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# ON DIVISIBILITY OF A RELATION OF THE FIBONACCI NUMBERS

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**Abstract:** Many interesting identities were found for the Fibonacci numbers $F_n$ defined by recurrence relation $F_{n+2} = F_n + F_{n+1}$ with $F_0 = 0$, $F_1 = 1$ and the Lucas numbers $L_n$ defined by the same recurrence but with the initial conditions $L_0 = 2$, $L_1 = 1$. In this paper we focus on the problems on divisibility of integers expressed by terms of sequences related to the Fibonacci and the Lucas numbers.

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## 1. Introduction

Various interesting relations hold for divisibility of the Fibonacci and the Lucas numbers. We can find most of them in monographies by Koshy [3] and Vajda [5], for example the relations $F_{2n-1} \mid F_{4n}+1$, $L_{2n} \mid F_{4n+1}+1$, $F_{2n} \mid F_{4n+2}-1$, $L_{2n+1} \mid F_{4n+3}-1$.

We will derive another divisibility relations using some famous formulas, in particular identities (85) and (86), which are given in [5] (see in Section 3 of this paper).

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## 2. Preliminary Lemmas

**Lemma 1.** *Let $a, n \geq 0$ be any integers. Let $k$ be any even integer. Then*

$$4\,|F_{3n+3} - F_{3n} - 2\ , \tag{1}$$

$$4\,|F_{3n} - 2n\ , \tag{2}$$

$$L_k + 2\,|L_{ak} + 2(-1)^{a+1}\ . \tag{3}$$

*Proof.* Using the defining formula of $F_n$ we have $F_{3n+3} - F_{3n} - 2 = 2(F_{3n+1} - 1)$. Thus, (1) is equivalent to the relation $2 \mid F_{3n+1} - 1$, which follows from relations $F_3 \mid F_n \Leftrightarrow 3 \mid n$ and $(F_{3n}, F_{3n+1}) = 1$ (see [3], Corollary 16.2 and Theorem 16.3).

We prove relation (2) by induction on $n$. It is clearly true for $n = 0$. Assume it is valid for any integer $n$, thus $4 \mid F_{3n} - 2n$. As

$$F_{3(n+1)} - 2(n+1) = (F_{3n} - 2n) + (F_{3n+3} - F_{3n} - 2)\ ,$$

we have that $4 \mid F_{3(n+1)} - 2(n+1)$ for every $n > 0$ with respect to (1).

To prove relation (3) we will consider these two cases:

First, $k \equiv 2 \bmod 4$. We use the identity $L_{2h} - 2(-1)^h = 5\,F_h^2$, where $h$ is any integer ((23) from [5]). Replacing $2h$ by $ak$, where $a$ is any integer, we have $L_{ak} + 2(-1)^{\frac{ak}{2}+1} = 5\,F_{\frac{ak}{2}}^2$. Setting $a = 1$ we have specially $L_k + 2 = 5\,F_{\frac{k}{2}}^2$. Thus

$$\frac{L_{ak} + 2(-1)^{a+1}}{L_k + 2} = \frac{5\,F_{\frac{ak}{2}}^2}{5\,F_{\frac{k}{2}}^2} = \left(\frac{F_{a\frac{k}{2}}}{F_{\frac{k}{2}}}\right)^2$$

and as $F_m \mid F_{mk}$, for any integer $m \neq 0$ (see [3], p. 196), the assertion follows.

Now, $k \equiv 0 \bmod 4$. We use the identities $L_{2n} + 2(-1)^n = L_n^2$ and $L_{2n} - 2(-1)^n = 5\,F_n^2$, where $n$ is any integer (see (17c) and (23) in [5]). Replacing $2n$ by $ak$, where $a$ is any integer we obtain

$$L_{ak} + 2 = L_{\frac{ak}{2}}^2 \quad \text{and} \quad L_{ak} - 2 = 5\,F_{\frac{ak}{2}}^2\ .$$

Setting $a = 1$ we have specially $L_k + 2 = L_{\frac{k}{2}}^2$. Thus

$$\frac{L_{ak} + 2(-1)^{a+1}}{L_k + 2} = \begin{cases} \frac{L_{ak}-2}{L_k+2} = \frac{5\,F_{\frac{ak}{2}}^2}{L_{\frac{k}{2}}^2} = 5\left(\frac{F_{a\frac{k}{2}}}{L_{\frac{k}{2}}}\right)^2\ , & a \equiv 0 \bmod 2\ , \\ \frac{L_{ak}+2}{L_k+2} = \frac{L_{\frac{ak}{2}}^2}{L_{\frac{k}{2}}^2} = \left(\frac{L_{a\frac{k}{2}}}{L_{\frac{k}{2}}}\right)^2\ , & a \equiv 1 \bmod 2 \end{cases}$$

and as $L_m \mid F_{am}$ for any even integer $a$ and $L_m \mid L_{am}$, for an odd $a$, the proof of this case is over. □

**Lemma 2.** *Let $n \geq 3$ be any positive integer and let $a$, $l$ be any integers. Then*

$$L_n \equiv F_{n-3} \bmod F_n \ , \tag{4}$$

$$L_{2nl+a} \equiv (-1)^{nl}\, L_a \bmod F_n \ . \tag{5}$$

*Proof.* Using identity (7a) in [5] we have $L_n - 2\,F_n = F_{n-3}$, hence $L_n \equiv F_{n-3} \bmod F_n$. From the relation $L_{k+m} - (-1)^m L_{k-m} = 5F_m F_k$ (see (17b) in [5]), setting $k = nl + a$ and $m = nl$ we obtain $L_{2nl+a} - (-1)^{nl} L_a = 5F_{nl+a}F_{nl}$ and congruence (5) is proved as $F_n \mid F_{nl}$. □

**Corollary 3.** *Let $l$ be any integer.*

*(i) If $n$ is any positive even integer then*

$$L_{2nl} \equiv 2 \bmod F_n \ , \qquad L_{2nl+n} \equiv F_{n-3} \bmod F_n \ .$$

*(ii) If $n$ is any positive odd integer then*

$$L_{4nl} \equiv 2 \bmod F_n \ , \qquad L_{4nl+n} \equiv F_{n-3} \bmod F_n \ ,$$

$$L_{4nl+2n} \equiv -2 \bmod F_n \ , \qquad L_{4nl+3n} \equiv -\, F_{n-3} \bmod F_n \ .$$

*Proof.* Using relation (5) we have for any integer $n \geq 3$, setting $l = 1$, $a = 0$ and $a = n$, respectively,

$$L_{2n} \equiv (-1)^n\, 2 \bmod F_n \ , \ L_{3n} \equiv (-1)^n\, L_n \equiv (-1)^n\, F_{n-3} \bmod F_n$$

and the assertion easily follows. □

## 3. Main Results

**Theorem 4.** *Let $k \neq 0$ be any even integer or $k = -1$ or $k = -3$. Then the relation*

$$(L_k + 2) \mid \frac{F_{kn}}{F_k} + n\,(-1)^n$$

*holds for any nonnegative integer $n$.*

*Proof.* We divide the proof of the assertion into the following cases:

(i) For $k = -1$ the divisor $L_k + 2$ is equal to 1, thus the assertion is obvious.

(ii) For $k = -3$ the divisor $L_k + 2$ is equal to $-2$, thus we have to prove that: $\frac{F_{-3n}}{F_{-3}} + n\,(-1)^n$ is even for any $n$. Using formula $F_{-n} = (-1)^{n+1} F_n$ (see

(2) in [5]), we have

$$\frac{F_{-3n}}{F_{-3}} + n\,(-1)^n = (-1)^n \frac{1}{2}(2n - F_{3n})$$

and this case follows from relation (2).

(iii) Let $k \neq 0$ be any even integer.

(a) For $n = 0$ and $n = 1$ the assertion clearly holds.

(b) For any integer $n \geq 2$ we use the identities

$$\frac{F_{kn}}{F_k} = \sum_{i=0}^{\frac{n-3}{2}} (-1)^{i\,k}\, L_{(n-2i-1)k} + (-1)^{\frac{k}{2}(n-1)} = \sum_{i=0}^{\frac{n}{2}-1} (-1)^{i\,k}\, L_{(n-2i-1)k} \ ,$$

which hold for an odd $n$ and an even $n$, respectively (see (85) and (86) from [5]). Denoting

$$\varepsilon(k,n) = \begin{cases} 0 \ , & n \equiv 0 \bmod 2 \ , \\ (-1)^{\frac{k}{2}(n-1)} \ , & n \equiv 1 \bmod 2 \ , \end{cases}$$

we can join the previous two identities into the relation

$$\frac{F_{kn}}{F_k} = \sum_{i=0}^{\lfloor \frac{n-2}{2} \rfloor} (-1)^{i\,k}\, L_{(n-2i-1)k} + \varepsilon(k,n) \ .$$

Thus, we have

$$\begin{aligned} \frac{F_{kn}}{F_k} + n\,(-1)^n &= \sum_{i=0}^{\lfloor \frac{n-2}{2} \rfloor} L_{(n-2i-1)k} + \varepsilon(k,n) + n\,(-1)^n \\ &= \sum_{i=0}^{\lfloor \frac{n-2}{2} \rfloor} \left(L_{(n-2i-1)k} + 2\,(-1)^n\right) = \sum_{i=0}^{\lfloor \frac{n-2}{2} \rfloor} \left(L_{(n-2i-1)k} + 2\,(-1)^{n-2i}\right) \quad (6) \end{aligned}$$

and the divisibility of the summand of the previous sum for any $i$ by $L_k + 2$ follows from (3). □

**Theorem 5.** *$F_{nF_n^k}$ is divisible by $F_n^{k+1}$ for all $n \geq 1$ and $k \geq 1$.*

*Proof.* We use induction on $k$. For $k = 0$ the proved divisibility is clear. Let us consider that it holds for $k \geq 1$, thus $F_n^{k+1} \mid F_{nF_n^k}$, and we will show that it holds for $k+1$ too. As $F_{nF_n^k} \mid F_{nF_n^{k+1}}$ we have that $F_n^{k+1} \mid F_{nF_n^{k+1}}$ with respect to the induction hypothesis.

It is sufficient to prove that $F_n \mid \frac{F_{nF_n^{k+1}}}{F_n^{k+1}}$ as it means $F_n \mid \frac{F_{nF_n^{k+1}}}{F_{nF_n^k}}$.

The base of our proof are relations from [5] (see relations (85) and (86)):

$$\frac{F_{ht}}{F_t} = \sum_{i=0}^{\frac{h-3}{2}} (-1)^{it} L_{(h-2i-1)t} + (-1)^{\frac{(h-1)t}{2}} \qquad \text{(for odd } h \geq 3\text{)} \ , \tag{7}$$

$$\frac{F_{ht}}{F_t} = \sum_{i=0}^{\frac{h}{2}-1} (-1)^{it} L_{(h-2i-1)t} \qquad \text{(for even } h \geq 2\text{)} \tag{8}$$

and we always set $t = nF_n^k$ and $h = F_n$.

Consider the following four cases:

(i) Let $n \equiv 2, 4 \bmod 6$, which means that $n$ is an even integer and $F_n$ is an odd integer.

From identity (7) and from the relation $L_{2nl} \equiv 2 \bmod F_n$ in Corollary 3 we have

$$\begin{aligned} \frac{F_{nF_n^{k+1}}}{F_{nF_n^k}} &= \sum_{i=0}^{\frac{F_n-3}{2}} (-1)^{inF_n^k} L_{(F_n-1-2i)\,nF_n^k} + (-1)^{\frac{(F_n-1)nF_n^k}{2}} \\ &= \sum_{i=0}^{\frac{F_n-3}{2}} L_{2n\left(\frac{F_n-1}{2}-i\right)\,F_n^k} + 1 \equiv \left(\frac{F_n-3}{2}+1\right) \cdot 2 + 1 \equiv 0 \bmod F_n \ . \end{aligned}$$

(ii) Let $n \equiv 0 \bmod 6$, which means that $n$ and $F_n$ are even integers.

From identity (8) and from the relation $L_{2nl} \equiv 2 \bmod F_n$ in Corollary 3 we have

$$\begin{aligned} \frac{F_{nF_n^{k+1}}}{F_{nF_n^k}} &= \sum_{i=0}^{\frac{F_n}{2}-1} (-1)^{inF_n^k} L_{(F_n-1-2i)\,nF_n^k} \\ &= \sum_{i=0}^{\frac{F_n}{2}-1} L_{2n(F_n-1-2i)\,\frac{F_n}{2}F_n^{k-1}} \equiv \frac{F_n}{2} \cdot 2 \equiv F_n \equiv 0 \bmod F_n \ . \end{aligned}$$

(iii) Let $n \equiv 1, 5 \bmod 6$, which means that $n$ and $F_n$ are odd integers.

We prove the assertion similarly as in case (i) from identity (7) and the congruences $F_{6k+1} \equiv F_{6k+5} \equiv 1 \bmod 4$, $L_{4nl} \equiv 2 \bmod F_n$, $L_{4nl+2n} \equiv -2 \bmod F_n$.

(iv) Let $n \equiv 3 \bmod 6$, which means that $n$ is an odd integer and $F_n$ is an even integer.

We obtain the assertion similarly as in case (ii) from identity (8) and the congruences $F_{6k+3} \equiv 2 \bmod 4$, $L_{4nl} \equiv 2 \bmod F_n$ and $L_{2n(2l+1)} \equiv -2 \bmod F_n$ (see Corollary 3). □

## 4. Concluding Remark

Our proof of Theorem 5 is based on derivation of some congruences for the Fibonacci and the Lucas numbers. Benjamin and Rouse (see [2], Corollary 4) chose a purely combinatorial approach to this problem using the number of ways to tile a board with squares and dominoes.

## References


[1] M. Bencze, Problem B-1000, *The Fibonacci Quarterly*, **43** (2005).

[2] A.T. Benjamin, J.A. Rouse, When does $F_m^L$ divides $F_n$? A combinatorial solution, *Applications of Fibonacci Numbers*, **10** (2006), 27-34.

[3] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, A Wiley-Interscience Publications, New York (2002).

[4] J. Pla, Problem B-994, *The Fibonacci Quarterly*, **43** (2005).

[5] S. Vajda, *Fibonacci and Lucas Numbers and the Golden Section*, Holstel Press (1989).