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UNSTABLE RULED SURFACES WITH LARGE DEGREE

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Abstract: Here we describe the irreducible families of the set of all degree $d \geq 4g-3$ abstract ruled surfaces $S \subset \mathbf{P}^{v-1}$ with a prescribed genus $g \geq 2$ curve as the normalization of a general hyperplane section.

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Let C be a smooth curve with genus $g \geq 2$ defined over an algebraically closed field. Here we study the scroll $S \subset \mathbf{P}^{v-1}$ with C as normalization of a general hyperplane section $d := \deg(S) \geq 4g-4$ and whose associated rank 2 vector bundle on C is unstable. Fix integers $d > 0$, $v > 0$, $s \leq 0$. Let $A(C; d, v; <)$ (resp. $A(C; d, v; \leq)$, resp. $A(C; d, v; s)$) denote the set of all pairs (E, V), where E is a rank 2 vector bundle on C which is not semistable (resp. not stable, resp. with index of stability $s(E) = s$) and V is a v-dimensional linear subspace of $H^0(C, E)$. We recall that $s(E) := \deg(E) - 2 \cdot \deg(L)$, where L is a maximal degree rank one subsheaf of E. For any linear subspace V of $H^0(C, E)$ let $\widetilde{V}$ denote the associated linear subspace of $H^0(\mathbf{P}(E), \mathcal{O}_{\mathbf{P}(E)}(1))$. $\dim(\widetilde{V}) = \dim(V)$. V spans E if and only if $\widetilde{V}$ spans $\mathcal{O}_{\mathbf{P}(E)}(1)$. Let $A(C; d, v; <)'$ (resp. $A(C; d, v; <)''$, resp. $A(C; d, v; <)'''$) denote the set of all pairs (E, V) with V spanning E) (resp. and with $(\mathcal{O}_{\mathbf{P}(E)}(1), \widetilde{V}$ inducing a birational morphism of $\mathbf{P}(E)$, resp. an embedding of $\mathbf{P}(E)$). We will also use a similar notation for

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the corresponding subsets of $A(C;d,v;\leq)$ and of $A(C;d,v;s)$. For all positive integers r,m let $W^r_m(C)'$ (resp. $W^r_m(C)''$, resp. $W^r_m(C)'''$) denote the set of all $L \in W^r_m(C)$ such that L is spanned (resp. spanned and inducing a birational map onto its image, resp. very ample).

Theorem 1. *Fix an integer $d \geq 4g-4$. The following algebraic subsets form a partition of $A(C;d,v;<)$. For all integers $m \geq 0$, $x > 0$ such that $d > 2m$, $x > 0$ such that $d-m+1-g+x \geq v$ let $B(C;d-m,m,x)$ be the set of all extensions of $M \in W^{x-1}_m(C)\backslash W^x_m(C)$ by an element $L \in \mathrm{Pic}^{d-m}(C)$. $h^0(C,E) = d-m+1-v$ for any such E. Let $V(C;d-m,m,x;v)$ denote the set of all pairs (E,V) with $E \in B(C;d-m,m,x)$ and V a v-dimensional linear subspace of $H^0(C,E)$. If $d \geq 4g+1$ to get $A(C;d,v;<)'$ (resp. $A(C;d,v;<)''$, resp. $A(C;d,v;<)'''$) just take $M \in W^{x-1}_m(C)'$ (resp. $M \in W^{x-1}_m(C)''$, resp. $M \in W^{x-1}_m(C)'''$) and then take only the linear spaces V spanning E (resp. $\widetilde{V}$ spanning and birationally very ample, resp. $\widetilde{V}$ spanning and inducing an embedding). If $d > 4g-4$, then the same description holds (with $m \leq d/2$ instead of $m < d/2$) for the sets $A(C;d,v;\leq)$, $A(C;d,v;\leq)'$, $A(C;d,v;\leq)'$ and $A(C;d,v;\leq)'''$.*

Remark 1. Fix an integer $s \leq 0$. To get $A(C;d,v;s)$, $A(C;d,v;s)'$, $A(C;d,v;s)''$ and $A(C;d,v;s)''$ in the statement of Theorem 1 just take only the integers m such that $d+s = 2m$.

Remark 2. In the statement of Theorem 1 assume either $g = 2$ or that C has general moduli. The Brill-Noether theory of C gives when $B(C;d-m,m,x)$, $V(C;d-m,m,x;v)$, $A(C;d,v;<)'$, $A(C;d,v;<)''$ and $A(C;d,v;<)'''$ are non-empty. In all non-emptyness cases we also get irreducibility, except when $\rho(g,x-1,m) = 0$. In these cases we could obtain the irredubility not for a fixed curve C, but moving C in a non-empty open subset of $\mathcal{M}_g$ (see [2]).

Here we also give a few new cases to the reducibility of the Hilbert scheme of scrolls in $\mathbf{P}^{d+1-2g}$ with sectional genus g, degree d and with a curve with general moduli as the normalization of a general hyperplane section (see Examples 3, 1 and 2). These example are a small addition to the example given in [1], Example 5.12.

Remark 3. Fix a smooth genus g curve C, $L, M \in \mathrm{Pic}(C)$ and an extension

$$0 \to L \xrightarrow{u} E \xrightarrow{v} M \to 0\,. \tag{1}$$

Set $S := \mathbf{P}(E)$ and let $\mathcal{O}_S(1)$ the tautological relatively ample line bundle on S. The surjection v in (1) induces a section $\sigma : C \to S$ of the ruling of S. Fix a linear subspace V of $H^0(C,E)$ such that $\widetilde{V}$ induces an embedding of S. Let

W be the linear subspace of $H^0(C, M)$ induced by V and the map v. Since $\widetilde{V}$ gives an embedding of $\sigma(C)$, the pair (M, W) induces an embedding of C. In particular M is very ample. If W induces an embedding of C, $h^1(C, L) = 0$ and $V \cap H^0(C, L)$ induces an embedding of C, then $\widetilde{V}$ induces an embedding of S.

Remark 4. Fix an exact sequence on C

$$0 \to L \to E \to M \to 0 \tag{2}$$

with $L, M \in \mathrm{Pic}(C)$, $\deg(L) \geq \deg(L)$ and $d := \deg(\det(E)) \geq 4g-4$. Hence $\deg(L) \geq d/2$. Thus $h^1(C, L) \leq 1$ and $h^1(C, L) = 1$ if and only if $d = 4g-4$ and $L \cong \omega_C$. Assume $d = 4g-4$ and $L \cong \omega_C$. Hence $h^1(C, M) \leq 1$. Hence $h^1(C, E) \leq 2$ and if $h^1(C, E) = 2$, then $M\omega_C$. We claim that (under the assumption $d = 4g-4$, $L \cong \omega_C$) $h^1(C, E) = 2$ if and only if $E \cong \omega_C \oplus \omega_C$. Since "if " part is obvious, we only need to prove that if $h^1(C, E) = 2$ and $L \cong M \cong \omega_C$, then (1) splits. Look at the Serre dual exact sequence:

$$0 \to M^* \otimes \omega_C \to E^* \otimes \omega_C \to L^* \otimes \omega_C \to 0\,. \tag{3}$$

By assumption $M^* \otimes \omega_C \cong L^* \otimes \omega_C \cong \mathcal{O}_C$. By Serre duality it is sufficient to prove that (2) splits if $h^0(C, E^* \otimes \omega_C) = 2$. This assertion follows from the computation of the coboundary map giving an extension class.

Remark 5. To cover the case $d = 4g-4$ of the description of $A(C; d, v; \leq)$ we only need to add the cases with $L \cong \omega_C$ (see Remark 4).

Proof of Theorem 1. The description of $A(C; d, v; <)$ and $A(C; d, v; \leq)$ follows from the first part of Remark 4. Remark 3 shows that our assumptions for $A(C; d, v; <)'$, $A(C; d, v; <)''$ and $A(C; d, v; <)'''$ are necessary conditions. Since $d - m \geq 2g+1$, any $L \in \mathrm{Pic}^{d-m}(C)$ is non-special and very ample. Hence these conditions are sufficient conditions, too. □

For any smooth genus g curve C let $H[d, C]$ denote the set of all non-degenerate degree d surface scrolls $S \subset \mathbf{P}^{d+1-2g}$ with C as the normalization of a general hyperplane section of S. Let $H'[d, g]$ be the union of all $H[d, C]$ with C with general moduli. $\mathcal{H}_{d,g}$ denote the ireducible component of $H[d, g]$ formed by the non-special scrolls. Notice that a scroll $S \in H[d, g]$ is non-special if and only it it is not a birational linear projection of a scroll in a high dimensional projective space.

Example 1. Fix integers $g \geq 3$, $d \geq 2g+2$, and set $a := \lfloor g/3 \rfloor$ and $e := g - 3a$. Hence $0 \leq e \leq 2$. Set $m := 2a + 2 + e$. Notice that $\rho(g, 2, m) = 3m - 6a - 2e - 6 = e$. Let C be a general curve with genus g. Let A be te set of all decomposable vector bundles on C $E = L \oplus M$ with L sufficiently general in $\mathrm{Pic}^{d-m}(C)$ (e.g. L non-special and very ample), $M \in W^2_m(C)$, M spanned, $h^0(C,M) = 3$ and whose associated linear system is birational onto its image. The generality of C gives $A \neq \emptyset$ and $\dim(A) = e + g$. We have $h^1(C,E) = h^1(C,M) = g + 2 - m = a$. The morphism associated to $H^0(C,E)$ maps $\mathbf{P}(E)$ birationally onto its image in $\mathbf{P}^{d+1+a-2g}$. The Grassmannian of all $(d + 2 - 2g)$-dimensional linear subspaces of $\mathbb{K}^{(d+2+a-2g)}$ has dimension $a(d + 2 - 2g)$. Since $\dim(M(C;2,d)) = 4g - 3$, projecting into $\mathbf{P}^{d+1-2g}$ the contruction gives an algebraic subset of $H[d,C]$ not lying in the fiber over C of the moduli map $\mathcal{H}_{d,g}\mathcal{M}_g$ (the non-special component) if $e + a(d+2-2g) \geq 3g-3$, i.e if $d \geq 2g + 7$ and $e = 0, 1$ or if $d \geq 2g + 8$ and $e = 2$. Moving C in $\mathcal{M}_g$ we get corresponding algebraic subsets of $H'[d,g]$ different from $\mathcal{H}_{d,g}$ if $d \geq 2g + 7$ and $e = 0, 1$ or if $d \geq 2g + 8$ and $e = 2$. If $e = 0$, then A has several irreducible components, but varying C in $\mathcal{M}_g$ we get a unique irreducible subset of $H'[d,g]$ (see [2]).

Example 2. Fix integers $g \geq 2$ and $d \geq 2g + 2$ and set $a := \lfloor g/2 \rfloor$ and $e := g - 2a$. Hence $0 \leq e \leq 1$. Set $m := a + 1 + e$. Notice that $\rho(g,1,m) = e$. Let C be a general curve with genus g. Let A be te set of all decomposable vector bundles on C $E = L \oplus M$ with L sufficiently general in $\mathrm{Pic}^{d-m}(C)$ (e.g. L non-special and very ample), $M \in W^1_m(C)$, M spanned and $h^0(C,M) = 2$. The generality of C gives $A \neq \emptyset$ and $\dim(A) = e + g$. We have $h^1(C,E) = h^1(C,M) = g + 1 - m = a$. The morphism associated to $H^0(C,E)$ maps $\mathbf{P}(E)$ birationally onto its image in $\mathbf{P}^{d+1+a-2g}$. The Grassmannian of all $(d+2-2g)$-dimensional linear subspaces $\mathbb{K}^{(d+2+a-2g)}$ has dimension $a(d + 2 - 2g)$. Since $\dim(M(C;2,d)) = 4g - 3$, projecting into $\mathbf{P}^{d+1-2g}$ the contruction gives an algebraic subset of $H[d,C]$ not lying in the fiber over C of the moduli map $\mathcal{H}_{d,g}\mathcal{M}_g$ (the non-special component) if $e + g + a(d + 2 - 2g) \geq 4g - 3$. Hence it is sufficient to assume $d \geq 2g + 5$.

Example 3. In the set-up of [1], Example 5.12, take all non-trivial extensions of M by L. In this way if $g - 2 > d - 2m$, we get a larger dimensional family: we add $g - 2 - d + 2m$ to the dimension of the family described in [1], Example 5.12. More precisely, set $l := \lfloor g/4 \rfloor$, $\epsilon := g - 4l$ amd $m := 3 + g - l$. Hence $0 \leq \epsilon \leq 3$ and $\rho(g,3,m) = \epsilon$. Let C be a general genus g curve. Varying C we get a component of $H[d,g]$ different from $\mathcal{H}_{d,g}$ if $g + \epsilon + \max\{0, g - 2 + d - 2m\} + l(d + 2 - 2g) \geq 4g + 3$. In this way we cover the case $d = 2g + 10$ and $\epsilon = 2, 3$ if $l \geq \epsilon + 2$.

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