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BIVARIATE OPERATORS WHICH INTERPOLATE SOME PARTIAL DERIVATIVES OF A FUNCTION ON THE VERTICES OF A SQUARE

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Abstract: In this article we study the operator

$$B_r = P_1'Q_r'' \oplus P_2'Q_{r-1}'' \oplus \cdots \oplus P_r'Q_1'',$$

where P_i, Q_j are univariate Birkhoff interpolation projectors with two nodes (see [1]) and P_i', Q_j'' are the corresponding parametric extensions (see [4]). The univariate projectors P_i, Q_j are chosen so that the bivariate projectors P_i', Q_j'' form the chains in a lattice of projectors, i.e.

$$P_1' \leq \cdots \leq P_r', \quad Q_1'' \leq \cdots \leq Q_r''.$$

We give the range space, the interpolation properties and the remainder term for this operator. The operator B_r interpolates some partial derivatives of a function on the vertices of the square. Some comparations about approximation order and information used with tensor product operator are given.

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1. Introduction

Let X, Y be the linear spaces on $\mathbb{R}$.

The linear operator P defined on space X is called projector if $P^2 = P$.

The operator $P^C = I - P$, where I is identity operator, is called the remainder projector of P.

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The range space of projector P is

$$\mathcal{R}(P) = \{Pf | f \in X\}\,.$$

The precision set of projector P is denoted by $\mathcal{P}(P)$.

The Boolean sum of two projectors is defined by

$$P \oplus Q = P + Q - PQ\,.$$

Proposition 1. *If P, Q are commutative projectors then we have*

$$\begin{aligned} \mathcal{R}(P \oplus Q) &= \mathcal{R}(P) + \mathcal{R}(Q)\,, \\ \mathcal{P}(P \oplus Q) &= \mathcal{P}(P) \cup \mathcal{P}(Q)\,. \end{aligned} \tag{1}$$

If P_1, P_2 are projectors on space X, we define relation "$\leq$":

$$P_1 \leq P_2 \Leftrightarrow P_1 P_2 = P_1\,. \tag{2}$$

Boolean methods in multivariate approximation were introduced by W.J. Gordon in [5] in 1969. These methods are extented by F.J. Delvos, H. Posdorf, W. Schempp in [3], [4].

We give the following result from [4].

Proposition 2. *Let $r \in \mathbb{N}$, $P_1,\ldots, P_r$ projectors on $C(X)$ and $Q_1,\ldots, Q_r$ projectors on $C(Y)$. Let $P'_1,\ldots, P'_r$, $Q''_1,\ldots, Q''_r$ be the corresponding parametric extension on $C(X \times Y)$. We assume that*

$$P'_1 \leq P'_2 \leq \cdots \leq P'_r, \quad Q''_1 \leq Q''_2 \leq \cdots \leq Q''_r \tag{3}$$

and

$$P'_k Q''_l = Q''_l, P'_k, \quad 1 \leq k, l \leq r\,.$$

We have that the operator

$$B_r = P'_1 Q''_r \oplus P'_2 Q''_{r-1} \oplus \cdots \oplus P'_r Q''_1 \tag{4}$$

is projector and it has representation

$$B_r = \sum_{m=1}^{r} P'_m Q''_{r+1-m} - \sum_{m=1}^{r-1} P'_m Q''_{r-m}\,. \tag{5}$$

Moreover, we have

$$B_r^C = P_r'^C + P_{r-1}'^C Q_1''^C + \cdots + P_1'^C Q_{r-1}''^C + Q_r''^C - (P_r'^C Q_1''^C + \cdots + P_1'^C Q_r''^C)\,. \tag{6}$$

In this article we study the projector B_r in the case when the projectors P_i and Q_j are some univariate Birkhoff polynomial interpolation projectors with two nodes.

2. Main Result

Let $n_k^1 \in \mathbb{N}$, $k=\overline{1,r}$, $n_l^2 \in N$, $l=\overline{1,r}$ and $p_1, p_2 \in \mathbb{N}$, with $p_1 < n_k^1 - 1$, $k=\overline{1,r}$ and $p_2 < n_l^2 - 1$, $l=\overline{1,r}$.

Assume that

$$\begin{array}{l} 1 \le n_1^1 \le n_2^1 \le \cdots \le n_r^1 \\ 1 \le n_1^2 \le n_2^2 \le \cdots \le n_r^2 \end{array} \tag{7}$$

Let the univariate Birkhoff interpolation projectors with two nodes, named (n, p)-interpolation projectors (see [1]), given by

$$(P_k f_1)(x) = \sum_{i=0}^{n_k^1-2} b_{1i}^k(x) f_1^{(i)}(a) + b_{2p_1}^k(x) f_1^{(p_1)}(b), \quad 1 \le k \le r$$

$$(Q_l f_2)(y) = \sum_{j=0}^{n_l^2-2} \widetilde{b}_{1j}^l(y) f_2^{(j)}(c) + \widetilde{b}_{2p_2}^l(y) f_2^{(p_2)}(d), \quad 1 \le l \le r$$

where the function $f_1 : [a,b] \to \mathbb{R}$ is assumed to be n_r^1 continously differentiable, $f_2 : [c,d] \to \mathbb{R}$ is assumed to be n_r^2 continously differentiable and $h = b-a = d-c$.

The cardinal functions b_{1i}^k, $k=\overline{1,r}$ and $b_{2p_1}^k$, $k=\overline{1,r}$ satisfy the conditions

$$\begin{cases} (b_{1i}^k)^{(j)}(a) = \delta_{ij}, \quad j = \overline{0, n_k^1 - 2}\ , \\ (b_{1i}^k)^{(p_1)}(b) = 0\,, \end{cases}$$

for $i = \overline{0, n_k^1 - 2}$ and respectively

$$\begin{cases} (b_{1p_1}^k)^{(j)}(a) = 0, \quad j = \overline{0, n_k^1 - 2}\ , \\ (b_{1p_1}^k)^{(p_1)}(b) = 1\,, \end{cases} \quad .$$

These functions are given by formulas

$$b_{1i}^k(x) = \frac{(x-a)^i}{i!}, \quad i = \overline{0, p_1 - 1}\,,$$

$$b_{1i}^k(x) = \frac{(x-a)^i}{i!} + \frac{(n_k^1 - p_1 - 1)!}{(n_k^1 - 1)!(i-p_1)!} \frac{(x-a)^{n_k^1-1}}{(b-a)^{n_k^1-i-1}}, \quad i = \overline{p_1, n_k^1 - 2}\,,$$

$$b_{2p_1}^k(x) = \frac{(n_k^1 - p_1 - 1)!}{(n_k^1 - 1)!} \frac{(x-a)^{n_k^1-1}}{(b-a)^{n_k^1-p_1-1}}\,.$$

We have analogous conditions and formulas for the functions $\widetilde{b}_{1j}^l$, $l=\overline{1,r}$, $j = \overline{0, n_l^2 - 2}$ and $\widetilde{b}_{2p_2}^l$, $l=\overline{1,r}$

If $f:[a,b]\times[c,d]\to\mathbb{R}$ $f\in C^{n_r^1,n_r^2}([a,b]\times[c,d])$, then the parametric extensions are given by

$$(P_k'f)(x,y)=\sum_{i=0}^{n_k^1-2} b_{1i}^k(x)f^{(i,0)}(a,y)+b_{2p_1}^k(x)f^{(p_1,0)}(b,y),\quad 1\le k\le r\,,$$

$$(Q_l''f)(x,y)=\sum_{j=0}^{n_l^2-2} \widetilde{b}_{1j}^l(y)f^{(0,j)}(x,c)+\widetilde{b}_{2p_2}^l(y)f^{(0,p_2)}(y,d),\quad 1\le l\le r.$$

Theorem 3. *The parametric extensions*

$$P_1',\dots,P_r',Q_1'',\dots,Q_r''$$

are bivariate interpolations projectors which form the chains, i.e.

$$P_1'\le\cdots\le P_r',\quad Q_1''\le\cdots\le Q_r''\,.$$

The proof of this theorem results using conditions (7).

The projectors P_k',Q_l'' are commutative

$$P_k'Q_l''=Q_l''P_k',\quad 1\le k,l\le r\,.$$

The projectors $P_1',\dots,P_r',Q_1'',\dots,Q_r''$ generate a distributive lattice on $C([a,b]\times[c,d])$. A special element of this lattice is

$$B_r=P_1'Q_r''\oplus\cdots\oplus P_r'Q_1'',\quad r\in\mathbb{N}\,. \tag{8}$$

Next we give some properties of projector B_r from (8).

Proposition 4. *The range space of projector B_r is given by*

$$\mathcal{R}(B_r)=\Pi_{n_1^1-1}\otimes\Pi_{n_r^2-1}+\cdots+\Pi_{n_r^1-1}\otimes\Pi_{n_1^2-1}. \tag{9}$$

Proof. Taking into account Proposition 1 we have

$$\mathcal{R}(B_r)=\mathcal{R}(P_1'Q_r'')+\cdots+\mathcal{R}(P_r'Q_1'')\,.$$

As

$$\mathcal{R}(P_k)=\Pi_{n_k^1-1},\quad 1\le k\le r\,,$$

$$\mathcal{R}(Q_n)=\Pi_{n_l^1-1},\quad 1\le l\le r\,,$$

it follows (9). □

Proposition 5. *The projector B_r satisfies interpolation properties*

$$(B_rf)^{(p,q)}(a,c)=f^{(p,q)}(a,c)$$

for $p=\overline{0,n_1^1-2}$ *and* $q=\overline{0,n_r^2-2}$, $p=\overline{n_1^1-1,n_2^1-2}$ *and* $q=\overline{0,n_{r-1}^2-2}$, $\dots$,

$p = \overline{n^1_{r-1} - 1, n^1_r - 2}$ and $q = \overline{0, n^2_1 - 2}$,

$$(B_r f)^{(p,q)}(a, d) = f^{(p,q)}(a, d)$$

for $p = \overline{0, n^1_r - 2}$ and $q = p_2$,

$$(B_r f)^{(p,q)}(b, c) = f^{(p,q)}(b, c)$$

for $p = p_1$ and $q = \overline{0, n^2_r - 2}$,

$$(B_r f)^{(p,q)}(b, d) = f^{(p,q)}(b, d)$$

for $p = p_1$ and $q = p_2$.

Proof. From Proposition 1 it follows that

$$\mathcal{P}(B_r) = \mathcal{P}(P'_1 Q''_r) \cup \cdots \cup \mathcal{P}(P'_r Q''_1)$$

and therefore we get the interpolation properties of B_r. □

Example 6. We take $r = 3$, $n^1_1 = n^2_1 = 2$, $n^1_2 = n^2_2 = 3$, $n^1_3 = n^2_3 = 4$, $p_1 = p_2 = 1$. The projector

$$B_3 = P'_1 Q''_3 \oplus P'_2 Q''_2 \oplus P'_3 Q''_1$$

has the following interpolation properties

$$(B_3 f)^{(p,q)}(a, c) = f^{(p,q)}(a, c)$$

$$p = 0, \quad q = \overline{0, 2}; \quad p = 1, \quad q = \overline{0, 1}; \quad p = 2, q = 0\,,$$

$$(B_3 f)^{(p,q)}(a, d) = f^{(p,q)}(a, d), \quad p = \overline{0, 2}, \quad q = 1\,,$$

$$(B_3 f)^{(p,q)}(b, c) = f^{(p,q)}(b, c), \quad p = 1, \quad q = \overline{0, 2}\,,$$

$$(B_3 f)^{(1,1)}(b, d) = f^{(1,1)}(b, d)\,.$$

Taking into account Proposition 2 we obtain the following representation for the interpolation function

$$B_r f(x, y) = \sum_{m=1}^{r} \sum_{i=0}^{n^1_m - 2} \sum_{j=0}^{n^2_{r+1-m} - 2} b^m_{1i}(x) \widetilde{b}^{r+1-m}_{1j}(y) f^{(i,j)}(a, c)$$

$$- \sum_{m=1}^{r-1} \sum_{i=0}^{n^1_m - 2} \sum_{j=0}^{n^2_{r-m} - 2} b^m_{1i}(x) \widetilde{b}^{r-m}_{1j}(y) f^{(i,j)}(a, c) + \sum_{m=1}^{r} \sum_{i=0}^{n^1_m - 2} b^m_{1i}(x) \widetilde{b}^{r+1-m}_{2p_2}(y) f^{(i,p_2)}(a, d)$$

$$- \sum_{m=1}^{r-1} \sum_{i=0}^{n^1_m - 2} b^m_{1i}(x) \widetilde{b}^{r-m}_{2p_2}(y) f^{(i,p_2)}(a, d) + \sum_{m=1}^{r} \sum_{j=0}^{n^2_{r+1-m} - 2} b^m_{2p_1}(x) \widetilde{b}^{r+1-m}_{1j}(y) f^{(p_1,j)}(b, c)$$

$$-\sum_{m=1}^{r-1}\sum_{j=0}^{n^2_{r-m}-2} b^m_{2p_1}(x)\widetilde{b}^{r-m}_{1j}(y)f^{(p_1,j)}(b,c)+\sum_{m=1}^{r} b^m_{2p_1}(x)\widetilde{b}^{r+1-m}_{2p_2}(y)f^{(p_1,p_2)}(b,d)$$
$$-\sum_{m=1}^{r-1} b^m_{2p_1}(x)\widetilde{b}^{r-m}_{2p_2}(y)f^{(p_1,p_2)}(b,d).$$

Proposition 7. *If $f\in C^{n^1_r,n^2_r}([a,b]\times[c,d])$ we can give the following representation for the remainder term*

$$f(x,y)-(B_rf)(x,y)=U_r(x)f^{(n^1_r,0)}(\sigma_r,y)+V_r(y)f^{(0,n^2_r)}(x,\tau_r) \tag{10}$$
$$-\sum_{i=1}^{r}U_i(x)V_{r+1-i}(y)f^{(n^1_i,n^2_{r+1-i})}(\xi_i,\eta_{r+1-i})+\sum_{i=1}^{r-1}U_i(x)V_{r-i}(y)f^{(n^1_i,n^2_{r-i})}(\sigma_i,\tau_{r-i})$$

where

$$U_k(x)=\frac{(x-a)^{n^1_k-1}}{(n^1_k-1)!}\left[\frac{x-a}{n^1_k}-\frac{b-a}{n^1_k-p_1}\right],\quad k=\overline{1,r}\,,$$
$$V_l(y)=\frac{(y-c)^{n^2_l-1}}{(n^2_l-1)!}\left[\frac{y-c}{n^2_l}-\frac{d-c}{n^2_l-p_2}\right],\quad k=\overline{1,r}\,,$$

and $\xi_i,\sigma_i\in[a,b]$ and $\eta_i,\tau_i\in[c,d], i=\overline{1,r}$

Proof. If $f_1\in C^{n^1_k}[a,b]$ and $f_2\in C^{n^2_l}[c,d]$, then:

$$(P^c_kf_1)(x)=f_1(x)-(P_kf_1)(x)=U_k(x)f_1^{(n^1_k)}(\xi_k),\quad 1\le k\le r\,,$$
$$(Q^c_lf_2)(y)=f_2(y)-(Q_lf_2)(y)=V_l(y)f_2^{(n^2_l)}(\eta_l),\quad 1\le l\le r\,.$$

Using formula (6) we get (10). □

Proposition 8. *Let $h=b-a=d-c$ and $q=\min\{n^1_{r-m}+n^2_m,\ 0\le m\le r\}$ with $\alpha_0=0$, $\beta_0=0$. Than we have*

$$f(x,y)-(B^B_rf)(x,y)=O(h^q),\quad h\to 0. \tag{11}$$

Proof. Taking into account

$$f_1(x)-(P_kf_1)(x)=O(h^{n^1_k}),\quad 1\le k\le r\,,$$
$$f_2(y)-(Q_lf_2)(y)=O(h^{n^2_l}),\quad 1\le l\le r\,,$$

and using formula (6) we get (11). □

3. Application

If we take $n_i^s = li + k$, for $s = 1, 2$ and $l = 1, k \geq 1$ or $l \geq 2, k \geq 0$ we have

$$q = \min\{rl + k, rl + 2k\} = rl + k\,.$$

The tensor product operator

$$P = P_r' Q_r''$$

has the same order of approximation, i.e. $rl+k$, and the cardinal of information set about function f used by operator

$$|\mathcal{P}(P)| = (lr + k)^2\,.$$

We have

$$|\mathcal{P}(B_r)| = (lr + k)^2 - l^2 \frac{r(r-1)}{2}$$

and for $r > 1$ we have

$$|\mathcal{P}(P)| < |\mathcal{P}(B_r)| \text{ and } \mathcal{P}(B_r) \subset \mathcal{P}(P)\,.$$

It follows that the projector B_r is more efficient than tensor product projector P.

We approximate the function $f : [0, h] \times [0, h] \to \mathbb{R}$ $f(x, y) = e^{-x^2-y^2}$, where $h = 1/n$, $n = 2, 4, 5, 10$ using the projectors B_r and $P_r' Q_r''$ for $r = 5$, $n_i^1 = n_i^2 = i + 1$, $i = \overline{1, 5}$, $p_1 = p_2 = 1$.

The cardinals of data sets about the approximate functions used by these operators: $|\mathcal{P}(B_5)| = 26$, $|\mathcal{P}(P_5' Q_5'')| = 36$.

The maximal values of absolute errors are listed in tabel. We notice that we obtain the same values of error for small domain.

n	h	$\max\|f - P_5' Q_5'' f\|$	$\max\|f - B_5 f\|$
2	0.5	$6 \cdot 10^{-4}$	$1 \cdot 10^{-3}$
4	0.25	$4 \cdot 10^{-5}$	$1.5 \cdot 10^{-4}$
5	0.2	$4 \cdot 10^{-6}$	$4 \cdot 10^{-6}$
10	0.1	$6 \cdot 10^{-8}$	$6 \cdot 10^{-8}$

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