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CHARACTERIZATION OF THE GENERATORS OF NORM PRINCIPAL RADICAL IDEAL OF NEST ALGEBRAS

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Abstract: Let $\mathcal{N}$ be complete nest of separable Hilbert space H, alg$\mathcal{N}$ be the nest algebra respect to nest $\mathcal{N}$, we characterize the generator of Jacobson radical $R_{\mathcal{N}}$ in several cases when it is a norm closed principal ideal of nest algebra alg$\mathcal{N}$.

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1. Introduction

Let H be a separable complex Hilbert space. Let $B(H)$ denote the algebra of all bounded linear operators on H. Let $\mathcal{K}$ be the ideal of compact operators of $B(H)$. For any vectors x and y in H, the rank-1 operator $\langle \cdot, y\rangle x$ will be denoted by $x \otimes y$.

A nest $\mathcal{N}$ in a Hilbert space H is a chain of self-adjoint projections ordered by inclusion of the corresponding range spaces. A nest is complete if it is closed in the strong operator topology. In this paper all nests will be complete. Let alg$\mathcal{N}$ denote the set of all operators in $B(H)$ that leave the range of each element of $\mathcal{N}$ invariant. This is an algebra under usual addition and multiplication and

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is called the *nest algebra* related to $\mathcal{N}$. We use dim$N$ to denote $\dim(NH)$ for any projection $N \in \mathcal{N}$, and we use $\bigvee\{N : N \in \mathcal{F}\}$ to denote the projection onto the subspace $\bigvee\{NH : N \in \mathcal{F}\}$, where $\mathcal{F}$ is a set of projections in $\mathcal{N}$ and $\bigwedge\{N : N \in \mathcal{F}\}$ to denote the projection onto $\bigwedge\{NH : N \in \mathcal{F}\}$. Ideals in nest algebras have played a significant role in the theory of these algebras. The first major result concerning ideals was Ringrose's characterization of the Jaconson radical $R_{\mathcal{N}}$ of a nest algebras alg$\mathcal{N}$ in terms of finite partitions of the nest [6]. Later, Larson showed that another radical-type ideal (called Larson's strong radical and denoted by $R^{\infty}_{\mathcal{N}}$) is significant in the study of similarities of nest algebras [4], and in [2], Hopenwasser studied systematically a large class of norm-closed two-sided ideals lying between the Jacobson radical and Larson's strong radical, in [2], Hopenwasser raised a question of whether the Jacobson radical may be a topologically principal ideal. John Orr answered this question by giving a necessary and sufficient conditions on the structure of the invariant nest of the algebra for the Jacobson radical to be topologically principal in [5] and X. Dai gave a different and independent proof in [1]. But, what kind of operators can be a generator of this norm-principal ideal or how to characterize the generators remains open. In this paper, we will talk about this question and answer this question in several cases.

2. Preliminaries and Lemmas

Throughout this paper, H will denote a separable Hilbert space. A *nest* is a totally ordered family of projections which contains 0 and I and which is closed in the strong operator topology. An *interval* is a projection E of the form $E = P - Q$, where P, $Q \in \mathcal{N}$ and $Q < P$. For each $N \in \mathcal{N}$ we define

$$N^+ = \bigwedge\{M \in \mathcal{N} | M > N\}, \quad I^+ = I\,,$$

$$N^- = \bigvee\{M \in \mathcal{N} | M < N\}, \quad 0^- = 0\,.$$

The intervals are ordered by the usual partial ordering of projections and the minimal intervals are called *atoms*. Atoms are of the form $M^+ - M$ or $M - M^-$ and the atoms of a nest are mutually orthogonal. The range of an atom is called a gap of the nest. A *partition* is a family $\varepsilon = \{E_n\}$ of intervals satisfying the properties: (i) $E_n E_m = 0$ if $n \neq m$, (ii) $\sum_n E_n = I$. Note that if P is a projection commuting with all the projections in the nest $\mathcal{N}$, then $P\mathcal{N} = \{PN | N \in \mathcal{N}\}$ is a nest on the space PH and alg $P\mathcal{N} = P$alg$\mathcal{N}P$. We say that a norm closed ideal of a topological algebra is *topologically principal* if there is an operator of the algebra for which the ideal is the smallest closed ideal containing this operator.

In this paper, we call such matrix as $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ a counter identity and which is denoted as I'. It is also applied for the infinite matrix, i.e., the infinite matrix (t_{ij}), $i, j \in Z$, where $t_{ij} = 1$ if $i < 0 < j$ and $-i = j$, $t_{ij} = 0$ for all the others is the counter identity. we will also call the operator a counter identity if its matrix under some orthonormal basis is a counter identity.

The following lemmas give us a criterion to determine if an operator in alg$\mathcal{N}$ is contained in $R_{\mathcal{N}}$ and a way to factor a positive invertible operator.

Lemma 2.1. (see Ringose Criterion, [6]) *An operator $A \in \text{alg}\mathcal{N}$ is in $R_{\mathcal{N}}$ if and only if A satisfies the property that given $\epsilon > 0$ there is a finite partition ε such that $\|ETE\| < \epsilon$ for all $E \in \varepsilon$.*

Lemma 2.2. *(see [4]) Let T be a positive invertible operator in $B(H)$ and let $\mathcal{N}$ be a complete nest in H. Then, there are invertible operators A, B in* alg$\mathcal{N}$ *with A^{-1}, B^{-1} not necessarily in* alg$\mathcal{N}$ *such that $T = A^*A = BB^*$.*

3. Main Results and Proofs

Theorem 3.1. *Let $\mathcal{N} = \{0 < N_1 < N_2 < I\}$ be a complete nest on separable Hilbert space H, and all the atoms are infinite dimensional, then an operator $G \in R_{\mathcal{N}}$ is a generator of $R_{\mathcal{N}}$ if and only if $G_{21} = N_1 G N_2$ and $G_{32} = N_2 G N_3$ are not compact.*

Proof. The necessary direction is trivial, we now prove the sufficient direction. For convenience, let $H_1 = N_1 H$, $H_2 = (N_2 - N_1)H$, $H_3 = N_2^{\perp} H$. Since G_{21} is not compact, there exists an infinite dimensional subspace $M \subset H_2$, such that G_{21} is bounded below on M, i.e., there exists $\alpha > 0$, such that for any $f \in M$, $\|G_{21} f\| \geq \alpha \|f\|$. Let Y be the range of G_{21} on M, then Y is closed subspace of H_1, and $G_{21}|_M : M \to Y$ is invertible. For any $B \in B(H_2, H_1)$, let Q be the projection: $H_2 \to M$, then $Q \in \text{alg}\mathcal{N}$. Let $T : H_1 \to H_1$ be defined by $Ty = TG_{21}Qx = Bx$ for some $x \in H_2$, if $y \in Y$ and $Ty = 0$ if $y \in Y^{\perp}$, so obviously T is well defined and it is linear. Since $\|Ty\| = 0$ if $y \in Y^{\perp}$ and if $y \in Y$, then $y = G_{21} g$ for some $g \in M$, so $\|y\| = \|G_{21} g\| \geq \alpha \|g\|$, but $\|Ty\| = \|TG_{21} g\| = \|TG_{21} Q g\| = \|Bg\| \leq \|B\| \|g\| \leq \|B\| \frac{1}{\alpha} \|y\|$, so T is also bounded, hence, $T \in B(H_1)$, of course $T \in \text{alg}\mathcal{N}$, and since $B = TG_{21}Q$, G_{21} is a generator of norm principal ideal $N_1 R_{\mathcal{N}} (N_2 - N_1)$. Similarly, G_{32} is a generator of norm principal ideal $(N_2 - N_1) R_{\mathcal{N}} N_2^{\perp}$. Lastly, we show that G_{31} will generate norm principal ideal $N_1R_\mathcal{N}N_2^\perp$. Let U_{21} be the partial isometry with initial space H_2, final space H_1. Let Q_i be projection from H onto H_i, for $i=1,2,3$, then $Q_1=U_{21}U_{21}^*$ and $U_{21}\in \mathrm{alg}\mathcal{N}$. Since $Q_1=N_1, Q_3=N_2^\perp$, so $N_1R_\mathcal{N}N_2^\perp = Q_1R_\mathcal{N}Q_3 = U_{21}(U_{21}^*R_\mathcal{N}Q_3 \subset \mathrm{alg}\mathcal{N}(N_2-N_1)R_\mathcal{N}N_2^\perp$, so $N_1R_\mathcal{N}N_2^\perp$ can be generated by G_{32} as well. Since $R_\mathcal{N}=N_1R_\mathcal{N}(N_2-N_1)+(N_2-N_1)R_\mathcal{N}N_2^\perp+N_1R_\mathcal{N}N_2^\perp$, so the Jocobson radical can be generated by G. □

In general, we have the following corollary.

Corollary 3.2. *Let $\mathcal{N}=\{0=P_0<P_1<\cdots<P_n=I\}$ be a finite nest whose atoms $E_k=P_k-P_{k-1}$, $k=1,2,\cdots,n$, are all infinite dimensional, then an operator $G\in R_\mathcal{N}$ is a generator of norm principal ideal $R_\mathcal{N}$ if and only if $E_{i-1}GE_i$ is not compact for $i=2,3,\cdots,n$.*

Theorem 3.3. *Let $\mathcal{N}=\{0=P_0<P_1<\cdots<P_n=I\}$ be a finite nest whose atoms $E_k=P_k-P_{k-1}$, $k=1,2,\cdots,n$, are all finite dimensional, then an operator $G\in R_\mathcal{N}$ is a generator of norm principal ideal $R_\mathcal{N}$ if and only if $E_{i-1}GE_i\neq 0$ for $i=2,3,\cdots,n$.*

Proof. We only prove the sufficient part, the necessary part is obvious. By Lemma 2.1, for any $T\in R_\mathcal{N}$, $T=\sum_{j<k}E_jTE_k$. It is sufficient to prove for fixed k, E_jTE_k, $j<k$ is generated by $E_{k-1}GE_k$. Because $E_{k-1}GE_k$ is a finite rank operator, without loss of generality, we can assume $E_{k-1}GE_k = x_{k-1}\otimes x_k$ with $x_{k-1}\in E_{k-1}H$, $x_k\in E_kH$ and $\|x_{k-1}\|=\|x_k\|=1$, then for any rank one operator in $E_{k-1}B(H)E_k$, say, $x\otimes y$, we have $x\otimes y = (x\otimes x_{k-1})(x_{k-1}\otimes x_k)(x_k\otimes y)$ and $x\otimes x_{k-1}, x_k\otimes y\in \mathrm{alg}\mathcal{N}$, so any rank one operator in $E_{k-1}B(H)E_k$ can be generated by $E_{k-1}GE_k$, but $E_{k-1}B(H)E_k$ are all finite operators, so $E_{k-1}TE_k$ can be generated by $E_{k-1}GE_k$. For any $j<k$, let $U_{k-1,j}$ be a partial isometry with initial space $E_{k-1}H$, final space is a subspace of E_jH, if $\dim E_{k-1}<\dim E_jH$, an unitary if $\dim E_{k-1}=\dim E_jH$, and a partial isometry with initial space be a subspace of $E_{k-1}H$, finial space E_jH, if $\dim E_{k-1}>\dim E_jH$. Then for any $S\in R_\mathcal{N}$, $SE_k=\sum_{j<k}E_jSE_k = \sum_{j<k-1}E_jSE_k+E_{k-1}SE_k = \sum_{j<k-1}U_{k-1,j}(U_{k-1,j}^*SE_k)+E_{k-1}SE_k$, since $U_{k-1,j}^*\in E_{k-1}B(H)E_k$ and $U_{k-1,j}\in \mathrm{alg}\mathcal{N}$ for $j<k-1$, so SE_k is generated by $E_{k-1}GE_k$. But $S=\sum_{k=1}^n SE_k$, so it can be generated by G. □

By Theorem 3.1 and Theorem 3.3, it is easy to get the following corollary and we leave it to our reader.

Corollary 3.4. *Let $\mathcal{N}$ be a complete finite nest of separable Hilbert space H with its atoms $\{E_i\}_{i=1}^n$, then an operator $G\in \mathrm{alg}\mathcal{N}$ is a generator of the* *radical* $R_{\mathcal{N}}$ *if and only if* $E_iGE_{i+1} \neq 0$ *for* $i = 1, 2, \cdots, n$, *and* E_mGE_n *is not compact for all* m, n *with* $\dim E_mH = \dim E_nH = +\infty$ *and* $m = \max\{k : k \leq n-1, \dim E_kH = +\infty\}$.

Theorem 3.5. *Let* $\mathcal{N}$ *be a complete nest which is order isomorphic to* $N \cup \{\infty\}$ *and whose atoms are finite dimensional. Let* $\{P_i\}_{i=1}^{\infty} \cup \{0, I\}$ *be the projections of the nest,* $\{E_n\}_{n=1}^{\infty}$ *are all its atoms, then an operator* $G \in R_{\mathcal{N}}$ *is a generator of the norm principal radical ideal* $R_{\mathcal{N}}$ *if and only if* $E_{i-1}GE_i \neq 0$ *for all* $i \geq 2$.

Proof. As before we just need to show the sufficient part. For any $n \in N$, by Theorem 3.3, GP_n is generator of $R_{\mathcal{N}}P_n$. Suppose $T \in R_{\mathcal{N}}$, for arbitrary $\varepsilon > 0$, by Lemma 2.1, there is an integer m such that $\|(I-P_n)T(I-P_n)\| < \frac{\varepsilon}{2}$, for any $n \geq m$. Since P_n is finite rank projection, P_nT is compact. Because $P_k \to I$ strongly, there is an integer l such that $\|P_nT(I-P_k)\| < \frac{\varepsilon}{2}$, for any $k \geq l$. Let $q = \max\{m, l\}$, then when $h \geq q$, we have $\|T(I-P_h)\| = \|P_hT(I-P_h) + (I-P_h)T(I-P_h)\| \leq \|P_hT(I-P_h)\| + \|(I-P_h)T(I-P_h)\| \leq \varepsilon$. Since TP_h is generated by G, and ε is arbitrary, so T is generated by G. □

Theorem 3.6. *Let* $\mathcal{N}$ *be a complete nest which is order isomorphic to* $N \cup \{\infty\}$ *and whose atoms are infinite dimensional. Let* $\{P_i\}_{i=1}^{\infty} \cup \{0, I\}$ *be the projections of the nest,* $\{E_n\}_{n=1}^{\infty}$ *are all its atoms, then an operator* $G \in R_{\mathcal{N}}$ *is a generator of the norm principal radical ideal* $R_{\mathcal{N}}$ *if* $E_{i-1}GE_i$ *is not compact, for all* $i \geq 2$ *and for any positive integer* n_0, *there exists* $i \geq n_0$, *such that* $(P_i - P_{i-1})G(I-P_i)$ *can be linear transformed to be counter identity* I'.

Proof. For any $T \in R_{\mathcal{N}}$, by Lemma 2.1, there is an positive integer k_0 such that $\|(I-P_i)T(I-P_i)\| < \varepsilon$ for all $i \geq k_0$, so there exists $m \geq k_0$, such that $\|(I-P_m)T(I-P_m)\| < \varepsilon$ and $(P_m - P_{m-1})G(I-P_m)$ can be linear transformed to be counter identity I', i.e., there exist operator sequence $\{A_i\}$, $\{B_i\}$ such that $A_i, B_i \in (P_m - P_{m-1})B(H)(I-P_m)$, and $(\Pi A_i)(P_m - P_{m-1})G(I-P_m)(\Pi B_i) = I'$. Now suppose that $(I-P_m)H = \mathrm{span}\{e_1, e_2, \cdots\}$, $(P_m - P_{m-1})H = \mathrm{span}\{\cdots, \beta_{-2}, \beta_{-1}\}$, and K be a Hilbert space with orthonormal basis $\{g_1, g_2, \cdots\}$. Let Q_n be the projection onto the subspace $K_n = \mathrm{span}\{g_k : k \leq n\}$, then $\mathcal{M} = \{Q_n\} \cup \{0, I\}$ is a complete nest of K. Let W and V be isometries mapping K into H defined by $Wg_i = e_i$, $Vg_i = \beta_{-i}$ for all $i \geq 1$, then it is easy to see that $W\mathrm{alg}\mathcal{M}W^* \subseteq \mathrm{alg}\mathcal{N}$, $V\mathrm{alg}\mathcal{M}V^* \subseteq (\mathrm{alg}\mathcal{N})^*$ and $V(\mathrm{alg}\mathcal{M})^*V^* \subseteq \mathrm{alg}\mathcal{N}$. The mapping $X \to V^*XW$ is a bijection of $B((I-P_m)H, (P_m - P_{m-1})H)$ onto $B(K)$, let $Y = V^*(P_m - P_{m-1})T(I-P_m)W$, since any operator can be expressed as linear combination of positive and invertible operators, without loss of generality, suppose Y is positive and invertible, by Lemma 2.2, there exists $A \in \text{alg}\mathcal{M}$ such that $Y = A^*A$. Since $V^*I'W = I$, $V^*(\Pi A_i)(P_m - P_{m-1})G(I - P_m)(\Pi B_i)W = V^*I'W = I$, so $Y = A^*IA = A^*V^*(\Pi A_i)(P_m - P_{m-1})G(I - P_m)(\Pi B_i)WA$, hence, $(P_m - P_{m-1})T(I-P_m) = (VA^*V^*)(\Pi A_i)(P_m-P_{m-1})G(I-P_m)(\Pi B_i)(WAW^*)$. But we know that $VA^*V^*, WAW^* \in \text{alg}\mathcal{N}$, so $(P_m - P_{m-1})T(I - P_m)$ is generated by $(P_m - P_{m-1})G(I - P_m)$. Since $P_kT(I - P_m) = \sum_{k=1}^{m}(P_k - P_{k-1})T(I - P_m)$, let U_{mk} be the isometry from $(P_m - P_{m-1})H$ to $(P_k - P_{k-1})H$, then for each k, U_{mk} belong to $\text{alg}\mathcal{N}$ and $U_{mk}U_{mk}^* = P_k - P_{k-1}$, so $(P_k - P_{k-1})T(I - P_m) = U_{mk}U_{mk}^*T(I - P_m) \in \text{alg}\mathcal{N}(P_m - P_{m-1})B(H)(I - P_m)$, so it is generated by $(P_m-P_{m-1})G(I-P_m)$, so $P_mT(I-P_m)$ is generated by $P_mG(I-P_m)$. Because $P_m\mathcal{N}$ is a finite nest with infinite dimensional atoms, by Theorem 3.1, P_mGP_m is a generator of P_mTP_m. Because $T = (I - P_m)T(I - P_m) + P_mT(I - P_m) + P_mTP_m$, and $\|(I - P_m)T(I - P_m)\| < \varepsilon$, so T is generated by G. □

Theorem 3.7. *Let $\mathcal{N}$ be a complete nest which is order isomorphic to $\{-\infty\} \cup Z \cup \{\infty\}$, with all its atoms are finite dimensional. Suppose its projections are $\{\cdots < P_{-2} < P_{-1} < P_0 < P_1 < P_2 < \cdots\}$, and its atoms are $\{E_n = P_n - P_{n-1}\}_{n\in Z}$, then an operator $G \in R_{\mathcal{N}}$ is a generator if $E_{n-1}GE_n \neq 0$ for all $n \in Z$ and there exists positive integer n_0 such that $P_nG(I - P_n)$ can be linear transformed to be the counter identity $I' \in P_nB(H)(I - P_n)$, for any $n \geq n_0$.*

Proof. For any $T \in R_{\mathcal{N}}$, by Lemma 2.1, for any $\varepsilon > 0$, there exists positive integer m_1 such that when $m \geq m_1$, $\|P_{-m}TP_{-m}\| < \varepsilon$ and $\|(I - P_m)T(I - P_m)\| < \varepsilon$. Because $P_{-n} \to 0$ strongly $(n \to +\infty)$, and $(P_{m_1} - P_{-m_1})T$ is compact, so $\|P_{-n}(P_{m_1} - P_{-m_1})T\| \to 0$ $(n \to +\infty)$, then there exists a positive integer m_2 such that $\|P_{-k}T(P_{m_1} - P_{-m_1})\| < \varepsilon$ for all $k \geq m_2$. Similarly, since $I - P_n \to 0$ strongly, there exists positive integer m_3 such that $\|(P_{m_1} - P_{-m_1})T(I - P_n)\| \to 0$ for any $n \geq m_3$. Let $m = \max\{m_1, m_2, m_3\}$, then, $\|P_{-m}TP_m\| = \|P_{-m}T(P_m - P_{-m}) + P_{-m}TP_{-m}\| \leq \|P_{-m}T(P_m - P_{-m})\| + \|P_{-m}TP_{-m}\| \leq 2\varepsilon$, and $\|(I - P_{-m})T(I - P_m)\| = \|(I - P_m)T(I - P_m) + (P_m - P_{-m})T(I - P_m)\| \leq \|(I - P_m)T(I - P_m)\| + \|(P_m - P_{-m})T(I - P_m)\| \leq 2\varepsilon$. By Theorem 3.1, $(P_m - P_{-m})G(P_m - P_{-m})$ generate $(P_m - P_{-m})T(P_m - P_{-m})$. Applying the similar argument of Theorem 3.6, we can show that $P_{-m}G(I-P_m)$ generate $P_{-m}T(I - P_m)$. Since $T = (P_m - P_{-m})TP_m + P_{-m}TP_m + P_{-m}T(I - P_m) + (I - P_m)T(I - P_m)$, so T is generated by G. □

Theorem 3.8. *Let $\mathcal{N}$ be a complete nest which is order isomorphic to $\{-\infty\} \cup Z \cup \{\infty\}$, with all its atoms are infinite dimensional. Suppose its* *projections are* $\{\cdots < P_{-2} < P_{-1} < P_0 < P_1 < P_2 < \cdots\}$, *and its atoms are* $\{E_n = P_n - P_{n-1}\}_{n\in Z}$, *then an operator* $G \in R_{\mathcal{N}}$ *is a generator if and only if* $E_{n-1}GE_n$ *is not compact, for all* $n \in Z$ *and there exists positive integer* n_0 *such that* $(P_n - P_{n-1})G(I - P_n)$ *is not compact, for any* $n \geq n_0$.

Proof. The necessary part is obvious, we now prove the sufficient part. Suppose $T \in R_{\mathcal{N}}$, by Lemma 2.1, for any $\varepsilon > 0$, there exist positive integer k such that $\|P_{-n}TP_{-n}\| < \varepsilon$ and $\|(I - P_k)T(I - P_k)\| < \varepsilon$, for any $n \geq k$, according to the given condition, there exist positive integer $m > k$ such that $\|P_{-m}TP_{-m}\| < \varepsilon$, $\|(I - P_m)T(I - P_m)\| < \varepsilon$ and $(P_m - P_{m-1})G(I - P_m)$ is not compact. Applying the similar technique as Theorem 3.1, we can prove that $(P_m - P_{-m})T(I - P_m)$ and $P_{-m}T(I - P_m)$ are generated by $(P_m - P_{m-1})G(I - P_m)$, $P_{-m}T(P_m - P_{-m})$ is generated by $P_{-m}G(P_m - P_{-m})$ and $(P_m - P_{-m})T(P_m - P_{-m})$ is generated by $(P_m - P_{-m})G(P_m - P_{-m})$. Since $T = P_{-m}TP_{-m} + (I - P_m)T(I - P_m) + P_{-m}T(I - P_m) + (P_m - P_{-m})T(I - P_m) + P_{-m}T(P_m - P_{-m}) + (P_m - P_{-m})T(P_m - P_{-m})$, so T is generated by G. □

Remark 3.9. In this paper, we just consider several types of nest with simple structure, even though, we just show the sufficient condition for the case of Theorems 3.6 and 3.7, although I conjecture that it is also necessary condition. For the more general situation, this problem about characterizing the generator of norm principal ideal $R_{\mathcal{N}}$ is very complex. It still challenge us.

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