$, then the result is true (Lemma 4). Now assume $t = 2$. If $L_1 = L_2$, then we apply the case $t = 2$ of Lemma 4. Now assume $L_1 \neq L_2$. Hence at least one of the integers $h^1(C, \omega_C \otimes L_1 \otimes L_2^*)$ and $h^1(C, \omega_C \otimes L_2 \otimes L_1^*)$ is zero. Exchanging if necessary the line bundles we may assume $h^1(C, \omega_C \otimes L_1 \otimes L_2^*) = 0$. Take a 2-dimensional linear subspace $V \subset H^0(C, L_2)$ spanning L_2. Apply Lemma 1 taking $E := L_2$ and $F := \omega_C \otimes L_1$. We get that the multiplication map $V \otimes H^0(C, L_1 \otimes \omega_C) \to H^0(C, L_1 \otimes L_2 \otimes \omega_C)$ is surjective. Hence the map $H^0(C, L_2) \otimes H^0(C, L_1 \otimes \omega_C) \to H^0(C, L_1 \otimes L_2 \otimes \omega_C)$ is surjective. Since μ_{L_1} is surjective (Lemma 4), we conclude the case $t = 2$. Now assume $t \geq 3$ and use induction on the integer t. We may order the line bundles $L_1, \dots, L_t$ so that $h^1(C, (\otimes_{i=1}^{t-1} L_i) \otimes L_t^*) = 0$. We fix a 2-dimensional linear subspace $V \subset H^0(C, L_t)$ spanning L_t. We apply Lemma 1 taking $E := L_t$ and

$F := (\otimes_{i=1}^{t-1} L_i) \otimes \omega_C)$ and then apply the case $t' := t-1$. □

Proposition 2. *Fix integers $r \ge 2$, $t \ge 1$, and r line bundles L_i, $1 \le i \le r$, on C. Assume $h^0(C, L_i) \ge 4$ for all i, that each L_i is spanned, and that all morphisms $h_{L_i} : C \to \mathbf{P}^{n_i}$, $n_i := h^0(C, L_i)$ are birational onto their images. Set $E := \oplus_{i=1}^r L_i$. Then $\mu_{t,E}$ and $\sigma_{t,E}$ are surjective.*

Proof. If $t = 1$, then the result is true by Lemma 4. If $t \ge 2$ use Lemmas 2 and 5 and Remark 4. □

If $r \ge 2$ the vector bundle $\oplus_{i=1}^r L_i$ is not stable. It is semistable if and only if $\deg(L_i) = \deg(L_1)$ for all i. Now we want to construct stable vector bundles such that $\mu_{t,E}$ and $\sigma_{t,E}$ are surjective and $h^1(C, E)$ is large. Instead of direct sums of line bundles we may use extensions of line bundles. We will not use the following result, which hover could be use to cover other sets of numerical data (g, t, r, d, h^1).

Proposition 3. *Fix $r \ge 2$ line bundles L_i, $1 \le i \le i \le r$, on C and set $F := \oplus_{i=1}^{r-1} L_i$. Let E be a general extension of L_r by F. Set $d_i := \deg(L_i)$. E is stable if one of the following set of conditions is satisfied.*

(a) $d_i = d_1$ if $2 \le i \le r-1$, $d_r = d_1 + 1$, and for any $M \in \mathrm{Pic}^{d_1}(C)$ at most g of the line bundles $L_1, \dots, L_{r-1}$ are isomorphic to M.

(b) $r = 2$, L_2 is arbitrary, $d_1 < d_2$, and for fixed L_2 the line bundle L_1 is general in $\mathrm{Pic}^{d_1}(C)$.

Proof. To prove part (a) we will only use that $g > 0$. Part (b) requires $g \ge 2$. Notice that $H^1(C, Hom(L_r, F)) = \oplus_{i=1}^{r-1} H^1(C, L_i \otimes L_r^*)$. Let G be any extension of L_r by F. Assume $d_i = d_1$ for $2 \le i \le r-1$, and $d_r = d_1 + 1$.

Claim. *G is stable if and only if no L_i, $1 \le i \le r-1$, is a direct factor of G.*

Proof of Claim. The "only if" part is obvious. Assume that G is not stable. Let $v : G \to L_r$ be the surjection induced by the extension. Take a proper subsheaf B of G with maximal slope. Hence $\mu(B) \ge \mu(G) = d_1 + 1/r$. Taking $\mathrm{rank}(B)$ maximal, we may assume that B is stable. Since B cannot be contained in F, the surjection v induces a non-zero map $u = v|B : B \to L_r$. Since $\mu(B) > d_r - 1$ and B is stable, u is surjective. Set $A := \mathrm{Ker}(u)$, $c := \deg(B)$ and $\rho := \mathrm{rank}(B)$. If $\rho = 1$, then u shows that L_r is a factor of G and that $E \cong F \oplus L_r$. Now assume $\rho \ge 2$. Hence A is a rank $\rho - 1$ subsheaf of F. Thus $\deg(A) \le (\rho - 1)d_1$ and we have eqaulity if and only if A is a direct factor of F. Since $c/\rho > \rho d_1$, $deg(A) = c - b_1 \ge (\rho - 1)d_1$. Hence A is a direct factor of F, say $F \cong A \oplus A'$ with $\mathrm{rank}(A') = r - \rho$. Since $B \cap F = A$, $B \cap A'$ is the zero-sheaf. Hence $B + A'$ is a rank r subsheaf of G which contains F and such that $u(A' + B) = L_r$. Hence $A' + B = E$. Since $B \cap A' = 0$, $G \cong A' \oplus B$, proving the claim.

Since $h^1(C, L_i \otimes L_r^*) = g$ for all $1 \le i \le r-1$ (Riemann-Roch and the assumption $d_r = d_i + 1$), we get part (a) and even a very precise description of the extensions of L_r by F which are stable. For instance, if the line bundles $L_1, \dots, L_{r-1}$ are pairwise non-isomorphic, then it is necessary and sufficient that all the components in $H^1(C, L_i \otimes L_r^*)$, $1 \le i \le r-1$, are non-zero. If s of the line bundles L_i are isomorphic to L_1, say the first s ones, it is sufficient that the first s components of the extensions are linearly independent and this condition may be satisfied if and only if $s \le g$. Now take the assumption of case (b). Up to a twist by L_2^* the openness of stability shows that it is sufficient to prove that for every integer $d < 0$ there is $N \in U(2, d)$ equipped with a surjection $N \to \mathcal{O}_C$. If $d \ge -g$ this is true, because there are rank 2 stable vector bundles on C with degree of stability $-d$ (see [8], Proposition 3.1). If $d < -g$, then use a dimensional count. □

Proposition 4. *Fix integers $r \ge 2$ such that $d \ge (r-1)(g+1)+1$. Write $a := \lceil (d+1)/r \rceil$. Fix any $L \in \mathrm{Pic}^a(C)$. Let F be a general element of $U(r-1, d-a)$. Let E be a general extension of L by E. Then $h^1(C, F) = 0$, $h^1(C, E) = h^1(C, L)$, and E is stable. If $\mu_{1,L}$ is surjective, then $\mu_{1,E}$ is surjective. If $d \ge (g+3)r+1$ and $\mu_{t,L}$ is surjective, then $\mu_{t,E}$ and $\sigma_{t,E}$ are surjective.*

Proof. Since $\mu(F) \ge g-1$, the generality of F gives $h^1(C, F) = 0$. Hence $h^1(C, E) = h^1(C, L)$. Every vector bundle on C is a flat limit of a family of stable vector bundles (see [8], Proposition 2.6, or, in arbitrary characteristic, [5], Corollary 2.2). A vector bundle G is stable if and only if $G \otimes R$ is stable for some $R \in \mathrm{Pic}(C)$. Since stability is an open condition, the previous sentences show that to prove that E is stable it is sufficient to prove the existence of a stable vector bundle, which is an extension of a line bundle of degree b by a rank $r-1$ vector bundle with degree $d-b$. Since $(d-b)/(r-1) < b$, this statement was proved by M. Teixidor i Bigas (see [10]). □

Remark 5. Fix vector bundles A, B on C and let E be any extension of B by E. If $h^1(C, A) = 0$, then $h^0(C, E) = h^0(C, A) + h^0(C, B)$ and $h^1(C, E) = h^1(C, B)$. Let $\epsilon \in h^1(C, A \otimes B^*)$ be the extension class which defines E. For al $\lambda \in \mathbb{K}$ let E_λ be the extension of B by A induced by $\lambda\epsilon \in h^1(C, A \otimes B^*)$. Hence $E_0 \cong A \oplus B$. Since the multiplication by λ induces an isomorphism of B if $\lambda \ne 0$, $E_\lambda \cong E$ for all $\lambda \in \mathbb{K}\backslash\{0\}$. The family $\{E_\lambda\}_{\lambda \in \mathbb{K}}$ is a flat family of vector bundles with constant cohomology. Fix any integer $t \ge 1$. In many cases (e.g. if both A and B are extensions of line bundles of positive degrees) also the cohomologies of the flat family $\{E_\lambda^{\otimes t} \otimes \omega_C\}_{\lambda \in \mathbb{K}}$. Hence if $\mu_{t,A\oplus B}$ (resp. $\sigma_{t,A\oplus B}$) is surjective, then $\mu_{t,E}$ (resp. $\sigma_{t,E}$) is surjective. Hence Propositions 2 and 4 and Lemma 3 give many stable vector bundles E such that $\mu_{t,E}$ and $\sigma_{t,E}$ are surjective.

Remark 6. If E is not spanned, but $E^{\otimes t} \otimes \omega_C$ (resp. $S^t(E) \otimes \omega_C$)) is spanned, then $\mu_{t,E}$ (resp. $\sigma_{t,E}$) is not surjective. $E^{\otimes t} \otimes \omega_C$ and.$S^t(E) \otimes \omega_C$ are spanned if either E is an iterated extension of line bundles of degree ≥ 2 or $t \geq 2$ and E is an iterated extension of line bundles of degree > 0.

Example 1. Assume that C is hyperelliptic. Let $R \in \mathrm{Pic}^2(C)$ denote the hyperelliptic line bundle. Let L be any line bundle on C such that $h^0(C, L) > 0$ and $h^1(C, L) > 0$. Set $d := \deg(L)$ and $k := h^0(C, L) - 1$. There is a unique degree $d-2k$ effective divisor D such that $h^0(C, \mathcal{O}_C(D)) = 1$ and $L \cong R^{\otimes k}(D)$. L is spanned if and only if $D = 0$, i.e. if and only if $d = 2k$. μ_L is surjective if and only if either $L \cong \mathcal{O}_C$ or $L \cong R$ or $L \cong \mathcal{O}_C(P)$ for some $P \in C$. Notice that $\dim(\mathrm{Coker}(\mu_{1,R^{\otimes k}}) = k - 1$ for all $1 \leq k \leq g - 1$. If $t \geq 2$ $\sigma_{t,L}$ is surjective if and only if $L \cong \mathcal{O}_C$.

Lemma 6. *Assume C not hyperelliptic. Fix an integer s such that $1 \leq s \leq g-2$. Let $S \subset C$ be a general subset such that $\sharp(S) = s$. Then $h^0(C, \omega_C(-S)) = g - s$ and $\omega_C(-S)$ is spanned. If $s \leq g - 3$, then the morphism $h_{\omega_C(-S)} : C \to \mathbf{P}^{g-s-1}$ induced by the complete linear system $|\omega_C(-S)|$ is birational onto its image.*

Proof. Since S is general, $h^0(C, \omega_C(-S)) = h^0(C, \omega_C) - \sharp(S)$. Let $h_{\omega_C} : C \to \mathbf{P}^{g-1}$ be the canonical embedding. A general hyperplane section of $h_{\omega_C}(C)$ is in linearly general position even if $\mathrm{char}(\mathbb{K}) > 0$, because the curve C is smooth and not a plane conic (see [9], Theorem 2.5). Since $h^0(C, \omega_C(-S)) = g - s$, $h_{\omega_C}(S)$ spans a linear subspace M of dimension $s-1$. Since S is general, M is a general $(s-1)$-dimensional linear subspace of $\mathbf{P}^{g-1}$ spanned by s points of $h_{\omega_C}(S)$. Since $s - 1 \leq g - 2$ and M is general, the linearly general position of $H \cap h_{\omega_C}(C)$ for a general hyperplane H containing M gives $h_{\omega_C}(S) = M \cap h_{\omega_C}(C)$ (scheme-theoretic intersection). This is equivalent to the spannedness of $\omega_C(-S)$. Now assume $s \leq g - 3$. Fix a general $P \in C$ and set $S' := S \cup \{P\}$. The first part gives the spannedness of $\omega_C(-S')$. Since $\omega_C(-S)$ is spanned, the spannedness of $\omega_C(-S')$ gives that $h_{\omega_C(-S)}$ is generically injective and separable, i.e. that it is birational onto its image. □

Proof of Proposition 1. Let $R \in \mathrm{Pic}^2(C)$ denote the hyperelliptic line bundle.

(i) Assume that $\mu_{1,E}$ is surjective. Since $d \geq r+1$, E is spanned (Remarks 2 and 3). Serre duality gives that any $a \in H^1(C, E)\backslash\{0\}$ induces a non-zero map $\phi_a : E \to \omega_C$. Set $M_a := \mathrm{Im}(\phi_a)$ and $N_a := \mathrm{Ker}(\phi_a)$. Since E is spanned, M_a is spanned. Hence there is an integer k_a such that $0 \leq k_a \leq g-1$ and $M_a \cong R^{\otimes k_a}$.

The map $H^0(C, E \otimes \omega_C) \to H^0(C, M_a \otimes \omega_C)$ induced by ϕ_a has cokernel of dimension at most $h^1(C, N_a \otimes \omega_C)$. First assume $h^1(C, N_a \otimes \omega_C) = 0$. We get that the natural map $H^0(C, E \otimes \omega_C) \to H^0(C, M_a \otimes \omega_C)$ is surjective. Hence the surjectivity of $\mu_{1,E}$ implies the surjectivity of μ_{1,M_a}. Hence $k_a \in \{0, 1\}$ (Example 1). Since E is semistable, we get $d \geq 2r$, contradiction. Now assume $h^1(C, N_a \otimes \omega_C) > 0$. First assume $r = 2$. In this case N_a is a line bundle of degree $d - 2k_a$. Since $2k_a \leq 2g - 2$, and $d \geq 2g - 1$, $\deg(N_a) > 0$. Hence $h^1(C, N_a \otimes \omega_C) = 0$, contradiction. Now assume $r \geq 3$. Since $k_a \geq 2$, we saw that $h^1(C, N_a \otimes \omega_C) > 0$. Any non-zero element of $H^1(C, N_a \otimes \omega_C)$ induces a non-zero map $\psi : N_a \to \mathcal{O}_C$. Set $A := \text{rank}(\psi)$. A is a rank $r - 2$ subsheaf of E with degree $\geq d - 2k_a \geq d - 2g + 2$. If E is stable (resp. semistable), then $r(d - 2g + 2) < (r - 2)d$ (resp. $r(d - 2g + 2) \leq (r - 2)d$), i.e. $d < r(g + 1)$ (resp. $d \leq r(g + 1)$), contradicting our assumptions and hence proving part (a).

(ii) Now we fix an integer $t \geq 2$ and take a, N_a, M_a as in (i). E is an extension of M_a by N_a and hence $E^{\otimes t}$ has a filtration whose graded subquotients are tensor powers of t copies of M_a, N_a and E. The last map in this filtration gives surjections $\mu' : E^{\otimes t} \to^{\otimes t}_{N_a}$. Since μ_{t,N_a} is not surjective, to prove part (b) it is sufficient to prove $h^1(C, \text{Ker}(\mu') \otimes \omega_C) == 0$. It is sufficient to prove $h^1(C, {N_a}^{\otimes x} \otimes E^{\otimes y} \otimes {M_a}^{\otimes z} \otimes \omega_C) = 0$ for all non-negative integers x, y, z such that $x + y + z = t$. Since $t \geq 2$, this is far easier thant part (i). □

Proof of Theorem 1. Since $d \geq r(g + 3)$, there are r integers $d_i \geq g + 3$ such that $d_1 + \cdots + d_r = d$. Fix a general $(L_1, \ldots, L_r) \in \text{Pic}^{d_1}(C) \times \cdots \times \text{Pic}^{d_r}(C)$. Set $F := \oplus_{i=1}^{r} L_i$. For any integer $x \geq g + 3$ a general $L \in \text{Pic}(C)^x(C)$ is non-special and very ample. Hence each μ_{t,L_i} is surjective (Lemma 4; in positive characteristic one may use the base-point-free-pencil trick for L_i if $d \geq 2g - 1$). Hence $\mu_{t,F}$ is surjective (Lemma 5). Every vector bundle on C is a flat limit of a family of stable vector bundles (see [2], Proposition 2.6, or, in arbitrary characteristic, [5], Corollary 2.2). Since $h^1(C, F) = 0$, a nearby stable vector bundle E has $h^1(C, E) = 0$ and $\mu_{t,E}$ is surjective. □

Proof of Theorem 2. If $r = s$, then set $G := \omega_C^{\oplus r}$. If $r > s$, then take $r - s$ pairwise non-isomorphic line bundles $M_i \in \text{Pic}^{2g-2}(C) \backslash \{\omega_C\}$, $1 \leq i \leq r - s$, and set $G := \omega_C^{\oplus r} \oplus \bigoplus_{i=1}^{r-s} M_i$. Let E be a general vectot bundle obtained from G making $r(2g - 2) - d$ general negative elementary transformations. E is stable (see [2], Thorem. 2.9). Since $d \geq r(g - 1) + s$ and these elementary transformations are general, $h^1(C, E) = h^1(C, G) = s$ (see [7], p. 101). The construction gives $\rho(E) = s$. To prove that $\mu_{t,E}$ is surjective if the negative elementary transformations are general, it is sufficient to prove the existence of a vector bundle F obtained from G making $r(2g - 2) - d$ negative elementary transformations such that $h^1(C, F) = s$ and $\mu_{t,F}$ is surjective. We may take as F a direct sum of r line bundles, each of them obtained from a factor of G making a general negative elementary transformation. For the factors ω_C we use Lemma 6. Then we apply Castelnuovo's Theorem to each factor of F (Lemma 4). □

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