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(E, M)-TOPOLOGICAL FUNCTORS ON TOPOLOGICAL ALGEBRAS

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Abstract: Conditions to preserve the topological relations among categories through compatible algebraic structures are investigated. Besides discussing a few specific examples, some related properties of topological algebra are studied.

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1. Introduction

Many categorical properties of topological algebras have been studied in the past decade. However, it seems that there is not much done on relationships among topological algebras for different topological categories. Nevertheless, H. -E. Porst has proved (see [13]), among other things, that the category ***Unif Grp*** of uniform groups is a full bireflective subcategory of the category ***Top Grp*** of topological groups.

In this paper, we will study conditions under which an (E, M)-topological functor between two categories can be carried over to the corresponding topological algebras. It turns out that this result has many interesting applications. For instance, it follows that "monotopological algebras" are extremal epireflective subcategories of topological algebras.

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In Section 2 we briefly review some concepts needed for our work. We will establish our main result and some of its consequences in Section 3.

2. Preliminaries

A *source* in a category $\boldsymbol{X}$ is a class $(X, f_i : X \to X_i)_{i\in I}$ of $\boldsymbol{X}$-morphisms from a fixed $\boldsymbol{X}$-object.

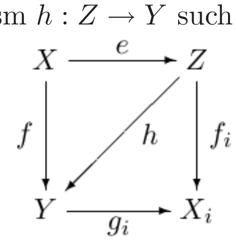
$\boldsymbol{X}$ is an *(E, M)-category* (see [8]) iff E is a class of epimorphisms in $\boldsymbol{X}$ and M is a class of sources in $\boldsymbol{X}$ such that:

(i) E and M are closed under composition with isomorphisms,

(ii) $\boldsymbol{X}$ is (E, M)-factorizable, i.e. for every source $(X, f_i : X \to X_i)_{i\in I}$ in $\boldsymbol{X}$ there exists an epimorphism $e : X \to Y$ belonging to E and a source $(Y, g_i : Y \to X_i)_{i\in I}$ in M such that $f_i = g_i \circ e$ for each $i \in I$,

and

(iii) $\boldsymbol{X}$ has the (E, M)-diagonalization property, i.e. if $f : X \to Y$ is an $\boldsymbol{X}$-morphism, $(Z, f_i : Z \to X_i)_{i\in I}$ is a source in $\boldsymbol{X}$, $e : X \to Z$ is E, and $(Y, g_i : Y \to X_i)_{i\in I}$ is a source in M with $f_i \circ e = g_i \circ f$ for each $i \in I$, then there exists a unique morphism $h : Z \to Y$ such that



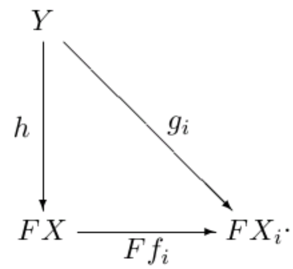
commutes for each $i \in I$.

A functor $T : \boldsymbol{Y} \to \boldsymbol{X}$ from a category $\boldsymbol{Y}$ into an (E, M)-category $\boldsymbol{X}$ is called *(E, M)-topological* iff for each family $(Y_i)_{i\in I}$ of $\boldsymbol{Y}$-objects and each source $(X, g_i : X \to TY_i)_{i\in I}$ in M there exists a T-initial source $(Y, f_i : Y \to Y_i)_{i\in I}$ in $\boldsymbol{Y}$ which T-lifts $(X, g_i : X \to TY_i)_{i\in I}$. In this case $\boldsymbol{Y}$ is said to be $(\mathbf{E}, \mathbf{M})$-*topological over* $\boldsymbol{X}$.

An (epi, monosource)-topological category over $\boldsymbol{Set}$ is called a *monotopological category*. An (iso, source) - topological category over $\boldsymbol{Set}$ is said to be a *topological category*. It turns out (see [1]) that such categories are precisely quotient reflective subcategories of topological categories.

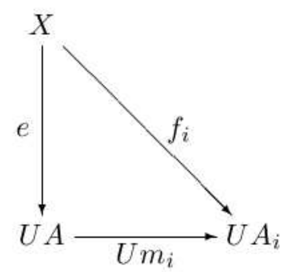
Suppose $F : \boldsymbol{X} \to \boldsymbol{Y}$ is a functor. Let X and Y be two objects in the categories $\boldsymbol{X}$ and $\boldsymbol{Y}$ respectively. A morphism $e : Y \to FX$ in $\boldsymbol{Y}$ *F-generates* X iff any two morphisms $r : X \to X'$ and $s : X \to X'$ in $\boldsymbol{X}$ with the property that $(Fr)e = (Fs)e$ are always equal.

A functor $F : \boldsymbol{X} \to \boldsymbol{Y}$ is called *topologically algebraic* iff for each family $(X_i)_{i \in I}$ of objects in $\boldsymbol{X}$ and for each source $(Y, g_i : Y \to FX_i)_{i \in I}$ in $\boldsymbol{Y}$, there exist an F-initial source $(X, f_i : X \to X_i)_{i \in I}$ in $\boldsymbol{X}$ and an F-generating morphism $h : Y \to FX$ in $\boldsymbol{Y}$ such that $(Ff_i) \circ h = g_i$ for all $i \in I$,



The above definition of a topologically algebraic functor is due to Y. H. Hong [11]. A topologically algebraic functor is faithful (see [11]) and is a generalization of a topological functor because any isomorphism is an F-generating morphism for any faithful functor F.

A functor $U : \boldsymbol{A} \to \boldsymbol{X}$ is called *(generating, monosource) - factorizable* if for every source $(X, f_i : X \to UA_i)_{i \in I}$ there exist a generating map $e : X \to UA$ and a monosource $(A, m_i : A \to A_i)_{i \in I}$ such that the diagram



commutes. A functor is said to be *essentially algebraic* [9] provided that it creates isomorphisms and is (generating, monosource) - factorizable.

There are several definitions for the term "algebraic functor" current in the literature, all of which are equivalent in some special categories, but not in general. We choose to adopt the following popular definition [10]. A functor $A : \boldsymbol{X} \to \boldsymbol{Y}$ is called *algebraic* iff A has a left adjoint and preserves and reflects regular epimorphisms.

A family $\Omega = (n_j)_{j\in J}$ of natural numbers indexed by some set J is called a *type*. The index set J is called the *order* of Ω. In the following, we let a type $\Omega = (n_j)_{j\in J}$ be fixed. A pair $(|A|, (\omega_j)_{j\in J})$ of a set $|A|$ and a family $\omega_j : |A|^{n_j} \to |A| (j \in J)$ of mappings is called an Ω-*algebra* (see, for example, [3]). For the sake of simplicity, we write A instead of the pair $(|A|, (\omega_j)_{j\in J})$ and $\omega_{j,A}$ for the n_j-*ary operation* ω_j on A. If the Ω-algebra A is clear from the context, we drop the suffix A in denoting its n_j-ary $(j \in J)$ operation. If A and B are Ω-algebras, then a mapping $f : |A| \to |B|$ is said to be an Ω-*morphism* $f : A \to B$ iff for each $j \in J$, $f \circ \omega_{j,A} = \omega_{j,B} \circ f^n$, where $n = n_j$ and $f^n : |A|^n \to |B|^n$ is the mapping with the obvious definition $(a_1, \ldots, a_n) \to (fa_1, \ldots, fa_n)$.

The symbol $\boldsymbol{Alg}(\Omega)$ denotes the category whose objects are Ω-algebras and whose morphisms are Ω-morphisms.

Let $\boldsymbol{X}$ be a construct with finite concrete powers and $\boldsymbol{A}$ be a subcategory of $\boldsymbol{Alg}(\Omega)$. By a *paired object* (from $\boldsymbol{X}$ and $\boldsymbol{A}$) is meant an ordered pair (X, A) where X and A are objects in $\boldsymbol{X}$ and $\boldsymbol{A}$ respectively with the same underlying set such that, for each $j \in J$, the $n(= n_j)$-ary operation $\omega_{j,A} : |A|^n \to |A|$ on A is an $\boldsymbol{X}$-morphism $\omega_{j,A} : X^n \to X$. In this case, we write $\omega_{j,X}$ for the $\boldsymbol{X}$-morphism from X^n to X whose underlying function is $\omega_{j,A}$. If (X, A) and (X', A') are two paired objects (from $\boldsymbol{X}$ and $\boldsymbol{A}$), then an $\boldsymbol{X}$-morphism $f : X \to X'$ that is also an $\boldsymbol{A}$-morphism $f : A \to A'$ is called a *paired morphism* (from $\boldsymbol{X}$ and $\boldsymbol{A}$) and is denoted by $f : (X, A) \to (X', A')$. The category of all paired objects (from $\boldsymbol{X}$ and $\boldsymbol{A}$) together with paired morphisms (from $\boldsymbol{X}$ and $\boldsymbol{A}$) is called the *paired category* (from $\boldsymbol{X}$ and $\boldsymbol{A}$). We denote this category by $\boldsymbol{X\,A}$.

In this work, we assume that all subcategories are isomorphism closed. The fact that the most of the natural subcategories fall into this class justifies our assumption. Unless otherwise stated, $\boldsymbol{X}$ and $\boldsymbol{Y}$ denote arbitrary constructs with finite concrete powers, and $\boldsymbol{A}$ represents any subcategory of $\boldsymbol{Alg}(\Omega)$. We write $|X|$ for the underlying set of an object X in a construct.

3. $(\boldsymbol{E}, \boldsymbol{M})$-Topological Functors

In this section we will show that topological (monotopological) functors between two constructs can be extended to topological (monotopological) functors between the corresponding paired categories under some assumptions.

Lemma 3.1. *Suppose $G : \boldsymbol{X} \to \boldsymbol{Y}$ is a concrete functor which preserves finite concrete powers, X is an $\boldsymbol{X}$-object, $Y := GX$, $((X_i, A_i))_{i\in I}$ is a family* *of $\boldsymbol{X}\,\boldsymbol{A}$-objects, $((Y, A), f_i : (Y, A) \to (GX_i, A_i))_{i \in I}$ is a source in $\boldsymbol{Y}\,\boldsymbol{A}$, and $(X, g_i : X \to X_i)_{i \in I}$ is a G-initial source in $\boldsymbol{X}$. If $f_i = g_i$ as functions for each $i \in I$, then (X, A) is an $\boldsymbol{X}\,\boldsymbol{A}$-object.*

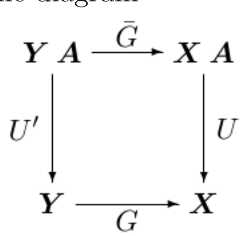
In particular, if $((X_i, A_i))_{i \in I}$ is a family of objects in the category $\boldsymbol{X}\,\boldsymbol{A}$, $(A, g_i : A \to A_i)_{i \in I}$ is a source in $\boldsymbol{A}$, and X is an $\boldsymbol{X}$-object having the same underlying set as A such that source $(X, g_i : X \to X_i)_{i \in I}$ is initial in $\boldsymbol{X}$, then the pair (X, A) lies in $\boldsymbol{X}\,\boldsymbol{A}$.

Lemma 3.2. *If $\boldsymbol{X}$ is monotopological and $\boldsymbol{A}$ is an essentially algebraic subcategory of $\boldsymbol{Alg}(\Omega)$, then $\boldsymbol{X}\,\boldsymbol{A}$ is an (extremal epi, monosource) - category.*

Proof. Since every category that has (epi, monosource) - factorizations is an (extremal epi, monosource) - category (see [1]), it remains to show that $\boldsymbol{X}\,\boldsymbol{A}$ is (epi, monosource) - factorizable. Let us assume that $((X, A), f_i : (X, A) \to (X_i, A_i))_{i \in I}$ is any source in $\boldsymbol{X}\,\boldsymbol{A}$. Since $\boldsymbol{A}$ is (epi, monosource) - factorizable (see [1]), there exists an $\boldsymbol{A}$-epimorphism $e : A \to A'$ and a monosource $\left(A', g_i : A' \to A_i\right)_{i \in I}$ such that $g_i \circ e = f_i$ for each $i \in I$. Since essentially algebraic functors preserve monosources (see [1]), $(|A'|, g_i : |A'| \to |A_i|)_{i \in I}$ is a monosource in $\boldsymbol{Set}$. Let X' be an $\boldsymbol{X}$-object with the same underlying set as A' initial with respect to the source $(X', g_i : X' \to X_i)_{i \in I}$. The pair (X', A') is an $\boldsymbol{X}\,\boldsymbol{A}$-object by Lemma 3.1, and the source $((X', A'), g_i : (X', A') \to (X_i, A_i))_{i \in I}$ is clearly a monosource in $\boldsymbol{X}\,\boldsymbol{A}$.

$e : X \to X'$ is an $\boldsymbol{X}$-morphism because $g_i \circ e (= f_i)$ is an $\boldsymbol{X}$-morphism for each $i \in I$. Since $e : A \to A'$ is an $\boldsymbol{A}$-epimorphism and the forgetful functor $\boldsymbol{X}\,\boldsymbol{A} \to \boldsymbol{A}$ is faithful, $e : (X, A) \to (X', A')$ is an $\boldsymbol{X}\,\boldsymbol{A}$-epimorphism (see [1]). □

We now prove our main theorem that an (E, M) - topological functor $G : \boldsymbol{Y} \to \boldsymbol{X}$ between two constructs with finite concrete powers can be extended to an (E', M') - topological functor $\bar{G} : \boldsymbol{Y}\,\boldsymbol{A} \to \boldsymbol{X}\,\boldsymbol{A}$ between the corresponding paired categories such that the diagram



commutes ($U' : \boldsymbol{Y}\,\boldsymbol{A} \to \boldsymbol{Y}$ and $U : \boldsymbol{X}\,\boldsymbol{A} \to \boldsymbol{X}$ are the forgetful functors) under some assumptions. More precisely:

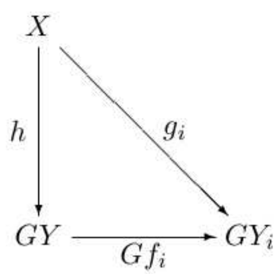
Theorem 3.3. *Suppose $G : \boldsymbol{Y} \to \boldsymbol{X}$ is a concrete (E, M) - topological functor and $\boldsymbol{X}\,\boldsymbol{A}$ is an (E', M') - category such that the forgetful functor $U : \boldsymbol{X}\,\boldsymbol{A} \to \boldsymbol{X}$ sends sources in M' to sources in M. Then the association $\bar{G} : \boldsymbol{Y}\,\boldsymbol{A} \to \boldsymbol{X}\,\boldsymbol{A}$, defined by*

$$\bar{G}(Y, A) = (GY, A),$$

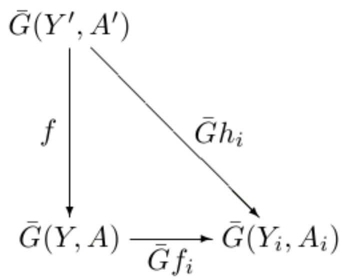
is an (E', M') - topological functor.

Proof. We first show that $\bar{G}$ is a functor. Let (Y, A) be a $\boldsymbol{Y}\,\boldsymbol{A}$-object. Since any (E, M) - topological functor preserves products (see [8]), $G(Y^n) = (GY)^n$ for every positive integer n. For any $j \in J$, the $n(= n_j)$-ary operation $\omega = \omega_{j,A}$ is a morphism from Y^n to Y, the set equalities $|GY| = |Y| = |A|$ and the function equality $G\omega = \omega$ are valid because G is concrete so that $\omega : (GY)^n \to GY$ is an $\boldsymbol{X}$-morphism, which means that (GY, A) is an $\boldsymbol{X}\,\boldsymbol{A}$-object. If $f : (Y, A) \to (Y', A')$ is a $\boldsymbol{Y}\,\boldsymbol{A}$-morphism then clearly $f : (GY, A) \to (GY', A')$ is an $\boldsymbol{X}\,\boldsymbol{A}$-morphism because $f = Gf$ is an $\boldsymbol{X}$-morphism between first components. Hence $\bar{G}$ is a functor.

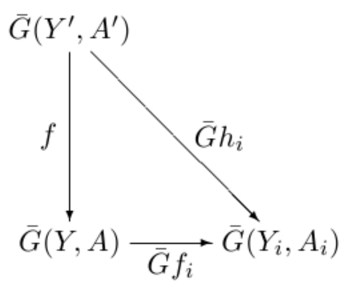
Now we show that $\bar{G}$ is an (E', M')-topological functor. Let $((Y_i, A_i))_{i\in I}$ be a family of objects in the category $\boldsymbol{Y}\,\boldsymbol{A}$ and $((X, A), g_i : (X, A) \to (GY_i, A_i))_{i\in I}$ be any source in M'. Since the forgetful functor $U : \boldsymbol{X}\,\boldsymbol{A} \to \boldsymbol{X}$ sends sources in M' to sources in M, $(X, g_i : X \to GY_i)_{i\in I}$ is a source in M. Since G is (E, M) - topological, there exists a G-initial source $(Y, f_i : Y \to Y_i)_{i\in I}$ that G-lifts $(X, g_i : X \to GY_i)_{i\in I}$, which implies that we can find an isomorphism $h : X \to GY$ such that the diagram



commutes. Since $\boldsymbol{A}$ is isomorphism closed and h is a bijection we can assume, without loss of generality, that $|Y| = |X|$, $GY = X$, and hence that $f_i = g_i$ as functions. By Lemma 3.1, (Y, A) is an $\boldsymbol{Y}\,\boldsymbol{A}$-object. We prove that the source $((Y, A), f_i : (Y, A) \to (Y_i, A_i))_{i\in I}$ is $\bar{G}$-initial and $\bar{G}$-lifts $((X, A), g_i : (X, A) \to (GY_i, A_i))_{i\in I}$. To this end, let us assume that $((Y', A'), h_i : (Y', A') \to (Y_i, A_i))_{i\in I}$ is any source in $\boldsymbol{Y}\,\boldsymbol{A}$ and $f : \bar{G}(Y', A') \to \bar{G}(Y, A)$ be a morphism such that the diagram



commutes for each $i \in I$. Using the fact that the above diagram commutes with first components and that Y has G-initial structure with respect to $(g_i)_{i \in I}$, we conclude that there exists a unique morphism $\bar{f} : Y' \to Y$ such that $G\bar{f} = f$ and $g_i \circ \bar{f} = h_i$ for all $i \in I$. Now $\bar{f} : (Y', A') \to (Y, A)$ is a $\boldsymbol{Y}\ \boldsymbol{A}$-morphism and the source $((Y, A), f_i : (Y, A) \to (Y_i, A_i))_{i \in I}$ $\bar{G}$-lifts the source $((X, A), g_i : (X, A) \to (GY_i, A_i))_{i \in I}$, because $\bar{f} = f$ (as functions) and the identity morphism $id : (X, A) \to (GY, A)$ is an isomorphism which makes the diagram



commute. □

Corollary 3.4. *Suppose* $\boldsymbol{Y}$ *is a construct with finite concrete powers,* $\boldsymbol{X}$ *is a monotopological category,* $G : \boldsymbol{Y} \to \boldsymbol{X}$ *is a concrete (extremal epi, monosource) - topological functor, and* $\boldsymbol{A}$ *is an essentially algebraic subcategory of* $\boldsymbol{Alg}(\Omega)$. *Then the association* $\bar{G} : \boldsymbol{Y}\ \boldsymbol{A} \to \boldsymbol{X}\ \boldsymbol{A}$, *defined by*

$$\bar{G}(Y, A) = (GY, A),$$

is an (extremal epi, monosource) - topological functor.

Proof. $\boldsymbol{X}\ \boldsymbol{A}$ is an (extremal epi, monosource) - category by Lemma 3.2. It is easy to show that the forgetful functor $U : \boldsymbol{X}\ \boldsymbol{A} \to \boldsymbol{X}$ preserves monosources. In fact, U is essentially algebraic (see [6]). Therefore, $\bar{G}$ is (extremal epi, monosource) - topological by Theorem 3.3. □

Corollary 3.5. *If* $\boldsymbol{Y}$ *is a full extremal epireflective subcategory of a monotopological category* $\boldsymbol{X}$ *and* $\boldsymbol{A}$ *is an essentially algebraic subcategory of*

$\boldsymbol{Alg}(\Omega)$, then $\boldsymbol{Y}\ \boldsymbol{A}$ is a full extremal epireflective subcategory of $\boldsymbol{X}\ \boldsymbol{A}$.

Proof. First we note that an embedding $\boldsymbol{W}$ into $\boldsymbol{Z}$ is (E, M)-topological iff $\boldsymbol{W}$ is a full E-reflective subcategory of Z (see [8]). This implies that the embedding $G : \boldsymbol{Y} \to \boldsymbol{X}$ is (extremal epi, monosource) - topological. Hence, by Corollary 3.4, the extension $\bar{G}$ is an (extremal epi, monosource) - topological. Since $\bar{G}$ is also an embedding, $\boldsymbol{Y}\ \boldsymbol{A}$ is a full extremal epireflective subcategory of $\boldsymbol{X}\ \boldsymbol{A}$ by the above observation. □

Since any monotopological category is a full extremal epireflective subcategory of a topological category, we can conclude that a monotopological category $\boldsymbol{Y}$ paired with an essentially algebraic subcategory $\boldsymbol{A}$ of $\boldsymbol{Alg}(\Omega)$ yields the paired category $\boldsymbol{Y}\ \boldsymbol{A}$ that is full extremal epireflective subcategory of a topological algebra $\boldsymbol{X}\ \boldsymbol{A}$ for some topological category $\boldsymbol{X}$ (for which $\boldsymbol{Y}$ is a full epireflective subcategory).

Corollary 3.6. *Suppose $\boldsymbol{Y}$ and $\boldsymbol{X}$ are constructs with finite concrete powers, and $G : \boldsymbol{Y} \to \boldsymbol{X}$ is a concrete topological functor. Then the association $\bar{G} : \boldsymbol{Y}\ \boldsymbol{A} \to \boldsymbol{X}\ \boldsymbol{A}$, defined by*

$$\bar{G}(Y, A) = (GY, A),$$

is a topological functor.

Proof. Since every category is an (iso, source) - category (see [1]), $\boldsymbol{X}\ \boldsymbol{A}$ is an (iso, source) - category. Clearly the forgetful functor $U : \boldsymbol{X}\ \boldsymbol{A} \to \boldsymbol{X}$ preserves sources. Hence the result follows from Theorem 3.3. □

It is well known that a semitopological functor is topological iff it has a full and faithful right adjoint. This implies that the restriction of a concretely topological functor $G : \boldsymbol{Y} \to \boldsymbol{X}$ to a reflective full subcategory $\boldsymbol{Z}$ of $\boldsymbol{Y}$ is topological provided the category $\boldsymbol{Z}$ contains all indiscrete objects with respect to the functor G (see reference [5]). In this case, $\bar{G} : \boldsymbol{Z}\ \boldsymbol{A} \to \boldsymbol{X}\ \boldsymbol{A}$ is topological from Corollary 3.6. Thus we have the following result.

Corollary 3.7. *Suppose $G : \boldsymbol{Y} \to \boldsymbol{X}$ is a concretely topological functor and $\boldsymbol{Z}$ is a full reflective subcategory of $\boldsymbol{Y}$ containing all indiscrete objects with respect to G. then $\boldsymbol{Z}\boldsymbol{A}$ is topological over $\boldsymbol{X}\ \boldsymbol{A}$.*

We now investigate some properties of the forgetful functors $T : \boldsymbol{X}\ \boldsymbol{A} \to \boldsymbol{A}$ and $F : \boldsymbol{X}\ \boldsymbol{A} \to \boldsymbol{Set}$.$T$ need not be (epi, monosource) - topological under the assumption that $\boldsymbol{X}$ is monotopological and $\boldsymbol{A}$ is an essentially algebraic subcategory of $\boldsymbol{Alg}(\Omega)$, because there are essentially algebraic categories that are not (epi, monosource) - categories (for example, the category of torsionfree Abelian groups with group homomorphisms is such one). Some properties of the forgetful functor T are listed in the following:

Corollary 3.8. *Suppose $\boldsymbol{X}$ is monotopological, $\boldsymbol{A}$ is a subcategory of $\boldsymbol{Alg}(\Omega)$, and $T : \boldsymbol{X}\,\boldsymbol{A} \to \boldsymbol{A}$ is the forgetful functor.*

(a) If $\boldsymbol{A}$ is an $(E$, monosource) - category for some E where monosources in $\boldsymbol{A}$ are point separating, then T is $(E$, monosource) - topological.

(b) If $\boldsymbol{A}$ is an essentially algebraic subcategory of $\boldsymbol{Alg}(\Omega)$, then T is (extremal epi, monosource) - topological.

(c) If $\boldsymbol{A}$ is an algebraic subcategory of $\boldsymbol{Alg}(\Omega)$, then $T : \boldsymbol{X}\,\boldsymbol{A} \to \boldsymbol{A}$ is (regular epi, monosource) - topological.

(d) If $\boldsymbol{X}$ is topological then T is topological.

Proof. Part (a) follows from Theorem 3.3 because the forgetful functor $G : \boldsymbol{X} \to \boldsymbol{Set}$ is (epi, monosource) - topological and the forgetful functor $U : \boldsymbol{A} \to \boldsymbol{Set}$ sends monosources in $\boldsymbol{A}$ to monosources in $\boldsymbol{Set}$.

Since an essentially algebraic subcategory of $\boldsymbol{Alg}(\Omega)$ is a (extremal epi, monosource) - category (see [1]), part (b) is immediate from (a) where E is taken to be the class of all extremal epimorphisms.

If $\boldsymbol{A}$ is algebraic, then $\boldsymbol{A}$ has regular factorizations (see [1]) and hence it is a (regular epi, monosource) - category (see [1]). Therefore, an application of part (a) with E as the class of all regular epimorphisms proves (c).

The final result follows from Corollary 3.6 because the forgetful functor $G : \boldsymbol{X} \to \boldsymbol{Set}$ is topological and $\bar{G} : \boldsymbol{X}\,\boldsymbol{A} \to \boldsymbol{Set}\,\boldsymbol{A}$ is nothing but the forgetful functor $T : \boldsymbol{X}\,\boldsymbol{A} \to \boldsymbol{A}$. □

Corollary 3.8 part (d) sheds some light on the cartesian closedness of topological algebras. Since cartesian closedness is preserved under topological functors, a topological algebra is cartesian closed only if its algebra part is cartesian closed (see [6]).

Corollary 3.9. *Suppose $\boldsymbol{X}$ is topological and $\boldsymbol{A}$ is a subcategory of $\boldsymbol{Alg}(\Omega)$.*

(a) If $\boldsymbol{A}$ has (epi, monosource) - factorizations, then $\boldsymbol{X}\,\boldsymbol{A}$ is an (extremal epi, monosource) - category.

(b) If $\boldsymbol{A}$ is an epireflective subcategory of $\boldsymbol{Alg}(\Omega)$, then $\boldsymbol{X}\,\boldsymbol{A}$ is a (regular epi, monosource) - category.

Proof. First note that if $\boldsymbol{Z}$ is topological over $\boldsymbol{W}$, then:

$\boldsymbol{Z}$ is (epi, monosource) - factorizable iff $\boldsymbol{W}$ is, and

$$\boldsymbol{Z} \text{ has regular factorizations iff } \boldsymbol{W} \text{ does}$$

(see [1]).

Suppose $\boldsymbol{A}$ has (epi, monosource) - factorizations. Since the forgetful functor $T : \boldsymbol{X}\,\boldsymbol{A} \to \boldsymbol{A}$ is topological, $\boldsymbol{X}\,\boldsymbol{A}$ is (epi, monosource) - factorizable by the above note. Since any category that is (epi, monosource) - factorizable is an (extremal epi, monosource) - category (see [1]), statement (a) follows.

To prove the second statement, assume that $\boldsymbol{A}$ is an epireflective subcategory of $\boldsymbol{Alg}(\Omega)$. Then $\boldsymbol{A}$ is closed under the formation of products and subalgebras (see [Hon, 74]). Consequently, $\boldsymbol{A}$ is algebraic (see [9], [11]), in particular, $\boldsymbol{A}$ has regular factorizations (see [1]). Hence $\boldsymbol{X}\,\boldsymbol{A}$ has regular factorizations by the above note. Thus $\boldsymbol{X}\,\boldsymbol{A}$ is a (regular epi, monosource) - category because any category having regular factorizations is one such (see [1]). □

The following result is known under stronger assumptions (see [10]).

Corollary 3.10. *If $\boldsymbol{X}$ is monotopological and $\boldsymbol{A}$ is an essentially algebraic subcategory of $\boldsymbol{Alg}(\Omega)$, then the forgetful functor $F : \boldsymbol{X}\,\boldsymbol{A} \to \boldsymbol{Set}$ is topologically algebraic.*

Proof. Note that the forgetful functor $U : \boldsymbol{A} \to \boldsymbol{Set}$ is essentially algebraic, and the forgetful functor $T : \boldsymbol{X}\,\boldsymbol{A} \to \boldsymbol{A}$ is (extremal epi, monosource) - topological by Corollary 3.8. Since the composite of an $(E$, monosource) - topological functor and an essentially algebraic functor is topologically algebraic (see [1]) and $F = U \circ T$, F is topologically algebraic. □

Our results can be applied to many topologically algebraic situations. However, for an illustration, we discuss the following examples.

Corollary 3.11. *If $\boldsymbol{A}$ is an essentially algebraic subcategory of $\boldsymbol{Alg}(\Omega)$, then*

(a) $\boldsymbol{Haus}\,\boldsymbol{A}$ is an extremal epireflective subcategory of $\boldsymbol{Top}\,\boldsymbol{A}$ (where $\boldsymbol{Haus}$ and $\boldsymbol{Top}$ are categories of Hausdorff spaces and topological spaces, respectively),

and

(b) $\boldsymbol{Pos}\,\boldsymbol{A}$ is an extremal epireflective subcategory of $\boldsymbol{Preord}\,\boldsymbol{A}$ (where $\boldsymbol{Pos}$ and $\boldsymbol{Preord}$ are categories of partially ordered sets and preordered sets, respectively).

Proof. Since $\boldsymbol{Haus}$ and $\boldsymbol{Pos}$ are full extremal epireflective subcategories of $\boldsymbol{Top}$ and $\boldsymbol{Preord}$ respectively (see [1]), the result follows from Corollary

3.5. □

Corollary 3.12. *$\boldsymbol{Cont}\,\boldsymbol{A}$ is topological over $\boldsymbol{X}\ \boldsymbol{A}$, where $\boldsymbol{Cont}$ is the category of contigual spaces, $\boldsymbol{X}$ is the category of generalized proximity spaces, and $\boldsymbol{A}$ is any subcategory of $\boldsymbol{Alg}(\Omega)$.*

Proof. Since $\boldsymbol{Cont}$ is topological over the category of generalized proximity spaces (see [2]), the result follows from Corollary 3.6. □

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References

[1] J. Adámek, H. Herrlich, G.E. Strecker, *Abstract and Concrete Categories*, John Wiley and Sons, Inc., New York (1990).

[2] H.L. Bentley, H. Herrlich, The forgetful functor $\boldsymbol{Cont} \rightarrow \boldsymbol{Prox}$ is topological, *Questiones Mathematicae*, **2** (1977), 45-57.

[3] P.M. Cohn, *Universal Algebra*, Harper and Row, Publishers, New York (1965).

[4] T.H. Fay, An axiomatic approach to categories of topological algebras, *Quaestiones Mathematicae*, **2** (1977), 113-137.

[5] T.H. Fay, G.C. L. Brümmer, K.A. Hardie, A characterization of topological functors, *Math. Colloq., Univ. Cape Town*, **12** (1979), 85-93.

[6] V.L. Gompa, Essentially algebraic functors and topological algebras, *Indian Journal of Mathematics* (1993).

[7] V.L. Gompa, Cartesian closedness and topological algebra, *International Journal of Pure and Applied Mathematics*, **21**, No. 3 (2005), 397 - 405.

[8] H. Herrlich, Topological functors, *General Topology and its Applications*, **4** (1974), 125-142.

[9] H. Herrlich, Essentially algebraic categories, *Quaestiones Mathematicae*, **9** (1986), 245-262.

[10] Y.H. Hong, *Studies on Categories of Universal Topological Algebras*, Doctoral Dissertation, McMaster University (1974).

[11] H. Herrlich, G.E. Strecker, *Category Theory*, Allyn and Bacon, Boston (1973).

[12] H.-E. Porst, On underlying functors in general and topological algebra, *Manuscripta Math.*, **20** (1977), 209-225.

[13] H.-E. Porst, How topological are topological groups, *Quaestiones Mathematicae*, **13** (1990), 291-299.

[14] W. Tholen, On Wyler's taut lift theorem, *General Topology and its Applications*, **8** (1978), 197-206.

[15] O. Wyler, On the categories of general topology and topological algebras, *Arc. Math.*, Basel, **22** (1971), 7-17.