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EXPECTED MEAN SQUARED ERROR OF ESTIMATORS FOR SYMMETRY AND ASYMMETRY MODELS FOR CONTINGENCY TABLES

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Abstract: For two-way contingency tables, Bishop et al [3, p. 313] measured the overall variability of the maximum likelihood estimators (MLEs) of cell probabilities for the independence and saturated (SA) models in terms of the expected mean squared error (risk).

This paper gives for square tables the risks of the MLEs of conditional cell probabilities under the symmetry (S) and conditional symmetry (CS) models on condition that an observation falls in one of the off-diagonal cells, and compares those risks for the S, CS and SA models. Moreover this paper gives for the multi-way tables the similar risk for the S model, and compares those risks for the S and SA models. It is shown that when the simpler model is correct, the overall variability for the simple model is smaller than for the more complicated model.

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1. Introduction

Consider an $R \times R$ contingency table. Let p_{ij} denote the probability that an observation will fall in the i-th row and j-th column of the table ($i = 1, \ldots, R; j = 1, \ldots, R$). Generally, we are interested in whether or not the independence between the row and column classifications holds. The independence (I) model is defined by

$$p_{ij} = p_{i+}p_{+j} \quad (i = 1, \ldots, R; j = 1, \ldots, R),$$

where $p_{i+} = \sum_{t=1}^{R} p_{it}$ and $p_{+j} = \sum_{s=1}^{R} p_{sj}$. However, for square contingency tables with the same row and column classifications, many observations tend to fall in (or near) the main diagonal cells. Thus, for such a case the independence between the rows and columns is unlikely to hold. So, we may be interested in whether or not there is a structure of symmetry in the square table.

The symmetry (S) model is defined by

$$p_{ij} = p_{ji} \quad (i \neq j);$$

see, e.g., Bowker [4], Bishop et al [3, p. 282], and Everitt [5, p. 142]. This model states that the probability that an observation will fall in cell (i, j) is equal to the probability that it falls in symmetric cell (j, i). Also, Bishop et al [3, p. 300] considered the S model for three-way tables. Bhapkar and Darroch [2], and Agresti [1, p. 440] considered the S model for multi-way tables; see also Tomizawa and Tahata [9], and Yamamoto et al [10].

McCullagh [6] considered the conditional symmetry (CS) model defined by

$$p_{ij} = \Delta p_{ji} \quad (i < j);$$

see also Everitt [5, p. 145] and Tomizawa [7], [8]. A special case of this model obtained by putting $\Delta = 1$ is the S model. This model states that the probability that an observation will fall in cell (i, j), $i < j$, is Δ times higher than the probability that it falls in cell (j, i).

Let x_{ij} denote the observed frequency in cell (i, j) of the table with $N = \sum\sum x_{ij}$. Assume that $\{x_{ij}\}$ have a multinomial distribution. Let $T_{ij}^{(M)}$ denote the maximum likelihood estimator (MLE) of cell probability p_{ij} under model M.

Bishop et al [3, p. 313] measured the overall variability of the MLEs of $\{p_{ij}\}$ for the I and saturated (SA) models in terms of the expected mean squared error (risk), i.e., for any model M,

$$R_M = \sum_{i=1}^{R}\sum_{j=1}^{R} E(T_{ij}^{(M)} - p_{ij})^2 = \sum_{i=1}^{R}\sum_{j=1}^{R} \left[\mathrm{Var}\,(T_{ij}^{(M)}) + \mathrm{Bias}\,(T_{ij}^{(M)})\right],$$

where

$$\mathrm{Var}\,(T_{ij}^{(M)}) = E[T_{ij}^{(M)} - E(T_{ij}^{(M)})]^2,$$
$$\mathrm{Bias}\,(T_{ij}^{(M)}) = [E(T_{ij}^{(M)}) - p_{ij}]^2.$$

See Appendix A for the details of risks of the MLEs for the I and SA models. Moreover, Bishop et al [3, p. 313] showed that the MLEs of $\{p_{ij}\}$ for the I model are always more precise than those for the SA model when the I model holds.

We are interested (i) in finding and comparing the risks of the MLEs for the S, CS, and SA models for square tables, and (ii) in finding and comparing the risks of the MLEs for the S and SA models for multi-way tables.

The purpose of this paper is (i) for square tables, to give the risks of the MLEs of conditional cell probabilities under the S, CS and SA models on condition that an observation falls in one of the off-diagonal cells, (ii) to compare those risks, and (iii) for multi-way tables, to give the similar risks for the S and SA models and (iv) to compare them.

2. Risks of MLEs for Models in Square Tables

2.1. Conditional Risk

Consider the $R \times R$ table. The MLEs of cell probabilities $\{p_{ij}\}$ under the S model are given by

$$T_{ij}^{(S)} = \begin{cases} \dfrac{x_{ij} + x_{ji}}{2N} & (i \neq j), \\ \dfrac{x_{ii}}{N} & (i = j). \end{cases}$$

Also, those under the CS model are given by

$$T_{ij}^{(CS)} = \begin{cases} \dfrac{B(x_{ij} + x_{ji})}{(B+C)N} & (i < j), \\ \dfrac{C(x_{ij} + x_{ji})}{(B+C)N} & (i > j), \\ \dfrac{x_{ii}}{N} & (i = j), \end{cases}$$

where

$$B = \sum\sum_{i<j} x_{ij}, \quad C = \sum\sum_{i>j} x_{ij}.$$

Moreover those under the SA model are given by

$$T_{ij}^{(SA)} = \frac{x_{ij}}{N} \quad (i = 1, \ldots, R; j = 1, \ldots, R).$$

We see that the MLEs of cell probabilities $\{p_{ii}\}$ on the main diagonal of the table under the S and CS models are identical to those under the SA model. Therefore the risks of the MLEs for the main diagonal cells for three models (i.e., S, CS and SA models) are the same. So, we are interested in the risks for the off-diagonal cells for these models. Also it is difficult to find the risk of $\{T_{ij}^{(CS)}\}$ for the CS model, i.e., R_{CS}.

Let q_{ij} denote the conditional probability that an observation will fall in cell (i, j) of the table on condition that it falls in one of the off-diagonal cells, i.e.,

$$q_{ij} = \frac{p_{ij}}{1 - \sum_{s=1}^{R} p_{ss}} \quad (i \neq j).$$

Using the $\{q_{ij}\}$, the S model may be expressed as

$$q_{ij} = q_{ji} \quad (i \neq j).$$

Similarly, the CS model may be expressed as

$$q_{ij} = \Delta q_{ji} \quad (i < j).$$

Therefore, we are now interested in considering the conditional risks of the MLEs of $\{q_{ij}\}$ under the SA, S and CS models on condition that an observation falls in one of the off-diagonal cells. The conditional risk is given by for any model M,

$$R_M^c = \sum\sum_{i \neq j} E(T_{ij}^{c(M)} - q_{ij})^2 = \sum\sum_{i \neq j} \left[\mathrm{Var}\,(T_{ij}^{c(M)}) + \mathrm{Bias}\,(T_{ij}^{c(M)}) \right],$$

with the MLEs $\{T_{ij}^{c(M)}\}$ of $\{q_{ij}\}$, $i \neq j$, on condition that an observation falls in one of the off-diagonal cells.

2.2. Case of Symmetry Model

The MLEs of $\{q_{ij}\}$ under the SA model on condition that an observation falls in one of the off-diagonal cells are given by

$$T_{ij}^{c(SA)} = \frac{x_{ij}}{N^*} \quad (i \neq j),$$

where

$$N^* = B + C = N - \sum_{s=1}^{R} x_{ss}.$$

Note that then N^* is fixed. For $\{T_{ij}^{c(SA)}\}$, the conditional bias term is always zero, and the conditional risk is

$$R_{SA}^c = \frac{1}{N^*} \sum\sum_{i \neq j} \Big(q_{ij}(1 - q_{ij}) \Big) = \frac{1}{N^*} \left(1 - \sum\sum_{i \neq j} q_{ij}^2 \right).$$

Next, the MLEs of $\{q_{ij}\}$ under the S model are given by

$$T_{ij}^{c(S)} = \frac{x_{ij} + x_{ji}}{2N^*} \quad (i \neq j).$$

We see

$$E(T_{ij}^{c(S)}) = \frac{1}{2}(q_{ij} + q_{ji}),$$

on condition that the observation falls in one of the off-diagonal cells. Thus $T_{ij}^{c(S)}$ is a conditional unbiased estimator of q_{ij} only when the S model holds (i.e., when $q_{ij} = q_{ji}$, $i \neq j$). For $T_{ij}^{c(S)}$ $(i \neq j)$, we find that the conditional bias term is

$$\text{Bias}\,(T_{ij}^{c(S)}) = \frac{1}{4}(q_{ij} - q_{ji})^2,$$

and the conditional variance term is

$$\text{Var}\,(T_{ij}^{c(S)}) = \frac{1}{4N^*} \left((q_{ij} + q_{ji}) - (q_{ij} + q_{ji})^2 \right).$$

Thus, the conditional risk for the S model is

$$R_S^c = \frac{1}{4} \sum\sum_{i \neq j} \left[\frac{1}{N^*} \left\{ (q_{ij} + q_{ji}) - (q_{ij} + q_{ji})^2 \right\} + (q_{ij} - q_{ji})^2 \right].$$

2.3. Case of Conditional Symmetry Model

We consider the conditional risk of the MLEs of $\{q_{ij}\}$ under the CS model on condition that an observation falls in one of the off-diagonal cells. Let T^* denote the $2 \times R(R-1)/2$ table constructed using $\{q_{ij}\}$, $i < j$, for cells in the upper right triangle of the table, and $\{q_{ij}\}$, $i > j$, for cells in the lower left triangle. Namely, the first row of table T^* is $(q_{12}, \ldots, q_{1R}, q_{23}, \ldots, q_{2R}, \ldots, q_{R-1,R})$ and the second row is $(q_{21}, \ldots, q_{R1}, q_{32}, \ldots, q_{R2}, \ldots, q_{R,R-1})$. The CS model indicates that each entry in the first row of the table T^* is the same multiple Δ of the corresponding entry in the second row. Therefore the CS model is equivalent to independence for $\{q_{ij}\}$, $i \neq j$, in the table T^*. So, the MLEs of $\{q_{ij}\}$ under the CS model are given by

$$T_{ij}^{c(CS)} = \begin{cases} \dfrac{B(x_{ij}+x_{ji})}{N^{*2}} & (i<j), \\ \dfrac{C(x_{ij}+x_{ji})}{N^{*2}} & (i>j). \end{cases}$$

We see

$$E(T_{ij}^{c(CS)}) = \left(\frac{N^*-1}{N^*}\right) Q_{ij}(q_{ij}+q_{ji}) + \frac{1}{N^*}q_{ij} \quad (i \neq j),$$

where

$$Q_{ij} = \begin{cases} Q_U & (i<j), \\ Q_L & (i>j), \end{cases}$$

$$Q_U = \sum_{i<j}\sum q_{ij}, \quad Q_L = \sum_{i>j}\sum q_{ij},$$

on condition that an observation falls in one of the off-diagonal cells. Thus $T_{ij}^{c(CS)}$ is a conditional unbiased estimator of q_{ij} only when the CS model holds, (i.e., when $q_{ij} = Q_{ij}(q_{ij}+q_{ji}),\ i \neq j$). For $T_{ij}^{c(CS)}$ $(i \neq j)$, the conditional bias and variance terms are given as

$$\text{Bias}\,(T_{ij}^{c(CS)}) = \left(\frac{N^*-1}{N^*}\right)^2 \left(Q_{ij}\left(q_{ij}+q_{ji}\right) - q_{ij}\right)^2,$$

and

$$\begin{aligned}
&\text{Var}\,(T_{ij}^{c(CS)}) \\
&= \frac{(N^*-1)(6-4N^*)}{N^{*3}} Q_{ij}^2(q_{ij}+q_{ji})^2 + 2\frac{(N^*-1)(N^*-4)}{N^{*3}} q_{ij}Q_{ij}(q_{ij}+q_{ji}) \\
&\quad + \frac{(N^*-1)(N^*-2)}{N^{*3}}\left(Q_{ij}(q_{ij}+q_{ji})^2 + Q_{ij}^2(q_{ij}+q_{ji})\right) + \frac{(N^*-2)}{N^{*3}} q_{ij}^2 \\
&\quad + \frac{(N^*-1)}{N^{*3}}\Big(2q_{ij}(q_{ij}+q_{ji}) + 2q_{ij}Q_{ij} + Q_{ij}(q_{ij}+q_{ji})\Big) + \frac{q_{ij}}{N^{*3}}.
\end{aligned}$$

Thus the conditional risk for the CS model is obtained by

$$R_{CS}^c = \sum_{i \neq j}\sum \left[\text{Var}\,(T_{ij}^{c(CS)}) + \text{Bias}\,(T_{ij}^{c(CS)})\right].$$

2.4. Comparisons Between Risks

From comparing the risks for the SA, S and CS models, we obtain the following theorems.

Theorem 1. *When the S model holds,*

$$R^c_{SA} - R^c_S = \frac{1}{2N^*}.$$

Theorem 2. *When the S model holds,*

$$R^c_{CS} - R^c_S = \frac{1}{2N^{*2}} + \frac{2(N^*-1)}{N^{*2}} \sum\sum_{i<j} q^2_{ij}.$$

Theorem 3. *When the S model holds,*

$$R^c_{SA} - R^c_{CS} = \frac{N^*-1}{2N^{*2}} \Big(1 - 4\sum\sum_{i<j} q^2_{ij}\Big).$$

Theorem 4. *When the CS model holds,*

$$R^c_{SA} - R^c_{CS} = \frac{N^*-1}{N^{*2}} \Big(1 - \sum\sum_{i<j}(q_{ij}+q_{ji})^2\Big)\Big(1-(Q^2_U+Q^2_L)\Big).$$

We see that:

(i) from Theorem 1, when the S model holds, $R^c_S < R^c_{SA}$,

(ii) from Theorem 2, when the S model holds, $R^c_S < R^c_{CS}$,

(iii) from Theorem 3, when the S model holds, $R^c_{CS} < R^c_{SA}$ because

$$\sum\sum_{i<j} q^2_{ij} < (\sum\sum_{i<j} q_{ij})^2 = \frac{1}{4},$$

and

(iv) from Theorem 4, when the CS model holds, $R^c_{CS} < R^c_{SA}$ because $\sum_{i<j}(q_{ij}+q_{ji})^2 < \sum_{i<j}(q_{ij}+q_{ji}) = 1$ and $Q^2_U + Q^2_L < Q_U + Q_L = 1$. Thus from Theorems 1, 2, 3 and 4, we see that when the simpler model is correct, the overall variability for the estimators based on the simple model is smaller than for the estimators based on the more complicated model.

3. Risk of MLEs for Symmetry Model in Multi-Way Tables

Consider an R^T table. Let $p_{i_1\ldots i_T}$ denote the probability that an observation falls in cell $(i_1,\ldots,i_T)$ of the table ($i_k = 1,\ldots,R; k = 1,\ldots,T$). Let $x_{i_1\ldots i_T}$ denote the observed frequency in cell $(i_1,\ldots,i_T)$ of the R^T table with $N = \sum\sum\ldots\sum x_{i_1\ldots i_T}$. Assume that $\{x_{i_1\ldots i_T}\}$ have a multinomial distribution. The symmetry (S^T) model is defined by

$$p_{i_1 \dots i_T} = p_{j_1 \dots j_T},$$

for $(j_1, \dots, j_T) \in D(i_1, \dots, i_T)$, where

$$D(i_1, \dots, i_T) = \{(j_1, \dots, j_T) | (j_1, \dots, j_T) \text{ is any permutation of } (i_1, \dots, i_T)\};$$

see Agresti [1, p. 440]. For example, when $T = 3$, the S^3 model is expressed as

$$p_{ijk} = p_{ikj} = p_{jik} = p_{jki} = p_{kji} = p_{kij} \quad (1 \le i, j, k \le R);$$

see, e.g., Bishop et al [3, p. 300].

We see that the MLEs of cell probabilities $\{p_{ss \dots s}\}$ on the main diagonal of the table under the S^T model (i.e., $\{x_{ss \dots s}/N\}, s = 1, \dots, R$) are equal to those under the saturated (SA^T) model. Therefore the risks of the MLEs for the main diagonal cells of the table for these two models are the same. Thus, as the case of two-way tables, we consider the conditional risks for these two models on condition that an observation falls in one of the off-diagonal cells.

Let $q_{i_1 \dots i_T}$ denote the conditional probability on condition that an observation falls in one of the off-diagonal cells, i.e.,

$$q_{i_1 \dots i_T} = \frac{p_{i_1 \dots i_T}}{1 - \sum_{s=1}^{R} p_{s \dots s}} \quad \big((i_1, \dots, i_T) \ne (s, \dots, s), s = 1, \dots, R\big).$$

Let

$$\begin{aligned} E_{(r_1, \dots, r_T)} \quad = \quad & \{(i_1, \dots, i_T) | i_t = 1, \dots, R; t = 1, \dots, T, \text{where the } r_1\text{'s} \\ & \text{elements of } (i_1, \dots, i_T) \text{ are equal}, r_2\text{'s elements are equal}, \\ & \dots, \text{and } r_T\text{'s elements are equal}\}, \end{aligned}$$

for $(r_1, \dots, r_T) \in S^*$, where

$$S^* = \{(r_1, \dots, r_T) | r_1 + r_2 + \dots + r_T = T,\ T - 1 \ge r_1 \ge r_2 \ge \dots \ge r_T \ge 0\}.$$

Note that

$$\begin{aligned} E_{(1, \dots, 1)} \quad = \quad & \{(i_1, \dots, i_T) | i_t = 1, \dots, R; t = 1, \dots, T, \text{where all elements of} \\ & (i_1, \dots, i_T) \text{ are different}\}, \end{aligned}$$

$$(s, s, \dots, s) \notin E(r_1, \dots, r_T) \text{ for } (r_1, \dots, r_T) \in S^*,\ s = 1, \dots, R,$$

and

$$\sum \sum_{(r_1, \dots, r_T) \in S^*} \cdots \sum \left(\sum \sum_{(i_1, \dots, i_T) \in E(r_1, \dots, r_T)} \cdots \sum q_{i_1 \dots i_T} \right) = 1.$$

The MLEs of $\{q_{i_1 \dots i_T}\}$ under the S^T model on condition that an observation falls in one of the off-diagonal cells are given by, for $(i_1, \ldots, i_T) \in E(r_1, \ldots, r_T)$,

$$T^{c(S^T)}_{i_1 \ldots i_T} = \frac{\sum\sum \cdots \sum_{(j_1, \ldots, j_T) \in D(i_1, \ldots, i_T)} x_{j_1 \ldots j_T}}{{}_T C_{r_1 \ldots r_T} N^*},$$

where

$${}_T C_{r_1 \ldots r_T} = \frac{T!}{r_1! \ldots r_T!},$$

$$N^* = \sum\sum_{not\ i_1 = \cdots = i_T} \cdots \sum x_{i_1 \ldots i_T} = N - \sum_{s=1}^{R} x_{s \ldots s}.$$

For example, when $T = 3$

$$T^{c(S^3)}_{ijk} = \begin{cases} \dfrac{x_{iik} + x_{iki} + x_{kii}}{3N^*} & (i = j \neq k), \\ \dfrac{x_{ijk} + x_{ikj} + x_{kji} + x_{jik} + x_{jki} + x_{kij}}{6N^*} & (i \neq j, i \neq k, j \neq k), \end{cases}$$

where

$$N^* = \sum\sum_{not\ i=j=k}\sum x_{ijk} = N - \sum_{s=1}^{R} x_{sss}.$$

In a similar manner to the case of two-way tables, we obtain the following results. For $(i_1, \ldots, i_T) \in E(r_1, \ldots, r_T)$, the conditional bias term on condition that an observation falls in one of the off-diagonal cells, is

$$\text{Bias}\,(T^{c(S^T)}_{i_1 \ldots i_T}) = \left(\frac{\sum\sum \cdots \sum_{(j_1, \ldots, j_T) \in D(i_1, \ldots, i_T)} q_{j_1 \ldots j_T}}{{}_T C_{r_1 \ldots r_T}} - q_{i_1 \ldots i_T} \right)^2,$$

and the conditional variance term is

$$\begin{aligned} \text{Var}\,(T^{c(S^T)}_{i_1 \ldots i_T}) &= \frac{1}{({}_T C_{r_1 \ldots r_T})^2 N^*} \\ &\times \left[\sum\sum \cdots \sum_{(j_1, \ldots, j_T) \in D(i_1, \ldots, i_T)} q_{j_1 \ldots j_T} - \left(\sum\sum \cdots \sum_{(j_1, \ldots, j_T) \in D(i_1, \ldots, i_T)} q_{j_1 \ldots j_T} \right)^2 \right]. \end{aligned}$$

For example, when $T = 3$ we find that the bias term is

$$\text{Bias}\,(T^{c(S^3)}_{ijk}) = \begin{cases} \left(\dfrac{q_{iik} + q_{iki} + q_{kii}}{3} - q_{ijk}\right)^2 & (i = j \neq k), \\ \left(\dfrac{q_{ijk} + q_{ikj} + q_{kji} + q_{jik} + q_{jki} + q_{kij}}{6} - q_{ijk}\right)^2 & \\ & (i \neq j, i \neq k, j \neq k), \end{cases}$$

and the variance term is

$$\mathrm{Var}\,(T_{ijk}^{c(S^3)}) = \begin{cases} \dfrac{1}{9N^*}\Big[(q_{iik}+q_{iki}+q_{kii}) - (q_{iik}+q_{iki}+q_{kii})^2\Big] \\ \hfill (i = j \neq k), \\ \dfrac{1}{36N^*}\Big[(q_{ijk}+q_{ikj}+q_{kji}+q_{jik}+q_{jki}+q_{kij}) \\ -\big(q_{ijk}+q_{ikj}+q_{kji}+q_{jik}+q_{jki}+q_{kij}\big)^2\Big] \\ \hfill (i \neq j, i \neq k, j \neq k). \end{cases}$$

Thus the conditional risk of MLEs of $\{q_{i_1\ldots i_T}\}$ for the S^T model on condition that an observation falls in one of the off-diagonal cells is obtained by

$$R_S^c = \sum\sum_{(r_1,\ldots,r_T)\in S^*}\cdots\sum \left[\sum\sum_{(i_1,\ldots,i_T)\in E(r_1,\ldots,r_T)}\cdots\sum \Big(\mathrm{Var}\,(T_{i_1\ldots i_T}^{c(S^T)}) + \mathrm{Bias}\,(T_{i_1\ldots i_T}^{c(S^T)})\Big)\right].$$

Also, the MLEs of $\{q_{i_1\ldots i_T}\}$ under the SA^T model are given by

$$T_{i_1\ldots i_T}^{c(SA^T)} = \frac{x_{i_1\ldots i_T}}{N^*} \quad \big((i_1,\ldots,i_T) \neq (s,\ldots,s), s = 1,\ldots,R\big).$$

The conditional risk is

$$\begin{aligned} R_{SA}^c = & \quad \frac{1}{N^*}\left(\sum\sum_{not\ i_1=\cdots=i_T}\cdots\sum q_{i_1\ldots i_T}(1-q_{i_1\ldots i_T})\right) \\ = & \quad \frac{1}{N^*}\left[1 - \sum\sum_{(r_1,\ldots,r_T)\in S^*}\cdots\sum\left(\sum\sum_{(i_1,\ldots,i_T)\in E(r_1,\ldots,r_T)}\cdots\sum q_{i_1\ldots i_T}^2\right)\right]. \end{aligned}$$

Noting that when the S^T model holds, for $(i_1,\ldots,i_T) \in E(r_1,\ldots,r_T)$,

$$\sum\sum_{(j_1,\ldots,j_T)\in D(i_1,\ldots,i_T)}\cdots\sum q_{j_1\ldots j_T} = {}_TC_{r_1\ldots r_T}\, q_{i_1\ldots i_T},$$

and the bias term for the S^T model equals zero, we obtain the following theorem.

Theorem 5. *For the R^T tables, when the S^T model holds,*

$$R_{SA}^c - R_S^c = \frac{1}{N^*}\sum\sum_{(r_1,\ldots,r_T)\in S^*}\cdots\sum\left[\frac{{}_TC_{r_1\ldots r_T}-1}{{}_TC_{r_1\ldots r_T}}\sum\sum_{(i_1,\ldots,i_T)\in E(r_1,\ldots,r_T)}\cdots\sum q_{i_1\ldots i_T}\right].$$

For example, from Theorem 5 with $T = 3$, when the S^3 model holds, we obtain

$$R_{SA}^c - R_S^c \quad = \frac{1}{N^*}\left(\frac{2}{3}\sum\sum_{i\neq k}(q_{iik}+q_{iki}+q_{kii}) + \frac{5}{6}\sum\sum_{i\neq j\neq k}\sum q_{ijk}\right)$$

$$= \frac{1}{N^*}\left(2\sum\sum_{i\neq k} q_{iik} + 5\sum\sum\sum_{i<j<k} q_{ijk}\right).$$

Note that Theorem 5 with $T = 2$ is identical to Theorem 1.

From Theorem 5 we find $R^c_S < R^c_{SA}$ when the S^T model holds. Thus we know that $\{T^{c(S^T)}_{i_1\ldots i_T}\}$ are always more precise than $\{T^{c(SA^T)}_{i_1\ldots i_T}\}$ when the S^T model holds.

4. Concluding Remarks

For square tables, we gave the conditional risks of the MLEs of conditional probabilities $\{q_{ij}\}$ under the S, CS, and SA models on condition that an observation falls in one of the off-diagonal cells. Also, for multi-way tables, we gave the conditional risks of the MLEs of conditional probabilities $\{q_{i_1\ldots i_T}\}$ under the S^T and SA^T models. By comparing the risks between the models, we obtained that when the simpler model is correct, the overall variability for the estimator based on the simple model is smaller than for the estimator based on the more complicated model.

Many readers may be interested in the unconditional risks of the MLEs (i.e., $\{T^{(S)}_{ij}\}$ and $\{T^{(CS)}_{ij}\}$) of cell probabilities $\{p_{ij}\}$ under the S and CS models. We point out that the risk of the MLEs of $\{p_{ij}\}$ for the S model could be obtained easily (in a similar way to R^c_S) as

$$\begin{aligned} R_S &= \frac{1}{4}\sum\sum_{i\neq j}\left[\frac{1}{N}\left\{(p_{ij}+p_{ji}) - (p_{ij}+p_{ji})^2\right\} + (p_{ij}-p_{ji})^2\right] \\ &\quad + \frac{1}{N}\sum_{i=1}^{R} p_{ii}(1-p_{ii}); \end{aligned}$$

but it would be difficult to obtain the unconditional risk for the CS model.

Finally we note that the CS model is defined for square tables, but it is difficult to consider the similar model for multi-way tables.

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References

[1] A. Agresti, *Categorical Data Analysis*, Second Edition, Wiley, New York (2002).

[2] V.P. Bhapkar, J.N. Darroch, Marginal symmetry and quasi symmetry of general order, *Journal of Multivariate Analysis*, **34** (1990), 173-184.

[3] Y.M.M. Bishop, S.E. Fienberg, P.W. Holland, *Discrete Multivariate Analysis: Theory and Practice*, The MIT Press, Cambridge, Massachusetts (1975).

[4] A.H. Bowker, A test for symmetry in contingency tables, *Journal of the American Statistical Association*, **43** (1948), 572-574.

[5] B.S. Everitt, *The Analysis of Contingency Tables*, Second Edition, Chapman and Hall, London (1992).

[6] P. McCullagh, A class of parametric models for the analysis of square contingency tables with ordered categories, *Biometrika*, **65** (1978), 413-418.

[7] S. Tomizawa, A decomposition of conditional symmetry model and separability of its test statistic for square contingency tables, *Sankhy ā, Series B*, **54** (1992), 36-41.

[8] S. Tomizawa, Diagonals-parameter symmetry model for cumulative probabilities in square contingency tables with ordered categories, *Biometrics*, **49** (1993), 883-887.

[9] S. Tomizawa, K. Tahata, The analysis of symmetry and asymmetry: Orthogonality of decomposition of symmetry into quasi-asymmetry and marginal symmetry for multi-way tables, *Journal de la Soci été Francaise de Statistique*, **148** (2007), 3-36.

[10] H. Yamamoto, T. Iwashita, S. Tomizawa, Decomposition of symmetry into ordinal quasi-symmetry and marginal equimoment for multi-way tables, *Austrian Journal of Statistics*, **36** (2007), 291-306.

Appendix A

For an $R \times R$ table, the MLEs of cell probabilities $\{p_{ij}\}$ under the I model are given by

$$T_{ij}^{(I)} = \frac{x_{i+}x_{+j}}{N^2} \quad (i = 1, \dots, R; j = 1, \dots, R),$$

where $x_{i+} = \sum_{t=1}^{R} x_{it}$ and $x_{+j} = \sum_{s=1}^{R} x_{sj}$. Those under the SA model are given by

$$T_{ij}^{(SA)} = \frac{x_{ij}}{N} \quad (i = 1, \dots, R; j = 1, \dots, R).$$

Referring Bishop et al [3, p. 314], we see that for $\{T_{ij}^{(SA)}\}$, the bias term is always zero, and the risk for $\{T_{ij}^{(SA)}\}$ is given by

$$R_{SA} = \frac{1}{N}\left(1 - \sum_{i=1}^{R}\sum_{j=1}^{R} p_{ij}^2\right).$$

Also, the risk for $\{T_{ij}^{(I)}\}$ is given by

$$R_I = \sum_{i=1}^{R}\sum_{j=1}^{R}\left[\mathrm{Var}\,(T_{ij}^{(I)}) + Bias(T_{ij}^{(I)})\right],$$

where

$$\begin{aligned}
\mathrm{Var}\,(T_{ij}^{(I)}) &= \frac{(N-1)(6-4N)}{N^3} p_{i+}^2 p_{+j}^2 + 2\frac{(N-1)(N-4)}{N^3} p_{ij}p_{i+}p_{+j} \\
&\quad + \frac{(N-1)(N-2)}{N^3}(p_{i+}p_{+j}^2 + p_{i+}^2 p_{+j}) + \frac{(N-2)}{N^3} p_{ij}^2 \\
&\quad + \frac{(N-1)}{N^3}(2p_{ij}p_{+j} + 2p_{ij}p_{i+} + p_{i+}p_{+j}) + \frac{p_{ij}}{N^3}, \\
\mathrm{Bias}\,(T_{ij}^{(I)}) &= \left(\frac{N-1}{N}\right)^2 (p_{i+}p_{+j} - p_{ij})^2, \\
& p_{i+} = \sum_{t=1}^{R} p_{it},\ p_{+j} = \sum_{s=1}^{R} p_{sj}.
\end{aligned}$$