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SOME U–STATISTICS IN GOODNESS–OF–FIT TESTS DERIVED FROM CHARACTERIZATIONS VIA RECORD VALUES

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Abstract: Using characterizations of continuous distributions via expected values of two functions of record values and U–statistics we construct tests of fit for continuous distributions.

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1. Introduction and Preliminaries

Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with cdf F and pdf f. For a fixed integer $k \geq 1$ we define the sequence $U_k(1), U_k(2), \ldots$ of k-th upper record times of $\{X_n,\, n \geq 1\}$ as follows:

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$U_k(1) = 1,$
$U_k(n+1) = \min\{j > U_k(n) : X_{j:j+k-1} > X_{U_k(n):U_k(n)+k-1}\}, \quad n \geq 1.$

Write

$$Y_n^{(k)} = X_{U_k(n):U_k(n)+k-1}, \qquad n \geq 1.$$

The sequence $\{Y_n^{(k)},\ n \geq 1\}$ is called the sequence of k-th (upper) record values of the above sequence. For convenience we also take $Y_0^{(k)} = 0$ and note that

$$Y_1^{(k)} = X_{1:k} = \min(X_1, \ldots, X_k)$$

(Dziubdziela and Kopociński [4]).

The k-th lower record times $L_k(n)$, $n \geq 1$, are defined as

$L_k(1) = 1,$
$L_k(n+1) = \min\{j > L_k(n) : X_{k:L_k(n)+k-1} > X_{k:j+k-1}\}, \quad n \geq 1,$

and the k-th lower record values as

$$Z_n^{(k)} = X_{k:L_k(n)+k-1}, \quad n \geq 1.$$

Note that $Z_1^{(k)} = X_{k:k} = \max(X_1, \ldots, X_k)$ (Szynal and Pawlas [21]).

The marginal and joint cdf and pdf of the k-th upper and lower record values are given in Dziubdziela and Kopociński [4], Nevzorov [20] and Pawlas and Szynal [21].

Characterizations of continuous distributions via expected values of two functions of record values were first established in Too and Lin [27] and Lin [8]. Some extensions were discussed in Grudzień and Szynal [5] and Malinowska et al [9]. We used them to construct goodness–of–fit tests for continuous distributions (cf. Morris and Szynal [13]–[19]). Goodness–of–fit tests using U–statistics and record values with $k = 2$ were presented in Morris and Szynal [13], [18], [19] and Morris et al [12]. In this paper we extend those results to the case when $k \geq 3$.

2. Characterization Conditions Involving k–th Record Values

We use characterization conditions as they were presented in Malinowska et al [9]. The presentation of those conditions in forms convenient to construct goodness–of–fit tests is contained in Morris and Szynal [16]. Here we recall some special cases.

Theorem 1. *For a given real number $r > -1$, $X \sim F$ iff*

$$\begin{cases} E\left(\log^r \frac{1}{1-F\left(Y_2^{(k)}\right)}\right) = \frac{\Gamma(r+2)}{k^r} \\ E\left(\log^{r+1} \frac{1}{1-F\left(Y_1^{(k)}\right)}\right) = \frac{\Gamma(r+2)}{k^{r+1}} \end{cases} \tag{2.1}$$

or

$$\begin{cases} E\left(\log^r \frac{1}{F\left(Z_2^{(k)}\right)}\right) = \frac{\Gamma(r+2)}{k^r} \\ E\left(\log^{r+1} \frac{1}{F\left(Z_1^{(k)}\right)}\right) = \frac{\Gamma(r+2)}{k^{r+1}}. \end{cases} \tag{2.1$'$}$$

We use the following implications.

Corollary 1. *If $X \sim F$ and F is continuous then*

$$\begin{cases} E\left(\log^r \frac{1}{1-F(X_{1:k})}\right) = \frac{\Gamma(r+1)}{k^r} \\ E\left(\log^{r+1} \frac{1}{1-F(X_{1:k})}\right) = \frac{\Gamma(r+2)}{k^{r+1}} \end{cases} \tag{2.2}$$

$$\begin{cases} E\left(\log^r \frac{1}{F(X_{k:k})}\right) = \frac{\Gamma(r+1)}{k^r} \\ E\left(\log^{r+1} \frac{1}{F(X_{k:k})}\right) = \frac{\Gamma(r+2)}{k^{r+1}} \end{cases} \tag{2.2$'$}$$

(cf. Morris and Szynal [16]).

3. Tests of $H : X \sim F$ when Parameters are Known

We consider the hypothesis $H : X \sim F$ when F is completely specified. From the sample $X_1, \ldots, X_n$ we construct tests of H based on (2.2) and (2.2$'$) for given $k \leq n$. To this end we estimate the expectations in (2.2) and (2.2$'$) by the U–statistics

$$\overline{V}_{n1}^{(r,k)} = \frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \log^r \frac{1}{1-F(X_{i:n})}$$

$$\overline{V}_{n2}^{(r,k)} = \frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \log^{r+1} \frac{1}{1-F(X_{i:n})}$$

and

$$\overline{V}_{n1}^{*(r,k)} = \frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \log^r \frac{1}{F(X_{n-i+1:n})}$$

$$\overline{V}_{n2}^{*(r,k)} = \frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \log^{r+1} \frac{1}{F(X_{n-i+1:n})}.$$

We first construct tests of $H_1 : X \sim \text{Exp}(1)$, with $F(x) = 1 - e^{-x}$, $x > 0$. Then under H_1, (2.2) and (2.2′) appear as

$$E\left[X_{1:k}^r\right] = \frac{\Gamma(r+1)}{k^r}, \qquad E\left[X_{1:k}^{r+1}\right] = \frac{\Gamma(r+2)}{k^{r+1}} \tag{3.1}$$

and

$$E\left[-\log\left(1 - e^{-X_{k:k}}\right)\right]^r = \frac{\Gamma(r+1)}{k^r}$$

$$E\left[-\log\left(1 - e^{-X_{k:k}}\right)\right]^{r+1} = \frac{\Gamma(r+2)}{k^{r+1}}.$$

Then the above estimates are

$$\overline{V}_{n1}^{(r,k)} = \frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} X_{i:n}^r, \qquad \overline{V}_{n2}^{(r,k)} = \frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} X_{i:n}^{r+1}$$

and

$$\overline{V}_{n1}^{*(r,k)} = \frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \left[-\log\left(1 - e^{-X_{n-i+1:n}}\right)\right]^r$$

$$\overline{V}_{n2}^{*(r,k)} = \frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \left[-\log\left(1 - e^{-X_{n-i+1:n}}\right)\right]^{r+1}.$$

Write

$$\overline{\boldsymbol{V}}_n^{(r,k)} = \begin{bmatrix} \overline{V}_{n1}^{(r,k)} \\ \overline{V}_{n2}^{(r,k)} \end{bmatrix}, \qquad \overline{\boldsymbol{V}}_n^{*(r,k)} = \begin{bmatrix} \overline{V}_{n1}^{*(r,k)} \\ \overline{V}_{n2}^{*(r,k)} \end{bmatrix}.$$

Then

$$\boldsymbol{\mu}_0 := \boldsymbol{\mu}_n^{(r,k)} = E\left[\overline{\boldsymbol{V}}_n^{(r,k)}\right] = \frac{\Gamma(r+1)}{k^{r+1}} \begin{bmatrix} k \\ r+1 \end{bmatrix}$$

$$\Sigma_n^{(r,k)} = \begin{bmatrix} \text{Var}\left(\overline{V}_{n1}^{(r,k)}\right) & \text{Cov}\left(\overline{V}_{n1}^{(r,k)}, \overline{V}_{n2}^{(r,k)}\right) \\ \text{Cov}\left(\overline{V}_{n1}^{(r,k)}, \overline{V}_{n2}^{(r,k)}\right) & \text{Var}\left(\overline{V}_{n2}^{(r,k)}\right) \end{bmatrix} := \begin{bmatrix} a_n^{(r,k)} & b_n^{(r,k)} \\ b_n^{(r,k)} & c_n^{(r,k)} \end{bmatrix}$$

where for $r > -1/2$ (see Section 6.2 in Appendix)

$$a_n^{(r,k)} := \mathrm{Var}\left(\overline{V}_{n1}^{(r,k)}\right)$$
$$= \frac{1}{\binom{n}{k}} \sum_{j=1}^{k-1} \binom{k}{j}\binom{n-k}{k-j} \left[2\frac{\Gamma(2r+2)}{k^r(k-j)^r} B_{\frac{k-j}{2k-j}}(r+1,r+1)\right.$$
$$\left. + j\frac{\Gamma(2r+1)}{(2k-j)^{2r+1}} - \frac{\Gamma^2(r+1)}{k^{2r}}\right] + \frac{1}{\binom{n}{k}}\left[\frac{\Gamma(2r+1)-\Gamma^2(r+1)}{k^{2r}}\right]$$
$$b_n^{(r,k)} := \mathrm{Cov}\left(\overline{V}_{n1}^{(r,k)}, \overline{V}_{n2}^{(r,k)}\right)$$
$$= \frac{1}{\binom{n}{k}} \sum_{j=1}^{k-1} \binom{k}{j}\binom{n-k}{k-j} \left[2(2k-j)\frac{\Gamma(2r+2)+\Gamma(2r+3)}{k^{r+1}(k-j)^{r+1}} B_{\frac{k-j}{2k-j}}(r+2,r+2)\right.$$
$$\left. + j\frac{\Gamma(2r+2)}{(2k-j)^{2r+2}} - \frac{\Gamma(r+1)\Gamma(r+2)}{k^{2r+1}}\right] + \frac{1}{\binom{n}{k}}\left[\frac{\Gamma(2r+2)-\Gamma(r+1)\Gamma(r+2)}{k^{2r+1}}\right]$$
$$c_n^{(r,k)} := \mathrm{Var}\left(\overline{V}_{n2}^{(r,k)}\right)$$
$$= \frac{1}{\binom{n}{k}} \sum_{j=1}^{k-1} \binom{k}{j}\binom{n-k}{k-j} \left[2\frac{\Gamma(2r+4)}{k^{r+1}(k-j)^{r+1}} B_{\frac{k-j}{2k-j}}(r+2,r+2)\right.$$
$$\left. + j\frac{\Gamma(2r+3)}{(2k-j)^{2r+3}} - \frac{\Gamma^2(r+2)}{k^{2r+2}}\right] + \frac{1}{\binom{n}{k}}\left[\frac{\Gamma(2r+3)-\Gamma^2(r+2)}{k^{2r+2}}\right]$$

where

$$B_x(\alpha,\beta) = \int_0^x t^{\alpha-1}(1-t)^{\beta-1}dt, \quad 0 < x < 1;\ \alpha,\beta > 0,$$

denotes the incomplete beta function.

Note that for $k > 1$

$$\Sigma^{(r,k)} := \lim_{n\to\infty} n\Sigma_n^{(r,k)} = \begin{bmatrix} \left(\sigma_1^{(r,k)}\right)^2 & \sigma_{12}^{(r,k)} \\ \sigma_{12}^{(r,k)} & \left(\sigma_2^{(r,k)}\right)^2 \end{bmatrix}$$

where

$$\left(\sigma_1^{(r,k)}\right)^2 = k^2\left[2\frac{\Gamma(2r+2)}{k^r(k-1)^r} B_{\frac{k-1}{2k-1}}(r+1,r+1) + \frac{\Gamma(2r+1)}{(2k-1)^{2r+1}} - \frac{\Gamma^2(r+1)}{k^{2r}}\right]$$

$$\sigma_{12}^{(r,k)} = k^2\left[2(2k-1)\frac{\Gamma(2r+2)+\Gamma(2r+3)}{k^{r+1}(k-1)^{r+1}} B_{\frac{k-1}{2k-1}}(r+2,r+2)\right.$$
$$\left. + \frac{\Gamma(2r+2)}{(2k-1)^{2r+2}} - \frac{\Gamma(r+1)\Gamma(r+2)}{k^{2r+1}}\right]$$

$$\left(\sigma_2^{(r,k)}\right)^2$$
$$= k^2\left[2\frac{\Gamma(2r+4)}{k^{r+1}(k-1)^{r+1}}B_{\frac{k-1}{2k-1}}(r+2,r+2)+\frac{\Gamma(2r+3)}{(2k-1)^{2r+3}}-\frac{\Gamma^2(r+2)}{k^{2r+2}}\right].$$

For $k=1$ we have

$$\Sigma^{(r,1)} := n\Sigma_n^{(r,1)} = \begin{bmatrix} \left(\sigma_1^{(r,1)}\right)^2 & \sigma_{12}^{(r,1)} \\ \sigma_{12}^{(r,1)} & \left(\sigma_2^{(r,1)}\right)^2 \end{bmatrix}$$

with

$$\left(\sigma_1^{(r,1)}\right)^2 = \Gamma(2r+1)-\Gamma^2(r+1), \qquad \left(\sigma_2^{(r,1)}\right)^2 = \Gamma(2r+3)-\Gamma^2(r+2)$$
$$\sigma_{12}^{(r,1)} = \Gamma(2r+2)-\Gamma(r+1)\Gamma(r+2).$$

Hence tests of H_1 based on the implication (3.1) can be obtained from

$$T_n^{(r,k)} = n\left(\overline{\boldsymbol{V}}_n^{(r,k)}-\boldsymbol{\mu}_0\right)'\left(\Sigma_n^{(r,k)}\right)^{-1}\left(\overline{\boldsymbol{V}}_n^{(r,k)}-\boldsymbol{\mu}_0\right)\xrightarrow{D}\chi^2(2). \tag{3.2}$$

If X has a continuous distribution with specified distribution function F then tests of H are obtained from $T_n^{(r,k)}$ in (3.2) by replacing the sample $X_1,\dots,X_n$ by $Y_1,\dots,Y_n$ with $Y=h(X)=\log\frac{1}{1-F(X)}$.

Since h is an increasing function then, referring to the definition of $\overline{V}_{n1}^{(r,k)}$, in this case $Y'_{1:k}=\min\left(Y_{i_1},Y_{i_2},\dots,Y_{i_k}\right)=h(X'_{1:k})$, where $X'_{i:k}=\min(X_{i_1},\dots,X_{i_k})$, $1\le i_1<i_2<\ldots<i_k\le n$, so that

$$\overline{V}_{n1}^{(r,k)} = \frac{1}{\binom{n}{k}}\sum{}'\log^r\frac{1}{1-F(X'_{1:k})} = \frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\log^r\frac{1}{1-F(X_{i:k})}$$

where $\sum'$ denotes summation over all $1\le i_1<i_2<\ldots<i_k\le n$.

The test statistic in (3.2) is thus

$$T_n^{(r,k)} = n\left(\overline{\boldsymbol{V}}_n^{(r,k)}-\boldsymbol{\mu}_0\right)'\left(\Sigma_n^{(r,k)}\right)^{-1}\left(\overline{\boldsymbol{V}}_n^{(r,k)}-\boldsymbol{\mu}_0\right)\xrightarrow{D}\chi^2(2)$$

where

$$\overline{V}_n^{(r,k)} = \begin{bmatrix}\overline{V}_{n1}^{(r,k)}\\ \overline{V}_{n2}^{(r,k)}\end{bmatrix} = \frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\begin{bmatrix}\log^r\frac{1}{1-F(X_{i:n})}\\ \log^{r+1}\frac{1}{1-F(X_{i:n})}\end{bmatrix}.$$

On the other hand $h_1(X)=\log\frac{1}{F(X)}$ is decreasing, so that $Y'_{1:k}=\min(Y_{i_1},Y_{i_2},\dots,Y_{i_k})=h_1\left(X'_{k:k}\right)$ where $X'_{k:k}=\max\left(X_{i_1},X_{i_2},\dots,X_{i_k}\right)$, $1\le i_1<i_2<\ldots<$

$i_k \leq n$. Then corresponding to $\overline{V}_{n1}^{(r,k)}$ we have the dual quantity

$$\overline{V}_{n1}^{*(r,k)} = \frac{1}{\binom{n}{k}} \Sigma' \log^r \frac{1}{F(X'_{k:k})} = \frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \log^r \frac{1}{F(X_{n-i+1:n})},$$

where Σ' denotes summation over all $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ and the dual test statistic is

$$T_n^{*(r,k)} = n \left(\overline{\boldsymbol{V}}_n^{*(r,k)} - \boldsymbol{\mu}_0\right)' \left(\Sigma_n^{(r,k)}\right)^{-1} \left(\overline{\boldsymbol{V}}_n^{*(r,k)} - \boldsymbol{\mu}_0\right) \xrightarrow{D} \chi^2(2) \tag{3.3}$$

where

$$\overline{\boldsymbol{V}}_n^{*(r,k)} = \begin{bmatrix} \overline{V}_{n1}^{*(r,k)} \\ \overline{V}_{n2}^{*(r,k)} \end{bmatrix} = \frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \begin{bmatrix} \log^r \frac{1}{F(X_{n-i+1:n})} \\ \log^{r+1} \frac{1}{F(X_{n-i+1:n})} \end{bmatrix}.$$

Taking into account that

$$\left(\Sigma_n^{(r,k)}\right)^{-1} = \frac{1}{\Delta_n^{(r,k)}} \begin{bmatrix} c_n^{(r,k)} & -b_n^{(r,k)} \\ -b_n^{(r,k)} & a_n^{(r,k)} \end{bmatrix},$$

where

$$\Delta_n^{(r,k)} = det\left(\Sigma_n^{(r,k)}\right),$$

we can write $T_n^{(r,k)}$ and $T_n^{*(r,k)}$ in the following extended forms

$$\begin{aligned} T_n^{(r,k)} = \frac{1}{\Delta_n^{(r,k)}} \Bigg[& c_n^{(r,k)} \left(\overline{V}_{n1}^{(r,k)} - \frac{\Gamma(r+1)}{k^r}\right)^2 \\ & -2b_n^{(r,k)} \left(\overline{V}_{n1}^{(r,k)} - \frac{\Gamma(r+1)}{k^r}\right) \left(\overline{V}_{n2}^{(r,k)} - \frac{\Gamma(r+2)}{k^{r+1}}\right) \\ & +a_n^{(r,k)} \left(\overline{V}_{n2}^{(r,k)} - \frac{\Gamma(r+2)}{k^{r+1}}\right)^2 \Bigg] \\ T_n^{*(r,k)} = \frac{1}{\Delta_n^{(r,k)}} \Bigg[& c_n^{(r,k)} \left(\overline{V}_{n1}^{*(r,k)} - \frac{\Gamma(r+1)}{k^r}\right)^2 \\ & -2b_n^{(r,k)} \left(\overline{V}_{n1}^{*(r,k)} - \frac{\Gamma(r+1)}{k^r}\right) \left(\overline{V}_{n2}^{*(r,k)} - \frac{\Gamma(r+2)}{k^{r+1}}\right) \\ & +a_n^{(r,k)} \left(\overline{V}_{n2}^{*(r,k)} - \frac{\Gamma(r+2)}{k^{r+1}}\right)^2 \Bigg]. \end{aligned}$$

We also consider tests using the statistics obtained from the following partitions:

$$\begin{aligned} T_n^{(r,k)} &= T_{n;c_1}^{(r,k)} + T_{n;c_2}^{(r,k)} = T_{n;c_3}^{(r,k)} + T_{n;c_4}^{(r,k)} \\ T_n^{*(r,k)} &= T_{n;c_1}^{*(r,k)} + T_{n;c_2}^{*(r,k)} = T_{n;c_3}^{*(r,k)} + T_{n;c_4}^{*(r,k)} \end{aligned}$$

where

$$T_{n;c_1}^{(r,k)} = \frac{1}{a_n^{(r,k)}} \left(\overline{V}_{n1}^{(r,k)} - \frac{\Gamma(r+1)}{k^r} \right)^2$$

$$T_{n;c_2}^{(r,k)} = \frac{1}{\Delta_n^{(r,k)} a_n^{(r,k)}} \cdot \left[a_n^{(r,k)} \overline{V}_{n2}^{(r,k)} - b_n^{(r,k)} \overline{V}_{n1}^{(r,k)} - \frac{\Gamma(r+1)}{k^{r+1}} \left((r+1) a_n^{(r,k)} - k b_n^{(r,k)} \right) \right]^2$$

$$T_{n;c_3}^{(r,k)} = \frac{1}{c_n^{(r,k)}} \left(\overline{V}_{n2}^{(r,k)} - \frac{\Gamma(r+2)}{k^{r+1}} \right)^2$$

$$T_{n;c_4}^{(r,k)} = \frac{1}{\Delta_n^{(r,k)} c_n^{(r,k)}} \cdot \left[c_n^{(r,k)} \overline{V}_{n1}^{(r,k)} - b_n^{(r,k)} \overline{V}_{n2}^{(r,k)} - \frac{\Gamma(r+1)}{k^{r+1}} \left(k c_n^{(r,k)} - (r+1) b_n^{(r,k)} \right) \right]^2$$

and $T_{n;c_1}^{*(r,k)}$, $T_{n;c_2}^{*(r,k)}$, $T_{n;c_3}^{*(r,k)}$, $T_{n;c_4}^{*(r,k)}$ are defined similarly.

4. Tests of $H : X \sim F$ when There are Unknown Parameters

We now study the hypothesis H that F has the form $F(x;\boldsymbol{\lambda})$ where $\boldsymbol{\lambda}(p \times 1)$ are unknown identifiable parameters with true value in the parameter space Λ, and we denote the pdf by $f(x;\boldsymbol{\lambda})$.

Here we define

$$\hat{\boldsymbol{V}}_n^{(r,k)} = \begin{bmatrix} \hat{V}_{n1}^{(r,k)} \\ \hat{V}_{n2}^{(r,k)} \end{bmatrix}, \qquad \hat{\boldsymbol{V}}_n^{*(r,k)} = \begin{bmatrix} \hat{V}_{n1}^{*(r,k)} \\ \hat{V}_{n2}^{*(r,k)} \end{bmatrix}$$

where

$$\hat{V}_{n1}^{(r,k)} := \overline{V}_{n1}^{(r,k)}\left(\hat{\boldsymbol{\lambda}}_n\right) = \frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \log^r \frac{1}{1 - F\left(X_{i:n};\hat{\boldsymbol{\lambda}}_n\right)}$$

$$\hat{V}_{n2}^{(r,k)} := \overline{V}_{n2}^{(r,k)}\left(\hat{\boldsymbol{\lambda}}_n\right) = \frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \log^{r+1} \frac{1}{1 - F\left(X_{i:n};\hat{\boldsymbol{\lambda}}_n\right)}$$

$$\hat{V}_{n1}^{*(r,k)} := \overline{V}_{n1}^{*(r,k)}\left(\hat{\boldsymbol{\lambda}}_n\right) = \frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \log^r \frac{1}{F\left(X_{n-i+1:n};\hat{\boldsymbol{\lambda}}_n\right)}$$

$$\hat{V}^{*(r,k)}_{n2} := \overline{V}^{*(r,k)}_{n2}\left(\hat{\boldsymbol{\lambda}}_n\right) = \frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \log^{r+1} \frac{1}{F\left(X_{n-i+1:n}; \hat{\boldsymbol{\lambda}}_n\right)}$$

where $\hat{\boldsymbol{\lambda}}_n$ is the MLE of $\boldsymbol{\lambda}$. We then construct tests of H as in Section 3, but using $F(x;\hat{\boldsymbol{\lambda}}_n)$ instead of $F(x)$. And we further assume regular estimation, for which

$$\sqrt{n}\left(\hat{\boldsymbol{\lambda}}_n - \boldsymbol{\lambda}\right) \xrightarrow{D} N\left(\mathbf{0}, I^{-1}(\boldsymbol{\lambda})\right)$$

where $I(\boldsymbol{\lambda})$ is the expected information matrix based on a single observation X.

Following the procedure presented for this case where a theorem of Pierce [23] is used (cf. Morris and Szynal [13]) the variance matrix of $\overline{\boldsymbol{V}}_n\left(\hat{\boldsymbol{\lambda}}_n\right)$ for large n is given approximately by

$$\begin{aligned}\Sigma^{(r,k)}_{n1} &= \Sigma^{(r,k)}_{n} - \frac{1}{n}\boldsymbol{B}^{(r,k)}_n \mathcal{I}^{-1}\left(\boldsymbol{B}^{(r,k)}_n\right)' = \Sigma^{(r,k)}_n - \boldsymbol{K}^{(r,k)}_n \\ &:= \begin{bmatrix} a^{(r,k)}_{n1} & b^{(r,k)}_{n1} \\ b^{(r,k)}_{n1} & c^{(r,k)}_{n1}\end{bmatrix} = \begin{bmatrix} a^{(r,k)}_{n} & b^{(r,k)}_{n} \\ b^{(r,k)}_{n} & c^{(r,k)}_{n}\end{bmatrix} - \begin{bmatrix} s^{(r,k)}_{n} & t^{(r,k)}_{n} \\ t^{(r,k)}_{n} & u^{(r,k)}_{n}\end{bmatrix}\end{aligned}$$

with

$$\boldsymbol{B}^{(r,k)}_n(2\times p) = E\left[\frac{\partial \overline{\boldsymbol{V}}^{(r,k)}_n(\boldsymbol{\lambda})}{\partial \boldsymbol{\lambda}}\right]$$

and

$$\boldsymbol{K}^{(r,k)}_n = \frac{1}{n}\boldsymbol{B}^{(r,k)}_n \mathcal{I}^{-1}\left(\boldsymbol{B}^{(r,k)}_n\right)'$$

where

$$E\frac{\partial \overline{V}^{(r,k)}_{n1}(\boldsymbol{\lambda})}{\partial \lambda_j} = \frac{1}{\binom{n}{k}}\Sigma' E\left[\frac{\partial}{\partial \lambda_j}\log^r \frac{1}{1-F(X'_{i:k};\boldsymbol{\lambda})}\right]$$

which after letting $Z = X_{1:k}$ has the form

$$\begin{aligned}E\left[\frac{\partial \overline{V}^{(r,k)}_{n1}(\boldsymbol{\lambda})}{\partial \lambda_j}\right] &= E\left[\frac{\partial}{\partial \lambda_j}\log^r \frac{1}{1-F(Z;\boldsymbol{\lambda})}\right] \\ &= rE\left[\left(\log^{r-1}\frac{1}{1-F(Z;\boldsymbol{\lambda})}\right)\frac{1}{1-F(Z,\boldsymbol{\lambda})}\frac{\partial F(Z;\boldsymbol{\lambda})}{\partial \lambda_j}\right].\end{aligned}$$

Now Z has pdf $k(1-F)^{k-1}f$, whence

$$\begin{aligned}&E\left[\frac{\partial \overline{V}^{(r,k)}_{n1}(\boldsymbol{\lambda})}{\partial \lambda_j}\right] \\ &\qquad = kr\int (1-F(x;\boldsymbol{\lambda}))^{k-2}\log^{r-1}\frac{1}{1-F(x;\boldsymbol{\lambda})}\frac{\partial F(x;\boldsymbol{\lambda})}{\partial \lambda_j} f(x;\boldsymbol{\lambda})dx\end{aligned}$$

$$= krE\left[(1-F(X;\boldsymbol{\lambda}))^{k-2}\log^{r-1}\frac{1}{1-F(X;\boldsymbol{\lambda})}\frac{\partial F(X;\boldsymbol{\lambda})}{\partial\lambda_j}\right].$$

Thus

$$\begin{aligned}\boldsymbol{B}_n^{(r,k)} &:= \boldsymbol{B}^{(r,k)}\\ &= k\begin{bmatrix} rb^{(r,k)}(\lambda_1) & rb^{(r,k)}(\lambda_2) & \dots & rb^{(r,k)}(\lambda_p)\\ (r+1)b^{(r+1,k)}(\lambda_1) & (r+1)b^{(r+1,k)}(\lambda_2) & \dots & (r+1)b^{(r+1,k)}(\lambda_p)\end{bmatrix}\end{aligned}$$

where

$$b^{(r,k)}(\lambda_j) = E\left[(1-F(X;\boldsymbol{\lambda}))^{k-2}\log^{r-1}\frac{1}{1-F(X;\boldsymbol{\lambda})}\frac{\partial F(X;\boldsymbol{\lambda})}{\partial\lambda_j}\right],$$

$j=1,\dots,p$. Thus corresponding to $T_n^{(r,k)}$ in (3.2) we have

$$\hat{T}_n^{(r,k)} = T_n^{(r,k)}\left(\hat{\boldsymbol{\lambda}}_n\right) = n\left(\hat{\boldsymbol{V}}_n^{(r,k)}-\boldsymbol{\mu}_0\right)'\left(\Sigma_{n1}^{(r,k)}\right)^{-1}\left(\hat{\boldsymbol{V}}_n^{(r,k)}-\boldsymbol{\mu}_0\right)\xrightarrow{D}\chi^2(2)$$

under H.

The discussion of the dual test corresponding to $T_n^{*(r,k)}$ in (3.3) is similar. We have

$$\hat{T}_n^{*(r,k)} = n\left(\hat{\boldsymbol{V}}_n^{*(r,k)}-\boldsymbol{\mu}_0\right)'\left(\Sigma_{n1}^{*(r,k)}\right)^{-1}\left(\hat{\boldsymbol{V}}_n^{*(r,k)}-\boldsymbol{\mu}_0\right)\xrightarrow{D}\chi^2(2)$$

under H with

$$\begin{aligned}\Sigma_{n1}^{*(r,k)} &= \Sigma_n^{(r,k)} - \frac{1}{n}\boldsymbol{B}_n^{*(r,k)}\mathcal{I}^{-1}\left(\boldsymbol{B}_n^{*(r,k)}\right)' = \Sigma_n^{(r,k)} - \boldsymbol{K}_n^{*(r,k)}\\ &= \begin{bmatrix} a_n^{(r,k)} & b_n^{(r,k)}\\ b_n^{(r,k)} & c_n^{(r,k)}\end{bmatrix} - \begin{bmatrix} s_n^{*(r,k)} & t_n^{*(r,k)}\\ t_n^{*(r,k)} & u_n^{*(r,k)}\end{bmatrix}\end{aligned}$$

where

$$\begin{aligned}\boldsymbol{B}_n^{*(r,k)} &:= \boldsymbol{B}^{*(r,k)}\\ &= -\begin{bmatrix} rb^{*(r,k)}(\lambda_1) & rb^{*(r,k)}(\lambda_2) & \dots & rb^{*(r,k)}(\lambda_p)\\ (r+1)b^{*(r+1,k)}(\lambda_1) & (r+1)b^{*(r+1,k)}(\lambda_2) & \dots & (r+1)b^{*(r+1,k)}(\lambda_p)\end{bmatrix}\end{aligned}$$

and

$$b^{*(r,k)}(\lambda_j) = E\left[F^{k-2}(X;\boldsymbol{\lambda})\log^{r-1}\frac{1}{F(X;\boldsymbol{\lambda})}\frac{\partial F(x;\boldsymbol{\lambda})}{\partial\lambda_j}\right],\quad j=1,\dots,p,$$

$$\boldsymbol{K}_n^{*(r,k)} = \frac{1}{n}\boldsymbol{B}^{*(r,k)}\mathcal{I}^{-1}\left(\boldsymbol{B}^{*(r,k)}\right)' = \begin{bmatrix} s_n^{*(r,k)} & t_n^{*(r,k)}\\ t_n^{*(r,k)} & u_n^{*(r,k)}\end{bmatrix}.$$

Finally, we note that

$$\begin{aligned}b^{(r,k)}(\lambda_j) &= E\left[(1-F(X;\boldsymbol{\lambda}))^{k-2}\log^{r-1}\frac{1}{1-F(X;\boldsymbol{\lambda})}\frac{\partial F(X;\boldsymbol{\lambda})}{\partial\lambda_j}\right]\\ &= \int_0^1 (1-y)^{k-2}\log^{r-1}\frac{1}{1-y}g_j(y;\boldsymbol{\lambda})dy\end{aligned}$$

and

$$\begin{aligned} b^{*(r,k)}(\lambda_j) &= E\left[F^{k-2}(X;\boldsymbol{\lambda})\log^{r-1}\frac{1}{F(X;\boldsymbol{\lambda})}\frac{\partial F(X;\boldsymbol{\lambda})}{\partial\lambda_j}\right] \\ &= \int_0^1 y^{k-2}\log^{r-1}\frac{1}{y}g_j(y;\boldsymbol{\lambda})dy \end{aligned}$$

where

$$\frac{\partial F(x;\boldsymbol{\lambda})}{\partial\lambda_j} = g_j(y;\boldsymbol{\lambda}) \qquad \text{and} \qquad y = F(x;\boldsymbol{\lambda}).$$

In the commonly occurring cases when $\boldsymbol{\lambda} = (\alpha)$ and $\boldsymbol{\lambda} = (\alpha,\beta)$ we write

$$I^{-1} := [i_{11}] \qquad \text{and} \qquad I^{-1} := \begin{bmatrix} i_{11} & i_{12} \\ i_{12} & i_{22} \end{bmatrix}$$

and then for $\boldsymbol{\lambda} = (\alpha)$ we have

$$\begin{aligned} s_n^{(r,k)} &= \frac{r^2k^2}{n}\left(b^{(r,k)}(\alpha)\right)^2 i_{11}, \quad u_n^{(r,k)} = \frac{(r+1)^2k^2}{n}\left(b^{(r+1,k)}(\alpha)\right)^2 i_{11} \\ t_n^{(r,k)} &= \frac{r(r+1)k^2}{n}b^{(r,k)}(\alpha)b^{(r+1,k)}(\alpha)i_{11} \end{aligned}$$

with

$$b^{(r,k)}(\alpha) = E\left[(1-F(X;\alpha))^{k-2}\log^{r-1}\frac{1}{1-F(X;\alpha)}\frac{\partial F(X;\alpha)}{\partial\alpha}\right]$$

and for $\boldsymbol{\lambda} = (\alpha,\beta)$ we have

$$\begin{aligned} s_n^{(r,k)} &= \frac{r^2k^2}{n}\left[\left(b^{(r,k)}(\alpha)\right)^2 i_{11} + 2b^{(r,k)}(\alpha)b^{(r,k)}(\beta)i_{12} + \left(b^{(r,k)}(\beta)\right)^2 i_{22}\right] \\ t_n^{(r,k)} &= \frac{r(r+1)k^2}{n}\left[b^{(r,k)}(\alpha)b^{(r+1,k)}(\alpha)i_{11} + \left(b^{(r,k)}(\alpha)b^{(r+1,k)}(\beta)\right.\right. \\ &\qquad \left.\left. + b^{(r+1,k)}(\alpha)b^{(r,k)}(\beta)\right) i_{12} + b^{(r,k)}(\beta)b^{(r+1,k)}(\beta)i_{22}\right], \\ u_n^{(r,k)} &= \frac{(r+1)^2k^2}{n}\left[\left(b^{(r+1,k)}(\alpha)\right)^2 i_{11} + 2b^{(r+1,k)}(\alpha)b^{(r+1,k)}(\beta)i_{12}\right. \\ &\qquad \left. + \left(b^{(r+1,k)}(\beta)\right)^2 i_{22}\right] \end{aligned}$$

with

$$\begin{aligned} b^{(r,k)}(\alpha) &= E\left[(1-F(X;\alpha,\beta))^{k-2}\log^{r-1}\frac{1}{1-F(X;\alpha,\beta)}\frac{\partial F(X;\alpha,\beta)}{\partial\alpha}\right] \\ b^{(r,k)}(\beta) &= E\left[(1-F(X;\alpha,\beta))^{k-2}\log^{r-1}\frac{1}{1-F(X;\alpha,\beta)}\frac{\partial F(X;\alpha,\beta)}{\partial\beta}\right]. \end{aligned}$$

Similar formulae can be written for $s_n^{*(r,k)}$, $t_n^{*(r,k)}$ and $u_n^{*(r,k)}$, i.e.

$$s_n^{*(r,k)} = \frac{r^2k^2}{n}\left(b^{*(r,k)}(\alpha)\right)^2 i_{11}, \quad u_n^{*(r,k)} = \frac{(r+1)^2k^2}{n}\left(b^{*(r+1,k)}(\alpha)\right)^2 i_{11},$$

$$t_n^{*(r,k)} = \frac{r(r+1)k^2}{n} b^{*(r,k)}(\alpha) b^{*(r+1,k)}(\alpha) i_{11}$$

with

$$b_k^{*(r)}(\alpha) = E(F(X;\alpha))^{k-2}\log^{r-1}\frac{1}{F(X;\alpha)}\frac{\partial F(X;\alpha)}{\partial\alpha}$$

and

$$s_n^{*(r,k)} = \frac{r^2k^2}{n}\left[\left(b^{*(r,k)}(\alpha)\right)^2 i_{11} + 2b^{*(r,k)}(\alpha)b^{*(r,k)}(\beta)i_{12} + \left(b^{*(r,k)}(\beta)\right)^2 i_{22}\right]$$

$$t_n^{*(r,k)} = \frac{r(r+1)k^2}{n}\Big[b^{*(r,k)}(\alpha)b^{*(r+1,k)}(\alpha)i_{11} + \Big(b^{*(r,k)}(\alpha)b^{*(r+1,k)}(\beta) + b^{*(r+1,k)}(\alpha)b^{*(r,k)}(\beta)\Big) i_{12} + b^{*(r,k)}(\beta)b^{*(r+1,k)}(\beta)i_{22}\Big]$$

$$u_n^{*(r,k)} = \frac{(r+1)^2k^2}{n}\left[\left(b^{*(r+1,k)}(\alpha)\right)^2 i_{11} + 2b^{*(r+1,k)}(\alpha)b^{*(r+1,k)}(\beta)i_{12} + \left(b^{*(r+1,k)}(\beta)\right)^2 i_{22}\right]$$

with

$$b^{*(r,k)}(\alpha) = E\left[F^{k-2}(X;\alpha,\beta)\log^{r-1}\frac{1}{F(X;\alpha,\beta)}\frac{\partial F(X;\alpha,\beta)}{\partial\alpha}\right]$$

$$b^{*(r,k)}(\beta) = E\left[F^{k-2}(X;\alpha,\beta)\log^{r-1}\frac{1}{F(X;\alpha,\beta)}\frac{\partial F(X;\alpha,\beta)}{\partial\beta}\right].$$

Write

$$\left(\Sigma_{n1}^{(r,k)}\right)^{-1} = \frac{1}{\Delta_{n1}^{(r,k)}}\begin{bmatrix} c_{n1}^{(r,k)} & -b_{n1}^{(r,k)} \\ -b_{n1}^{(r,k)} & a_{n1}^{(r,k)}\end{bmatrix}, \qquad \Delta_{n1}^{(r,k)} = det\left(\Sigma_{n1}^{(r,k)}\right)$$

$$\left(\Sigma_{n1}^{*(r,k)}\right)^{-1} = \frac{1}{\Delta_{n1}^{*(r,k)}}\begin{bmatrix} c_{n1}^{*(r,k)} & -b_{n1}^{*(r,k)} \\ -b_{n1}^{*(r,k)} & a_{n1}^{*(r,k)}\end{bmatrix}, \qquad \Delta_{n1}^{*(r,k)} = det\left(\Sigma_{n1}^{*(r,k)}\right).$$

Then the test statistics $\hat{T}_n^{(r,k)}$ and $\hat{T}_n^{*(r,k)}$ we write in extended forms

$$\hat{T}_n^{(r,k)} = \frac{1}{\Delta_{n1}^{(r,k)}}\Bigg[c_{n1}^{(r,k)}\left(\hat{V}_{n1}^{(r,k)} - \frac{\Gamma(r+1)}{k^r}\right)^2 - 2b_{n1}^{(r,k)}\left(\hat{V}_{n1}^{(r,k)} - \frac{\Gamma(r+1)}{k^r}\right)\left(\hat{V}_{n2}^{(r,k)} - \frac{\Gamma(r+2)}{k^{r+1}}\right)$$

$$+a_{n1}^{(r,k)}\left(\hat{V}_{n2}^{(r,k)}-\frac{\Gamma(r+2)}{k^{r+1}}\right)^2\Bigg]$$

$$\hat{T}_n^{*(r,k)}=\frac{1}{\Delta_{n1}^{*(r,k)}}\Bigg[c_{n1}^{*(r,k)}\left(\hat{V}_{n1}^{*(r,k)}-\frac{\Gamma(r+1)}{k^r}\right)^2 -2b_{n1}^{*(r,k)}\left(\hat{V}_{n1}^{*(r,k)}-\frac{\Gamma(r+1)}{k^r}\right)\left(\hat{V}_{n2}^{*(r,k)}-\frac{\Gamma(r+2)}{k^{r+1}}\right) +a_{n1}^{*(r,k)}\left(\hat{V}_{n2}^{*(r,k)}-\frac{\Gamma(r+2)}{k^{r+1}}\right)^2\Bigg]$$

and we use the partitions

$$\hat{T}_n^{(r,k)}=\hat{T}_{n;c_1}^{(r,k)}+\hat{T}_{n;c_2}^{(r,k)}=\hat{T}_{n;c_3}^{(r,k)}+\hat{T}_{n;c_4}^{(r,k)}$$
$$\hat{T}_n^{*(r,k)}=\hat{T}_{n;c_1}^{*(r,k)}+\hat{T}_{n;c_2}^{*(r,k)}=\hat{T}_{n;c_3}^{*(r,k)}+\hat{T}_{n;c_4}^{*(r,k)}$$

where

$$\hat{T}_{n;c_1}^{(r,k)}=\frac{1}{a_{n1}^{(r,k)}}\left(\hat{V}_{n1}^{(r,k)}-\frac{\Gamma(r+1)}{k^r}\right)^2$$

$$\hat{T}_{n;c_2}^{(r,k)}=\frac{1}{\Delta_{n1}^{(r,k)}a_{n1}^{(r,k)}} \cdot\left[a_{n1}^{(r,k)}\hat{V}_{n2}^{(r,k)}-b_{n1}^{(r,k)}\hat{V}_{n1}^{(r,k)}-\frac{\Gamma(r+1)}{k^{r+1}}\left((r+1)a_{n1}^{(r,k)}-kb_{n1}^{(r,k)}\right)\right]^2$$

$$\hat{T}_{n;c_3}^{(r,k)}=\frac{1}{c_{n1}^{(r,k)}}\left(\hat{V}_{n2}^{(r,k)}-\frac{\Gamma(r+2)}{k^{r+1}}\right)^2$$

$$\hat{T}_{n;c_4}^{(r,k)}=\frac{1}{\Delta_{n1}^{(r,k)}c_{n1}^{(r,k)}} \cdot\left[c_{n1}^{(r,k)}\hat{V}_{n1}^{(r,k)}-b_{n1}^{(r,k)}\hat{V}_{n2}^{(r,k)}-\frac{\Gamma(r+1)}{k^{r+1}}\left(kc_{n1}^{(r,k)}-(r+1)b_{n1}^{(r,k)}\right)\right]^2$$

and $\hat{T}_{n;c_1}^{*(r,k)}$, $\hat{T}_{n;c_2}^{*(r,k)}$, $\hat{T}_{n;c_3}^{*(r,k)}$, $\hat{T}_{n;c_4}^{*(r,k)}$ are defined similarly.

5. Special Cases

5.1. Exponential Distribution: $X \sim \mathrm{Exp}(\alpha)$

Here

$$F(x;\alpha) = 1 - e^{-\alpha x}, \quad x > 0, \quad \Lambda = \{\alpha : \alpha > 0\}$$

$$\frac{\partial F}{\partial \alpha} = x e^{-\alpha x}, \quad \mathcal{I}^{-1} = \alpha^2, \quad \hat{\alpha}_n = \frac{1}{\overline{X}_n}, \quad \overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$\begin{aligned} b^{(r,k)}(\alpha) &= E\left[(1 - F(X;\alpha))^{k-2} \log^{r-1} \frac{1}{1 - F(X;\alpha)} \frac{\partial F(X;\alpha)}{\partial \alpha}\right] \\ &= \frac{1}{\alpha}\int_0^1 (1-y)^{k-1} \log^{r-1} \frac{1}{1-y} dy \\ &= \frac{1}{\alpha}\frac{\Gamma(r+1)}{k^{r+1}} \qquad \text{for } k > 0 \text{ and } r > -1 \\ s_n^{(r,k)} &= \frac{r^2 \Gamma^2(r+1)}{n k^{2r}}, \qquad u_n^{(r,k)} = \frac{(r+1)^2 \Gamma^2(r+2)}{n k^{2r+2}} \\ t_n^{(r,k)} &= \frac{r(r+1)\Gamma(r+1)\Gamma(r+2)}{n k^{2r+1}}. \end{aligned}$$

Sometimes we use the following expressions:

$$\begin{aligned} s_n^{(r,k)} &= \frac{(\Gamma(r+2) - \Gamma(r+1))^2}{n k^{2r}}, \qquad u_n^{(r,k)} = \frac{(\Gamma(r+3) - \Gamma(r+2))^2}{n k^{2r+2}} \\ t_n^{(r,k)} &= \frac{(\Gamma(r+2) - \Gamma(r+1))(\Gamma(r+3) - \Gamma(r+2))}{n k^{2r+1}}. \end{aligned}$$

Now

$$\begin{aligned} b^{*(r,k)} &= E\left[(F(X;\alpha))^{k-2} \log^{r-1} \frac{1}{F(X;\alpha)} \frac{\partial F(X;\alpha)}{\partial \alpha}\right] \\ &= \frac{1}{\alpha}\int_0^1 y^{k-2}(1-y) \log^{r-1} \frac{1}{y} \log \frac{1}{1-y} dy \\ &= \frac{1}{\alpha}\frac{\Gamma(r+1)}{r}\left[\frac{1}{k^r} - \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+k)^r}\right], \quad k > 0,\ r > -1,\ r \neq 0 \\ &= \frac{1}{\alpha}\frac{\Gamma(r+1)}{r} A^{(r,k)}, \end{aligned}$$

with

$$A^{(r,k)} = \frac{1}{k^r} - \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+k)^r}.$$

Letting

$$I^{(r,k)} = \int_0^1 y^{k-2}(1-y)\log^{r-1}\frac{1}{y}\log\frac{1}{1-y}dy$$

we use

$$s_n^{*(r,k)} = \frac{r^2k^2}{n}\left(I^{(r,k)}\right)^2 = \frac{k^2}{n}\Gamma^2(r+1)\left(A^{(r,k)}\right)^2,$$

$$t_n^{*(r,k)} = \frac{r(r+1)k^2}{n}I^{(r,k)}\cdot I^{(r+1,k)} = \frac{k^2}{n}\Gamma(r+1)\Gamma(r+2)A^{(r,k)}A^{(r+1,k)}$$

$$u_n^{*(r,k)} = \frac{(r+1)^2k^2}{n}\left(I^{(r+1,k)}\right)^2 = \frac{k^2}{n}\Gamma^2(r+2)\left(A^{(r+1,k)}\right)^2.$$

In simulations we use $s_n^{*(r,k)}$, $t_n^{*(r,k)}$ and $u_n^{*(r,k)}$ with the integral representations.

Then

$$a_{n1}^{(r,k)} = a_n^{(r,k)} - \frac{r^2}{n}\frac{\Gamma^2(r+1)}{k^{2r}}$$

$$= \frac{1}{\binom{n}{k}}\sum_{j=1}^{k-1}\binom{k}{j}\binom{n-k}{k-j}\left[2\frac{\Gamma(2r+2)}{k^r(k-j)^r}B_{\frac{k-j}{2k-j}}(r+1,r+1)\right.$$

$$\left.+j\frac{\Gamma(2r+1)}{(2k-j)^{2r+1}} - \frac{\Gamma^2(r+1)}{k^{2r}}\right] + \frac{1}{\binom{n}{k}}\left[\frac{\Gamma(2r+1)-\Gamma^2(r+1)}{k^{2r}}\right]$$

$$- \frac{(\Gamma(r+2)-\Gamma(r+1))^2}{nk^{2r}}$$

$$b_{n1}^{(r,k)} = b_n^{(r,k)} - \frac{r(r+1)}{n}\frac{\Gamma(r+1)\Gamma(r+2)}{k^{2r+1}}$$

$$= \frac{1}{\binom{n}{k}}\sum_{j=1}^{k-1}\binom{k}{j}\binom{n-k}{k-j}\cdot\left[2(2k-j)\frac{\Gamma(2r+2)+\Gamma(2r+3)}{k^{r+1}(k-j)^{r+1}}B_{\frac{k-j}{2k-j}}(r+2,r+2)\right.$$

$$\left.+j\frac{\Gamma(2r+2)}{(2k-j)^{2r+2}} - \frac{\Gamma(r+1)\Gamma(r+2)}{k^{2r+1}}\right]$$

$$+ \frac{1}{\binom{n}{k}}\left[\frac{\Gamma(2r+2)-\Gamma(r+1)\Gamma(r+2)}{k^{2r+1}}\right]$$

$$- \frac{(\Gamma(r+2)-\Gamma(r+1))(\Gamma(r+3)-\Gamma(r+2))}{nk^{2r+1}}$$

$$c_{n1}^{(r,k)} = c_n^{(r,k)} - \frac{(r+1)^2}{n}\frac{\Gamma^2(r+2)}{k^{2r+2}}$$

$$= \frac{1}{\binom{n}{k}} \sum_{j=1}^{k-1} \binom{k}{j} \binom{n-k}{k-j} \left[2 \frac{\Gamma(2r+4)}{k^{r+1}(k-j)^{r+1}} B_{\frac{k-j}{2k-j}}(r+2,r+2) \right.$$

$$\left. + j \frac{\Gamma(2r+3)}{(2k-j)^{2r+3}} - \frac{\Gamma^2(r+2)}{k^{2r+2}} \right] + \frac{1}{\binom{n}{k}} \left[\frac{\Gamma(2r+3) - \Gamma^2(r+2)}{k^{2r+2}} \right]$$

$$- \frac{(\Gamma(r+3) - \Gamma(r+2))^2}{nk^{2r+2}}$$

$$a_{n1}^{*(r,k)} = a_n^{(r,k)} - \frac{1}{n} k^2 \Gamma^2(r+1) \left(A^{(r,k)} \right)^2$$

$$b_{n1}^{*(r,k)} = b_n^{(r,k)} - \frac{1}{n} k^2 \Gamma(r+1)\Gamma(r+2) A^{(r,k)} A^{(r+1,k)}$$

$$c_{n1}^{*(r,k)} = c_n^{(r,k)} - \frac{1}{n} k^2 \Gamma^2(r+2) \left(A^{(r,k)} \right)^2 .$$

In particular cases we have

$$a_{n1}^{(r,1)} = \frac{1}{n} \left[\Gamma(2r+1) - \Gamma^2(r+1) - (\Gamma(r+2) - \Gamma(r+1))^2 \right]$$

$$b_{n1}^{(r,1)} = \frac{1}{n} \left[\Gamma(2r+2) - \Gamma(r+1)\Gamma(r+2) \right.$$

$$\left. - (\Gamma(r+2) - \Gamma(r+1)) (\Gamma(r+3) - \Gamma(r+2)) \right]$$

$$c_{n1}^{(r,1)} = \frac{1}{n} \left[\Gamma(2r+3) - \Gamma^2(r+2) - (\Gamma(r+3) - \Gamma(r+2))^2 \right]$$

$$a_{n1}^{(r,2)} = \frac{4}{n} \left[\frac{1}{2^{r-1}} \Gamma(2r+2) B_{\frac{1}{3}}(r+1,r+1) + \frac{\Gamma(2r+1)}{3^{2r+1}} \right.$$

$$\left. - \frac{4\Gamma^2(r+1) + (\Gamma(r+2) - \Gamma(r+1))^2}{2^{2r+2}} \right]$$

$$- \frac{4}{n(n-1)} \left[\frac{1}{2^{r-1}} \Gamma(2r+2) B_{\frac{1}{3}}(r+1,r+1) \right.$$

$$\left. + \Gamma(2r+1) \left(\frac{1}{3^{2r+1}} - \frac{1}{2^{2r+1}} \right) - \frac{\Gamma^2(r+1)}{2^{2r+1}} \right]$$

$$b_{n1}^{(r,2)} = \frac{4}{n} \left[\frac{3}{2^r} (\Gamma(2r+2) + \Gamma(2r+3)) B_{\frac{1}{3}}(r+2,r+2) + \frac{\Gamma(2r+2)}{3^{2r+2}} \right.$$

$$\left. - \frac{4\Gamma(r+1)\Gamma(r+2) + (\Gamma(r+2) - \Gamma(r+1)) (\Gamma(r+3) - \Gamma(r+2))}{2^{2r+3}} \right]$$

$$- \frac{4}{n(n-1)} \left[\frac{3}{2^r} (\Gamma(2r+2) + \Gamma(2r+3)) B_{\frac{1}{3}}(r+2,r+2) \right.$$

$$+\Gamma(2r+2)\left(\frac{1}{3^{2r+2}}-\frac{1}{2^{2r+2}}\right)-\frac{\Gamma(r+1)\Gamma(r+2)}{2^{2r+2}}\Bigg]$$

$$c_{n1}^{(r,2)}=\frac{4}{n}\Bigg[\frac{1}{2^r}\Gamma(2r+4)B_{\frac{1}{3}}(r+2,r+2)+\frac{\Gamma(2r+3)}{3^{2r+3}}$$
$$-\frac{4\Gamma^2(r+2)+\left(\Gamma(r+3)-\Gamma(r+2)\right)^2}{2^{2r+4}}\Bigg]$$
$$-\frac{4}{n(n-1)}\Bigg[\frac{1}{2^r}\Gamma(2r+4)B_{\frac{1}{3}}(r+2,r+2)$$
$$+\Gamma(2r+3)\left(\frac{1}{3^{2r+3}}-\frac{1}{2^{2r+3}}\right)-\frac{\Gamma^2(r+2)}{2^{2r+3}}\Bigg]$$

$$a_{n1}^{(r,3)}=\frac{9}{n}\Bigg[\frac{1}{3^r}\Gamma(2r+2)\frac{1}{2^{r-1}}B_{\frac{2}{5}}(r+1,r+1)+\frac{\Gamma(2r+1)}{5^{2r+1}}$$
$$-\frac{9\Gamma^2(r+1)+\left(\Gamma(r+2)-\Gamma(r+1)\right)^2}{3^{2r+2}}\Bigg]$$
$$-\frac{18}{n(n-1)}\Bigg[\frac{2}{3^r}\Gamma(2r+2)$$
$$\cdot\left(\frac{1}{2^{r-1}}B_{\frac{2}{5}}(r+1,r+1)-B_{\frac{1}{4}}(r+1,r+1)\right)$$
$$+2\Gamma(2r+1)\left(\frac{1}{5^{2r+1}}-\frac{1}{4^{2r+1}}\right)-\frac{\Gamma^2(r+1)}{3^{2r}}\Bigg]$$
$$+\frac{6}{n(n-1)(n-2)}\Bigg[\frac{2}{3^{r-1}}\Gamma(2r+2)$$
$$\cdot\left(\frac{1}{2^r}B_{\frac{2}{5}}(r+1,r+1)-B_{\frac{1}{4}}(r+1,r+1)\right)$$
$$+3\Gamma(2r+1)\left(\frac{1}{5^{2r+1}}-\frac{2}{4^{2r+1}}+\frac{1}{3^{2r+1}}\right)-\frac{\Gamma^2(r+1)}{3^{2r}}\Bigg]$$

$$b_{n1}^{(r,3)}=\frac{9}{n}\Bigg[\frac{2}{3^{r+1}}\left(\Gamma(2r+2)+\Gamma(2r+3)\right)\frac{5}{2^{r+1}}B_{\frac{2}{5}}(r+2,r+2)+\frac{\Gamma(2r+2)}{5^{2r+2}}$$
$$-\frac{9\Gamma(r+1)\Gamma(r+2)+\left(\Gamma(r+2)-\Gamma(r+1)\right)\left(\Gamma(r+3)-\Gamma(r+2)\right)}{3^{2r+3}}\Bigg]$$
$$-\frac{18}{n(n-1)}\Bigg[\frac{2}{3^{r+1}}$$

$$\cdot\left(\Gamma(2r+2)+\Gamma(2r+3)\right)\left(\frac{5}{2^r}B_{\frac{2}{5}}(r+2,r+2)-4B_{\frac{1}{4}}(r+2,r+2)\right)$$
$$+2\Gamma(2r+2)\left(\frac{1}{5^{2r+2}}-\frac{1}{4^{2r+2}}\right)-\frac{\Gamma(r+1)\Gamma(r+2)}{3^{2r+1}}\Bigg]$$
$$+\frac{6}{n(n-1)(n-2)}\left[\frac{2}{3^r}\left(\Gamma(2r+2)+\Gamma(2r+3)\right)\right.$$
$$\cdot\left(\frac{5}{2^{r+1}}B_{\frac{2}{5}}(r+2,r+2)-4B_{\frac{1}{4}}(r+2,r+2)\right)$$
$$\left.+3\Gamma(2r+2)\left(\frac{1}{5^{2r+2}}-\frac{2}{4^{2r+2}}+\frac{1}{3^{2r+2}}\right)-\frac{\Gamma(r+1)\Gamma(r+2)}{3^{2r+1}}\right]$$

$$c_{n1}^{(r,3)}=\frac{9}{n}\left[\frac{1}{3^{r+1}}\Gamma(2r+4)\frac{1}{2^r}B_{\frac{2}{5}}(r+2,r+2)+\frac{\Gamma(2r+3)}{5^{2r+3}}\right.$$
$$\left.-\frac{9\Gamma^2(r+2)+\left(\Gamma(r+3)-\Gamma(r+2)\right)^2}{3^{2r+4}}\right]$$
$$-\frac{18}{n(n-1)}\left[\frac{2}{3^{r+1}}\Gamma(2r+4)\right.$$
$$\cdot\left(\frac{1}{2^r}B_{\frac{2}{5}}(r+2,r+2)-B_{\frac{1}{4}}(r+2,r+2)\right)$$
$$\left.+2\Gamma(2r+3)\left(\frac{1}{5^{2r+3}}-\frac{1}{4^{2r+3}}\right)-\frac{\Gamma^2(r+2)}{3^{2r+2}}\right]$$
$$+\frac{6}{n(n-1)(n-2)}\left[\frac{2}{3^r}\Gamma(2r+4)\right.$$
$$\cdot\left(\frac{1}{2^{r+1}}B_{\frac{2}{5}}(r+2,r+2)-B_{\frac{1}{4}}(r+2,r+2)\right)$$
$$\left.+3\Gamma(2r+3)\left(\frac{1}{5^{2r+3}}-\frac{2}{4^{2r+3}}+\frac{1}{3^{2r+3}}\right)-\frac{\Gamma^2(r+2)}{3^{2r+2}}\right]$$

$$a_{n1}^{(r,4)}=\frac{16}{n}\left[\frac{2}{4^r}\Gamma(2r+2)\frac{1}{3^r}B_{\frac{3}{7}}(r+1,r+1)+\frac{\Gamma(2r+1)}{7^{2r+1}}\right.$$
$$\left.-\frac{16\Gamma^2(r+1)+\left(\Gamma(r+2)-\Gamma(r+1)\right)^2}{4^{2r+2}}\right]$$
$$-\frac{72}{n(n-1)}\left[\frac{2}{4^r}\Gamma(2r+2)\right.$$
$$\cdot\left(\frac{2}{3^r}B_{\frac{3}{7}}(r+1,r+1)-\frac{1}{2^r}B_{\frac{1}{3}}(r+1,r+1)\right)$$

$$+2\Gamma(2r+1)\left(\frac{1}{7^{2r+1}}-\frac{1}{6^{2r+1}}\right)-\frac{\Gamma^2(r+1)}{4^{2r}}\Bigg]$$

$$+\frac{96}{n(n-1)(n-2)}\left[\frac{2}{4^r}\Gamma(2r+2)\left(\frac{1}{3^{r-1}}B_{\frac{3}{7}}(r+1,r+1)\right.\right.$$

$$\left.-\frac{3}{2^r}B_{\frac{1}{3}}(r+1,r+1)+B_{\frac{1}{5}}(r+1,r+1)\right)$$

$$+3\Gamma(2r+1)\left(\frac{1}{7^{2r+1}}-\frac{2}{6^{2r+1}}+\frac{1}{5^{2r+1}}\right)-\frac{\Gamma^2(r+1)}{4^{2r}}\Bigg]$$

$$-\frac{24}{n(n-1)(n-2)(n-3)}\left[\frac{2}{4^r}\Gamma(2r+2)\left(\frac{4}{3^r}B_{\frac{3}{7}}(r+1,r+1)\right.\right.$$

$$\left.-\frac{6}{2^r}B_{\frac{1}{3}}(r+1,r+1)+4B_{\frac{1}{5}}(r+1,r+1)\right)$$

$$+4\Gamma(2r+1)\left(\frac{1}{7^{2r+1}}-\frac{3}{6^{2r+1}}+\frac{3}{5^{2r+1}}-\frac{1}{4^{2r+1}}\right)-\frac{\Gamma^2(r+1)}{4^{2r}}\Bigg]$$

$$b_{n1}^{(r,4)}=\frac{16}{n}\left[\frac{2}{4^{r+1}}\left(\Gamma(2r+2)+\Gamma(2r+3)\right)\frac{1}{3^r}B_{\frac{3}{7}}(r+2,r+2)+\frac{\Gamma(2r+2)}{7^{2r+2}}\right.$$

$$\left.-\frac{16\Gamma(r+1)\Gamma(r+2)+\left(\Gamma(r+2)-\Gamma(r+1)\right)\left(\Gamma(r+3)-\Gamma(r+2)\right)}{4^{2r+3}}\right]$$

$$-\frac{72}{n(n-1)}\left[\frac{2}{4^{r+1}}\left(\Gamma(2r+2)+\Gamma(2r+3)\right)\left[\frac{2}{3^r}B_{\frac{3}{7}}(r+2,r+2)\right.\right.$$

$$\left.-\frac{1}{2^r}B_{\frac{1}{3}}(r+2,r+2)\right]+2\Gamma(2r+2)\left(\frac{1}{7^{2r+2}}-\frac{1}{6^{2r+2}}\right)$$

$$\left.-\frac{\Gamma(r+1)\Gamma(r+2)}{4^{2r+1}}\right]$$

$$+\frac{96}{n(n-1)(n-2)}\left[\frac{2}{4^{r+1}}\left(\Gamma(2r+2)+\Gamma(2r+3)\right)\right.$$

$$\cdot\left(\frac{1}{3^{r-1}}B_{\frac{3}{7}}(r+2,r+2)-\frac{3}{2^r}B_{\frac{1}{3}}(r+2,r+2)+B_{\frac{1}{5}}(r+2,r+2)\right)$$

$$\left.+3\Gamma(2r+2)\left(\frac{1}{7^{2r+2}}-\frac{2}{6^{2r+2}}+\frac{3}{5^{2r+2}}\right)-\frac{\Gamma(r+1)\Gamma(r+2)}{4^{2r+1}}\right]$$

$$-\frac{24}{n(n-1)(n-2)(n-3)}\left[\frac{2}{4^{r+1}}\left(\Gamma(2r+2)+\Gamma(2r+3)\right)\right.$$

$$\cdot\left(\frac{4}{3^r}B_{\frac{3}{7}}(r+2,r+2)-\frac{6}{2^r}B_{\frac{1}{3}}(r+2,r+2)+4B_{\frac{1}{5}}(r+2,r+2)\right)$$

$$+4\Gamma(2r+2)\left(\frac{1}{7^{2r+2}}-\frac{3}{6^{2r+2}}+\frac{3}{5^{2r+2}}-\frac{1}{4^{2r+2}}\right)-\frac{\Gamma(r+1)\Gamma(r+2)}{4^{2r+1}}\Bigg]$$

$$c_{n1}^{(r,4)}=\frac{16}{n}\Bigg[\frac{2}{4^{r+1}}\Gamma(2r+4)\frac{1}{3^{r+1}}B_{\frac{3}{7}}(r+2,r+2)+\frac{\Gamma(2r+3)}{7^{2r+3}}$$
$$-\frac{16\Gamma^2(r+2)+(\Gamma(r+3)-\Gamma(r+2))^2}{4^{2r+4}}\Bigg]$$
$$-\frac{72}{n(n-1)}\Bigg[\frac{2}{4^{r+1}}\Gamma(2r+4)$$
$$\cdot\left(\frac{2}{3^{r+1}}B_{\frac{3}{7}}(r+2,r+2)-\frac{1}{2^{r+1}}B_{\frac{1}{3}}(r+2,r+2)\right)$$
$$+2\Gamma(2r+3)\left(\frac{1}{7^{2r+3}}-\frac{1}{6^{2r+3}}\right)-\frac{\Gamma^2(r+2)}{4^{2r+2}}\Bigg]$$
$$+\frac{96}{n(n-1)(n-2)}\Bigg[\frac{2}{4^{r+1}}\Gamma(2r+4)\Bigg(\frac{1}{3^r}B_{\frac{3}{7}}(r+2,r+2)$$
$$-\frac{3}{2^{r+1}}B_{\frac{1}{3}}(r+2,r+2)+B_{\frac{1}{5}}(r+2,r+2)\Bigg)$$
$$+3\Gamma(2r+3)\left(\frac{1}{7^{2r+3}}-\frac{2}{6^{2r+3}}+\frac{1}{5^{2r+3}}\right)-\frac{\Gamma^2(r+2)}{4^{2r+2}}\Bigg]$$
$$-\frac{24}{n(n-1)(n-2)(n-3)}\Bigg[\frac{2}{4^{r+1}}\Gamma(2r+4)\Bigg(\frac{4}{3^{r+1}}B_{\frac{3}{7}}(r+2,r+2)$$
$$-\frac{6}{2^{r+1}}B_{\frac{1}{3}}(r+2,r+2)+4B_{\frac{1}{5}}(r+2,r+2)\Bigg)$$
$$+4\Gamma(2r+3)\left(\frac{1}{7^{2r+3}}-\frac{3}{6^{2r+3}}+\frac{3}{5^{2r+3}}-\frac{1}{4^{2r+3}}\right)-\frac{\Gamma^2(r+2)}{4^{2r+2}}\Bigg].$$

Note that the case $r=1$ for $k=1$ is excluded. Some special cases are as follows

$$a_{n1}^{(1,2)}=\frac{1}{12n}+\frac{1}{6n(n-1)},\quad b_{n1}^{(1,2)}=\frac{1}{18n}+\frac{4}{9n(n-1)},$$
$$c_{n1}^{(1,2)}=\frac{2}{27n}+\frac{7}{6n(n-1)}$$
$$a_{n1}^{(1,3)}=\frac{4}{45n}+\frac{1}{5n(n-1)}+\frac{1}{15n(n-1)(n-2)}$$
$$b_{n1}^{(1,3)}=\frac{44}{675n}+\frac{47}{150n(n-1)}+\frac{67}{450n(n-1)(n-2)}$$

$$
\begin{aligned}
c_{n1}^{(1,3)} &= \frac{628}{10125n} + \frac{1289}{2250n(n-1)} + \frac{2629}{6750n(n-1)(n-2)}\\
a_{n1}^{(1,4)} &= \frac{9}{112n} + \frac{3}{14n(n-1)} + \frac{6}{35n(n-1)(n-2)}\\
&+ \frac{3}{70n(n-1)(n-2)(n-3)}\\
b_{n1}^{(1,4)} &= \frac{39}{784n} + \frac{47}{196n(n-1)} + \frac{319}{1225n(n-1)(n-2)}\\
&+ \frac{106}{1225n(n-1)(n-2)(n-3)}\\
c_{n1}^{(1,4)} &= \frac{205}{5488n} + \frac{2549}{8232n(n-1)} + \frac{117317}{257250n(n-1)(n-2)}\\
&+ \frac{206357}{1029000n(n-1)(n-2)(n-3)}.
\end{aligned}
$$

For the direct tests we have

$$
\begin{aligned}
\hat{T}_n^{(r,k)} = \frac{1}{\Delta_{n1}^{(r,k)}} \Bigg[& c_{n1}^{(r,k)} \left(\frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \left(\frac{X_{i:n}}{\overline{X}_n} \right)^r - \frac{\Gamma(r+1)}{k^r} \right)^2 \\
& -2b_{n1}^{(r,k)} \left(\frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \left(\frac{X_{i:n}}{\overline{X}_n} \right)^r - \frac{\Gamma(r+1)}{k^r} \right) \\
& \times \left(\frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \left(\frac{X_{i:n}}{\overline{X}_n} \right)^{r+1} - \frac{\Gamma(r+2)}{k^{r+1}} \right) \\
& +a_{n1}^{(r,k)} \left(\frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \left(\frac{X_{i:n}}{\overline{X}_n} \right)^{r+1} - \frac{\Gamma(r+2)}{k^{r+1}} \right)^2 \Bigg] \\
&= \hat{T}_{n;c_1}^{(r,k)} + \hat{T}_{n;c_2}^{(r,k)} = \hat{T}_{n;c_3}^{(r,k)} + \hat{T}_{n;c_4}^{(r,k)}
\end{aligned}
$$

where

$$
\hat{T}_{n;c_1}^{(r,k)} = \frac{1}{a_{n1}^{(r,k)}} \left[\frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \left(\frac{X_{i:n}}{\overline{X}_n} \right)^r - \frac{\Gamma(r+1)}{k^r} \right]^2
$$

$$
\begin{aligned}
\hat{T}_{n;c_2}^{(r,k)} = \frac{1}{\Delta_{n1}^{(r,k)} a_{n1}^{(r,k)}} \Bigg[& a_{n1}^{(r,k)} \frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \left(\frac{X_{i:n}}{\overline{X}_n} \right)^{r+1} \\
-b_{n1}^{(r,k)} \frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} & \left(\frac{X_{i:n}}{\overline{X}_n} \right)^r - \left(a_{n1}^{(r,k)} \frac{\Gamma(r+2)}{k^{r+1}} - b_{n1}^{(r,k)} \frac{\Gamma(r+1)}{k^r} \right) \Bigg]^2
\end{aligned}
$$

$$\hat{T}_{n;c_3}^{(r,k)} = \frac{1}{c_{n1}^{(r,k)}}\left[\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\left(\frac{X_{i:n}}{\overline{X}_n}\right)^{r+1} - \frac{\Gamma(r+2)}{k^{r+1}}\right]^2$$

$$\hat{T}_{n;c_4}^{(r,k)} = \frac{1}{\Delta_{n1}^{(r,k)}c_{n1}^{(r,k)}}\left[c_{n1}^{(r,k)}\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\left(\frac{X_{i:n}}{\overline{X}_n}\right)^{r}\right.$$
$$\left.-b_{n1}^{(r,k)}\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\left(\frac{X_{i:n}}{\overline{X}_n}\right)^{r+1} - \left(c_{n1}^{(r,k)}\frac{\Gamma(r+1)}{k^r} - b_{n1}^{(r,k)}\frac{\Gamma(r+2)}{k^{r+1}}\right)\right]^2.$$

Then the dual tests we write as follows

$$\hat{T}_{n}^{*(r,k)} = \frac{1}{\Delta_{n1}^{*(r,k)}}\left[c_{n1}^{*(r,k)}\left(\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\log^r\frac{1}{1-\exp\left(X_{n-i+1}/\overline{X}_n\right)}\right.\right.$$
$$\left.-\frac{\Gamma(r+1)}{k^r}\right)^2$$
$$-2b_{n1}^{*(r,k)}\left(\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\log^r\frac{1}{1-\exp\left(X_{n-i+1}/\overline{X}_n\right)} - \frac{\Gamma(r+1)}{k^r}\right)$$
$$\times\left(\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\log^{r+1}\frac{1}{1-\exp\left(X_{n-i+1}/\overline{X}_n\right)} - \frac{\Gamma(r+2)}{k^{r+1}}\right)$$
$$\left.+a_{n1}^{*(r,k)}\left(\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\log^{r+1}\frac{1}{1-\exp\left(X_{n-i+1}/\overline{X}_n\right)} - \frac{\Gamma(r+2)}{k^{r+1}}\right)^2\right]$$
$$= \hat{T}_{n;c_1}^{*(r,k)} + \hat{T}_{n;c_2}^{*(r,k)} = \hat{T}_{n;c_3}^{*(r,k)} + \hat{T}_{n;c_4}^{*(r,k)}$$

where

$$\hat{T}_{n;c_1}^{*(r,k)} = \frac{1}{a_{n1}^{*(r,k)}}\left[\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\log^r\frac{1}{1-\exp\left(X_{n-i+1}/\overline{X}_n\right)} - \frac{\Gamma(r+1)}{k^r}\right]^2$$

$$\hat{T}_{n;c_2}^{*(r,k)} = \frac{1}{\Delta_{n1}^{*(r,k)}a_{n1}^{*(r,k)}}$$
$$\cdot\left[a_{n1}^{*(r,k)}\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\log^{r+1}\frac{1}{1-\exp\left(X_{n-i+1}/\overline{X}_n\right)}\right.$$
$$-b_{n1}^{*(r,k)}\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\log^{r}\frac{1}{1-\exp\left(X_{n-i+1}/\overline{X}_n\right)}$$

$$- \left(a_{n1}^{*(r,k)} \frac{\Gamma(r+2)}{k^{r+1}} - b_{n1}^{*(r,k)} \frac{\Gamma(r+1)}{k^r} \right) \Bigg]^2$$

$$\hat{T}_{n;c_3}^{*(r,k)} = \frac{1}{c_{n1}^{*(r,k)}} \cdot \left[\frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \log^{r+1} \frac{1}{1-\exp\left(X_{n-i+1}/\overline{X}_n\right)} - \frac{\Gamma(r+2)}{k^{r+1}} \right]^2$$

$$\hat{T}_{n;c_4}^{*(r,k)} = \frac{1}{\Delta_{n1}^{*(r,k)} c_{n1}^{*(r,k)}} \cdot \Bigg[c_{n1}^{*(r,k)} \frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \log^{r} \frac{1}{1-\exp\left(X_{n-i+1}/\overline{X}_n\right)}$$
$$- b_{n1}^{*(r,k)} \frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \log^{r+1} \frac{1}{1-\exp\left(X_{n-i+1}/\overline{X}_n\right)}$$
$$- \left(c_{n1}^{*(r,k)} \frac{\Gamma(r+1)}{k^r} - b_{n1}^{*(r,k)} \frac{\Gamma(r+2)}{k^{r+1}} \right) \Bigg]^2.$$

5.2. Rayleigh Distribution: $X \sim \mathrm{Ral}(\alpha)$

Here

$$F(x) := F(x;\alpha) = 1 - e^{-\alpha x^2}, \qquad f(x) = 2\alpha x e^{-\alpha x^2},$$
$$x > 0; \quad \Lambda = \{\alpha : \alpha > 0\}$$
$$\frac{\partial F}{\partial \alpha} = x^2 e^{-\alpha x^2}, \qquad \mathcal{I}^{-1}(\alpha) = \alpha^2,$$
$$\hat{\alpha}_n = 1/\overline{X_n^2}, \qquad \overline{X_n^2} = \frac{1}{n}\sum_{j=1}^n X_j^2.$$

The quantities $a_{n1}^{(r,k)}$, $b_{n1}^{(r,k)}$, $c_{n1}^{(r,k)}$ and $a_{n1}^{*(r,k)}$, $b_{n1}^{*(r,k)}$, $c_{n1}^{*(r,k)}$ are as for the exponential distribution and the tests statistics are as follows

$$\hat{T}_n^{(r,k)} = \frac{1}{\Delta_{n1}^{(r,k)}} \Bigg[c_{n1}^{(r,k)} \left(\frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \left(\frac{X_{i:n}^2}{\overline{X_n^2}} \right)^r - \frac{\Gamma(r+1)}{k^r} \right)^2$$
$$-2b_{n1}^{(r,k)} \left(\frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \left(\frac{X_{i:n}^2}{\overline{X_n^2}} \right)^r - \frac{\Gamma(r+1)}{k^r} \right)$$

$$\cdot\left(\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\left(\frac{X_{i:n}^2}{\overline{X_n^2}}\right)^{r+1}-\frac{\Gamma(r+2)}{k^{r+1}}\right)$$

$$+a_{n1}^{(r,k)}\left(\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\left(\frac{X_{i:n}^2}{\overline{X_n^2}}\right)^{r+1}-\frac{\Gamma(r+2)}{k^{r+1}}\right)^2\Bigg]$$

$$=\hat{T}_{n;c_1}^{(r,k)}+\hat{T}_{n;c_2}^{(r,k)}=\hat{T}_{n;c_3}^{(r,k)}+\hat{T}_{n;c_4}^{(r,k)}$$

where

$$\hat{T}_{n;c_1}^{(r,k)}=\frac{1}{a_{n1}^{(r,k)}}\left[\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\left(\frac{X_{i:n}^2}{\overline{X_n^2}}\right)^{r}-\frac{\Gamma(r+1)}{k^{r}}\right]^2$$

$$\hat{T}_{n;c_2}^{(r,k)}=\frac{1}{\Delta_{n1}^{(r,k)}a_{n1}^{(r,k)}}\left[a_{n1}^{(r,k)}\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\left(\frac{X_{i:n}^2}{\overline{X_n^2}}\right)^{r+1}\right.$$
$$-b_{n1}^{(r,k)}\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\left(\frac{X_{i:n}^2}{\overline{X_n^2}}\right)^{r}$$
$$\left.-\left(a_{n1}^{(r,k)}\frac{\Gamma(r+2)}{k^{r+1}}-b_{n1}^{(r,k)}\frac{\Gamma(r+1)}{k^{r}}\right)\right]^2$$

$$\hat{T}_{n;c_3}^{(r,k)}=\frac{1}{c_{n1}^{(r,k)}}\left[\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\left(\frac{X_{i:n}^2}{\overline{X_n^2}}\right)^{r+1}-\frac{\Gamma(r+2)}{k^{r+1}}\right]^2$$

$$\hat{T}_{n;c_4}^{(r,k)}=\frac{1}{\Delta_{n1}^{(r,k)}c_{n1}^{(r,k)}}\left[c_{n1}^{(r,k)}\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\left(\frac{X_{i:n}^2}{\overline{X_n^2}}\right)^{r}\right.$$
$$-b_{n1}^{(r,k)}\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\left(\frac{X_{i:n}^2}{\overline{X_n^2}}\right)^{r+1}$$
$$\left.-\left(c_{n1}^{(r,k)}\frac{\Gamma(r+1)}{k^{r}}-b_{n1}^{(r,k)}\frac{\Gamma(r+2)}{k^{r+1}}\right)\right]^2.$$

The dual tests are as follows

$$\hat{T}_n^{*(r,k)}=\frac{1}{\Delta_{n1}^{*(r,k)}}$$
$$\cdot\left[c_{n1}^{*(r,k)}\left(\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\log^r\frac{1}{1-\exp\left(X_{n-i+1}^2/\overline{X_n^2}\right)}-\frac{\Gamma(r+1)}{k^r}\right)^2\right.$$

$$-2b_{n1}^{*(r,k)}\left(\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\log^{r}\frac{1}{1-\exp\left(X_{n-i+1}^{2}/\overline{X_n^2}\right)}-\frac{\Gamma(r+1)}{k^{r}}\right)$$

$$\times\left(\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\log^{r+1}\frac{1}{1-\exp\left(X_{n-i+1}^{2}/\overline{X_n^2}\right)}-\frac{\Gamma(r+2)}{k^{r+1}}\right)$$

$$+a_{n1}^{*(r,k)}\left(\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\log^{r+1}\frac{1}{1-\exp\left(X_{n-i+1}^{2}/\overline{X_n^2}\right)}-\frac{\Gamma(r+2)}{k^{r+1}}\right)^{2}\Bigg]$$

$$=\hat{T}_{n;c_1}^{*(r,k)}+\hat{T}_{n;c_2}^{*(r,k)}=\hat{T}_{n;c_3}^{*(r,k)}+\hat{T}_{n;c_4}^{*(r,k)}$$

where

$$\hat{T}_{n;c_1}^{*(r,k)}=\frac{1}{a_{n1}^{*(r,k)}}\times\left[\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\log^{r}\frac{1}{1-\exp\left(X_{n-i+1}^{2}/\overline{X_n^2}\right)}-\frac{\Gamma(r+1)}{k^{r}}\right]^{2}$$

$$\hat{T}_{n;c_2}^{*(r,k)}=\frac{1}{\Delta_{n1}^{*(r,k)}a_{n1}^{*(r,k)}}\times\Bigg[a_{n1}^{*(r,k)}\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\log^{r+1}\frac{1}{1-\exp\left(X_{n-i+1}^{2}/\overline{X_n^2}\right)}$$

$$-b_{n1}^{*(r,k)}\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\log^{r}\frac{1}{1-\exp\left(X_{n-i+1}^{2}/\overline{X_n^2}\right)}$$

$$-\left(a_{n1}^{*(r,k)}\frac{\Gamma(r+2)}{k^{r+1}}-b_{n1}^{*(r,k)}\frac{\Gamma(r+1)}{k^{r}}\right)\Bigg]^{2}$$

$$\hat{T}_{n;c_3}^{*(r,k)}=\frac{1}{c_{n1}^{*(r,k)}}\times\left[\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\log^{r+1}\frac{1}{1-\exp\left(X_{n-i+1}^{2}/\overline{X_n^2}\right)}-\frac{\Gamma(r+2)}{k^{r+1}}\right]^{2}$$

$$\hat{T}_{n;c_4}^{*(r,k)}=\frac{1}{\Delta_{n1}^{*(r,k)}c_{n1}^{*(r,k)}}$$

$$\times \Bigg[c_{n1}^{*(r,k)} \frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \log^{r} \frac{1}{1-\exp\left(X_{n-i+1}^2/\overline{X_n^2}\right)}$$

$$- b_{n1}^{*(r,k)} \frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \log^{r+1} \frac{1}{1-\exp\left(X_{n-i+1}^2/\overline{X_n^2}\right)}$$

$$- \left(c_{n1}^{*(r,k)} \frac{\Gamma(r+1)}{k^r} - b_{n1}^{*(r,k)} \frac{\Gamma(r+2)}{k^{r+1}} \right) \Bigg]^2 .$$

5.3. Normal Distribution: $X \sim N(\mu, \sigma^2)$

Here

$$f(x;\mu,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$

$$-\infty < x < \infty, \quad \Lambda = \{(\mu,\sigma^2) : \mu \in \mathbb{R},\ \sigma^2 > 0\}$$

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

$$\frac{\partial F}{\partial \mu} = -f(x), \qquad \frac{\partial F}{\partial \sigma^2} = \frac{1}{2\sigma^2}(x-\mu)f(x)$$

$$\hat{\mu}_n = \overline{X}_n, \qquad \hat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2, \qquad S_n = \sqrt{\hat{\sigma}_n^2}$$

and

$$\mathcal{I}^{-1} = \begin{bmatrix} \sigma^2 & 0 \\ 0 & 2\sigma^4 \end{bmatrix}.$$

Hence

$$b^{(r,k)}(\mu)$$

$$= E\left[\left(1 - F(X;\mu,\sigma^2)\right)^{k-2} \log^{r-1} \frac{1}{1-F(X;\mu,\sigma^2)} \frac{\partial F(X;\mu,\sigma^2)}{\partial \mu} \right]$$

$$= -\int_{-\infty}^{\infty} \left(1 - F(x;\mu,\sigma^2)\right)^{k-2} \log^{r-1} \frac{1}{1-F(X;\mu,\sigma^2)} f^2(x)dx$$

$$= -\frac{1}{\sigma} E\left[(1-\Phi(Z))^{k-2} \log^{r-1} \frac{1}{1-\Phi(Z)} \phi(Z) \right] = -\frac{1}{\sigma} E_1^{(r,k)}$$

where

$$E_1^{(r,k)} = E\left[\phi(Z)\,(1-\Phi(Z))^{k-2} \log^{r-1} \frac{1}{1-\Phi(Z)} \right], \quad Z \sim N(0,1),$$

and ϕ, Φ denote pdf and cdf of Z, respectively.

Similarly

$$b^{(r,k)}\left(\sigma^2\right) = -\frac{1}{2\sigma^2}E_2^{(r,k)}$$

where

$$E_2^{(r,k)} = E\left[Z\phi(Z)\left(1-\Phi(Z)\right)^{k-2}\log^{r-1}\frac{1}{1-\Phi(Z)}\right].$$

Referring to the dual tests, we see similarly that

$$b^{*(r,k)}(\mu) = b^{(r,k)}(\mu), \qquad b^{*(r,k)}(\sigma^2) = -b^{(r,k)}(\sigma^2).$$

Then

$$\boldsymbol{B}^{(r,k)} = -k\begin{bmatrix} r\frac{1}{\sigma}E_1^{(r,k)} & r\frac{1}{2\sigma^2}E_2^{(r,k)} \\ (r+1)\frac{1}{\sigma}E_1^{(r+1,k)} & (r+1)\frac{1}{2\sigma^2}E_2^{(r+1,k)} \end{bmatrix}$$

and

$$\frac{1}{n}\boldsymbol{B}^{(r,k)}\mathcal{I}^{-1}\left(\boldsymbol{B}^{(r,k)}\right)' = \begin{bmatrix} s_n^{(r)} & t_n^{(r)} \\ t_n^{(r)} & u_n^{(r)} \end{bmatrix}$$

with

$$s_n^{(r,k)} = \frac{k^2r^2}{n}\left[\left(E_1^{(r,k)}\right)^2 + \frac{1}{2}\left(E_2^{(r,k)}\right)^2\right]$$

$$t_n^{(r,k)} = \frac{k^2r(r+1)}{n}\left[E_1^{(r,k)}E_1^{(r+1,k)} + \frac{1}{2}E_2^{(r,k)}E_2^{(r+1,k)}\right]$$

$$u_n^{(r,k)} = \frac{k^2(r+1)^2}{n}\left[\left(E_1^{(r+1,k)}\right)^2 + \frac{1}{2}\left(E_2^{(r+1,k)}\right)^2\right]$$

and

$$s_n^{*(r,k)} = s_n^{(r,k)}, \quad t_n^{*(r,k)} = t_n^{(r,k)}, \quad u_n^{*(r,k)} = u_n^{(r,k)}.$$

Then we use

$$a_{n1}^{(r,k)} = a_{n1}^{*(r,k)} = a_n^{(r,k)} - \frac{k^2r^2}{n}\left[\left(E_1^{(r,k)}\right)^2 + \frac{1}{2}\left(E_2^{(r,k)}\right)^2\right]$$

$$b_{n1}^{(r,k)} = b_{n1}^{*(r,k)} = b_n^{(r,k)} - \frac{k^2r(r+1)}{n}\left[E_1^{(r,k)}E_1^{(r+1,k)} + \frac{1}{2}E_2^{(r,k)}E_2^{(r+1,k)}\right]$$

$$c_{n1}^{(r,k)} = c_{n1}^{*(r,k)} = c_n^{(r,k)} - \frac{k^2(r+1)^2}{n}\left[\left(E_1^{(r+1,k)}\right)^2 + \frac{1}{2}\left(E_2^{(r+1,k)}\right)^2\right].$$

The test–statistics for the direct tests are

$$\hat{T}_n^{(r,k)} = \frac{1}{\Delta_{n1}^{(r,k)}}$$

$$
\begin{aligned}
&\times \Bigg[c_{n1}^{(r,k)} \left(\frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \log^r \frac{1}{1-\Phi\left(\frac{X_{i:n}-\overline{X}_n}{S_n}\right)} - \frac{\Gamma(r+1)}{k^r} \right)^2 \\
&-2b_{n1}^{(r,k)} \left(\frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \log^r \frac{1}{1-\Phi\left(\frac{X_{i:n}-\overline{X}_n}{S_n}\right)} - \frac{\Gamma(r+1)}{k^r} \right) \\
&\times \left(\frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \log^{r+1} \frac{1}{1-\Phi\left(\frac{X_{i:n}-\overline{X}_n}{S_n}\right)} - \frac{\Gamma(r+2)}{k^{r+1}} \right) \\
&+a_{n1}^{(r,k)} \left(\frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \log^{r+1} \frac{1}{1-\Phi\left(\frac{X_{i:n}-\overline{X}_n}{S_n}\right)} - \frac{\Gamma(r+2)}{k^{r+1}} \right)^2 \Bigg] \\
&= \hat{T}_{n;c_1}^{(r,k)} + \hat{T}_{n;c_2}^{(r,k)} = \hat{T}_{n;c_3}^{(r,k)} + \hat{T}_{n;c_4}^{(r,k)}
\end{aligned}
$$

where

$$
\hat{T}_{n;c_1}^{(r,k)} = \frac{1}{a_{n1}^{(r,k)}} \left[\frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \log^r \frac{1}{1-\Phi\left(\frac{X_{i:n}-\overline{X}_n}{S_n}\right)} - \frac{\Gamma(r+1)}{k^r} \right]^2
$$

$$
\begin{aligned}
\hat{T}_{n;c_2}^{(r,k)} = \frac{1}{\Delta_{n1}^{(r,k)} a_{n1}^{(r,k)}} \Bigg[& a_{n1}^{(r,k)} \frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \log^{r+1} \frac{1}{1-\Phi\left(\frac{X_{i:n}-\overline{X}_n}{S_n}\right)} \\
&-b_{n1}^{(r,k)} \frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \log^{r} \frac{1}{1-\Phi\left(\frac{X_{i:n}-\overline{X}_n}{S_n}\right)} \\
&- \left(a_{n1}^{(r,k)} \frac{\Gamma(r+2)}{k^{r+1}} - b_{n1}^{(r,k)} \frac{\Gamma(r+1)}{k^r} \right) \Bigg]^2
\end{aligned}
$$

$$
\begin{aligned}
\hat{T}_{n;c_3}^{(r,k)} &= \frac{1}{c_{n1}^{(r,k)}} \\
&\times \left[\frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \log^{r+1} \frac{1}{1-\Phi\left(\frac{X_{i:n}-\overline{X}_n}{S_n}\right)} - \frac{\Gamma(r+2)}{k^{r+1}} \right]^2
\end{aligned}
$$

$$
\hat{T}_{n;c_4}^{(r,k)} = \frac{1}{\Delta_{n1}^{(r,k)} c_{n1}^{(r,k)}} \Bigg[c_{n1}^{(r,k)} \frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \log^{r} \frac{1}{1-\Phi\left(\frac{X_{i:n}-\overline{X}_n}{S_n}\right)}
$$

$$-b_{n1}^{(r,k)}\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\log^{r+1}\frac{1}{1-\Phi\left(\frac{X_{i:n}-\overline{X}_n}{S_n}\right)}$$
$$-\left(c_{1n}^{(r,k)}\frac{\Gamma(r+1)}{k^r}-b_{n1}^{(r,k)}\frac{\Gamma(r+2)}{k^{r+1}}\right)\Bigg]^2.$$

Similarly, for the dual tests

$$\hat{T}_n^{*(r,k)}=\frac{1}{\Delta_{n1}^{(r,k)}}$$
$$\times\Bigg[c_{n1}^{(r,k)}\left(\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\log^{r}\frac{1}{\Phi\left(\frac{X_{n-i+1:n}-\overline{X}_n}{S_n}\right)}-\frac{\Gamma(r+1)}{k^r}\right)^2$$
$$-2b_{n1}^{(r,k)}\left(\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\log^{r}\frac{1}{\Phi\left(\frac{X_{n-i+1:n}-\overline{X}_n}{S_n}\right)}-\frac{\Gamma(r+1)}{k^r}\right)$$
$$\times\left(\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\log^{r+1}\frac{1}{\Phi\left(\frac{X_{n-i+1:n}-\overline{X}_n}{S_n}\right)}-\frac{\Gamma(r+2)}{k^{r+1}}\right)$$
$$+a_{n1}^{(r,k)}\left(\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\log^{r+1}\frac{1}{\Phi\left(\frac{X_{n-i+1:n}-\overline{X}_n}{S_n}\right)}-\frac{\Gamma(r+2)}{k^{r+1}}\right)^2\Bigg]$$
$$=\hat{T}_{n;c_1}^{*(r,k)}+\hat{T}_{n;c_2}^{*(r,k)}=\hat{T}_{n;c_3}^{*(r,k)}+\hat{T}_{n;c_4}^{*(r,k)}$$

where

$$\hat{T}_{n;c_1}^{*(r,k)}=\frac{1}{a_{n1}^{(r,k)}}\left[\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\log^{r}\frac{1}{\Phi\left(\frac{X_{n-i+1:n}-\overline{X}_n}{S_n}\right)}-\frac{\Gamma(r+1)}{k^r}\right]^2$$

$$\hat{T}_{n;c_2}^{*(r,k)}=\frac{1}{\Delta_{n1}^{(r,k)}a_{n1}^{(r,k)}}\Bigg[a_{n1}^{(r,k)}\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\log^{r+1}\frac{1}{\Phi\left(\frac{X_{n-i+1:n}-\overline{X}_n}{S_n}\right)}$$
$$-b_{n1}^{(r,k)}\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\log^{r}\frac{1}{\Phi\left(\frac{X_{n-i+1:n}-\overline{X}_n}{S_n}\right)}$$
$$-\left(a_{n1}^{(r,k)}\frac{\Gamma(r+2)}{k^{r+1}}-b_{n1}^{(r,k)}\frac{\Gamma(r+1)}{k^r}\right)\Bigg]^2$$

$$\hat{T}^{*(r,k)}_{n;c_3} = \frac{1}{c^{(r,k)}_{n1}} \left[\frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \log^{r+1} \frac{1}{\Phi\left(\frac{X_{n-i+1:n}-\overline{X}_n}{S_n}\right)} - \frac{\Gamma(r+2)}{k^{r+1}} \right]^2$$

$$\hat{T}^{*(r,k)}_{n;c_4} = \frac{1}{\Delta^{(r,k)}_{n1} c^{(r,k)}_{n1}} \left[c^{(r,k)}_{n1} \frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \log^{r} \frac{1}{\Phi\left(\frac{X_{n-i+1:n}-\overline{X}_n}{S_n}\right)} \right.$$
$$- b^{(r,k)}_{n1} \frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i}{k-1} \log^{r+1} \frac{1}{\Phi\left(\frac{X_{n-i+1:n}-\overline{X}_n}{S_n}\right)}$$
$$\left. - \left(c^{(r,k)}_{n1} \frac{\Gamma(r+1)}{k^r} - b^{(r,k)}_{n1} \frac{\Gamma(r+2)}{k^{r+1}} \right) \right]^2 .$$

6. Appendix

Here we present concepts, formulae and results (cf. Abramowitz and Stegun [1], Andrews et al [2], Srivastava and Choi [25], Temme [26]) used in the paper.

6.1. The Shifted Factorial. Special Functions

The shifted factorial called also Pochhammer's symbol is defined for $a \in \mathbb{C}$, $n \in \mathbb{N}$ as follows

$$(a)_0 = 1,$$
$$(a)_n = a(a+1)\cdots(a+n-1) = (a+n-1)(a)_{n-1}, \quad n = 1, 2, \ldots$$
$$(-a)_n = (-1)^n (a-n+1)_n. \qquad \text{(cf. Temme [26], p. 72)}$$

Let $p, q = 0, 1, 2, \ldots$ with $p \le q+1$. The hypergeometric function is defined by

$${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{z^n}{n!}$$

(cf. Temme [26], p. 124).

For $p = 2$ and $q = 1$ we have the Gauss hypergeometric function

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},$$

and for $p = 1$ and $q = 1$ we have the Kummer function:

$$ {}_1F_1(a;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!} $$

which is denoted by $M(a,c,z)$.

We take as the definition of the gamma function the Euler integral of the second kind

$$ \Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt, \qquad Re\, z > 0 \qquad \text{(cf. Temme [26], p. 44).} $$

The beta function is the Euler integral of the first kind

$$ B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1} dt, \qquad Re\, p > 0, \quad Re\, q > 0 \qquad \text{(cf. Temme [26], p. 44).} \tag{6.1} $$

The beta function in (6.1) can be expressed in terms of the gamma function:

$$ B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \qquad \text{(cf. Temme [26], p. 45).} $$

The incomplete gamma functions are:

$$ \gamma(a,z) = \int_0^z t^{a-1} e^{-t} dt, \qquad \Gamma(a,z) = \int_z^{\infty} t^{a-1} e^{-t} dt \qquad \text{(cf. Temme [26], p. 185).} $$

The incomplete beta functions are:

$$ B_x(p,q) = \int_0^x t^{p-1}(1-t)^{q-1} dt, \qquad 0 < x < 1, $$

$$ I_x(p,q) = \frac{1}{B(p,q)} \int_0^x t^{p-1}(1-t)^{q-1} dt \qquad \text{(cf. Temme [26], p. 128).} $$

The following formulae will be applied

$$ {}_2F_1(a,b;c;z) = {}_2F_1(b,a;c;z) \quad \text{(cf. Srivastava and Choi [25], p. 46)} $$

$$ \frac{d}{dz}\left[e^{-z} z^{c-1}\ {}_1F_1(a;c;z)\right] = -(1-c) z^{c-2}\ {}_1F_1(c-a;c-1;-z) \qquad \text{(cf. Slater et al [24], (2.1.12))} \tag{6.2} $$

$$ \begin{aligned} \gamma(a,z) &= \frac{z^a}{a} e^{-z} {}_1F_1(1;a+1;z) \\ &= \frac{z^a}{a} {}_1F_1(a;a+1;-z) \qquad \text{(cf. Temme [26], p. 278)} \end{aligned} \tag{6.3} $$

$$B_x(p,q) = \frac{x^p}{p}{}_2F_1(p,1-q;p+1;x) = \frac{x^p}{p}(1-x)^{q-1}{}_2F_1\left(1,1-q;p+1;\frac{x}{1-x}\right) = \frac{x^p}{p}(1-x)^q{}_2F_1(p+q,1;p+1;x) \tag{6.4}$$

(cf. Temme [26], p. 128)

$$\int_0^\infty e^{-st}t^{\alpha-1}{}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;xt)dt = \frac{\Gamma(\alpha)}{s^\alpha}{}_{p+1}F_q\left(a_1,\dots,a_p,\alpha;b_1,\dots,b_q;\frac{x}{s}\right) \tag{6.5}$$

(cf. Andrews et al [2], p. 115)

when $p \le q$, $Re\, s > 0$, $Re\, \alpha > 0$.

$$I_x(p,q) = (1-x)I_x(p,q-1) + xI_x(p-1,q) \tag{6.6}$$

(cf. Srivastava and Choi [25], p. 12).

6.2. Covariance Matrix

Referring to Section 3, to investigate the covariance matrix

$$\Sigma_n^{(r,k)} = \begin{bmatrix} \text{Var}\left(\overline{V}_{n1}^{(r,k)}\right) & \text{Cov}\left(\overline{V}_{n1}^{(r,k)},\overline{V}_{n2}^{(r,k)}\right) \\ \text{Cov}\left(\overline{V}_{n1}^{(r,k)},\overline{V}_{n2}^{(r,k)}\right) & \text{Var}\left(\overline{V}_{n2}^{(r,k)}\right) \end{bmatrix}$$

when $k > 1$ we use Hoeffding's Theorem (cf. Lee [7], p. 12–17).

Theorem 2. *Suppose that $h_1(x_1,\dots,x_k)$ and $h_2(x_1,\dots,x_k)$ are given symmetric functions, X is a continuous variate, and $X_1,\dots,X_n$ are i.i.d. as X, where $n \ge k$. We denote the associated U–statistics by*

$$S_1 = \frac{1}{\binom{n}{k}}\Sigma' h_1(X_{i_1},\dots,X_{i_k}) \qquad \text{and} \qquad S_2 = \frac{1}{\binom{n}{k}}\Sigma' h_2(X_{i_1},\dots,X_{i_k})$$

where Σ' is defined in Section 3, and for $j = 1,\dots,k$ we define the functions

$$h_{1j}(x_1,\dots,x_j) = E\left[h_1\left(x_1,\dots,x_j,X_{j+1},\dots,X_k\right)\right]$$

and

$$h_{2j}(x_1,\dots,x_j) = E\left[h_2\left(x_1,\dots,x_j,X_{j+1},\dots,X_k\right)\right].$$

Then

$$\text{Cov}(S_1,S_2) = \frac{1}{\binom{n}{k}}\sum_{j=1}^k \binom{k}{j}\binom{n-k}{k-j}\zeta_j \tag{6.7}$$

where

$$\zeta_j = \mathrm{Cov}\left(h_{1j}(X_1, \ldots, X_j), h_{2j}(X_1, \ldots, X_j)\right).$$

$\mathrm{Var}(S_1)$ *is included here as the special case when* $h_1 = h_2$.

Referring to Section 3, to determine $\mathrm{Cov}\left(\overline{V}_{n1}^{(r,k)}, \overline{V}_{n2}^{(r,k)}\right)$ we apply the theorem with $X \sim \mathrm{Exp}(1)$ and

$$h_1(x_1, \ldots, x_k) = (\min(x_1, \ldots, x_k))^r,$$
$$h_2(x_1, \ldots, x_k) = (\min(x_1, \ldots, x_k))^{r+1}.$$

We consider first the term ζ_k in (6.7). Since

$$h_{1k}(x_1, \ldots, x_k) = E[h_1(x_1, \ldots, x_k)] = x_{1:k}^r$$

and

$$h_{2k}(x_1, \ldots, x_k) = E[h_2(x_1, \ldots, x_k)] = x_{1:k}^{r+1}$$

then

$$\begin{aligned}\zeta_k &= \mathrm{Cov}\left(X_{1:k}^r, X_{1:k}^{(r+1)}\right) \\ &= E\left[X_{1:k}^{2r+1}\right] - E\left[X_{1:k}^r\right] E\left[X_{1:k}^{r+1}\right].\end{aligned}$$

Now $X \sim \mathrm{Exp}(1)$ so $X_{1:k} \sim \mathrm{Exp}(k)$, whence

$$E\left[X_{1:k}^{2r+1}\right] = k \int_0^\infty x^{2r+1} e^{-kx} dx = \frac{\Gamma(2r+2)}{k^{2r+1}}$$

and so

$$\zeta_k = \frac{\Gamma(2r+2) - \Gamma(r+1)\Gamma(r+2)}{k^{2r+1}}.$$

Similarly

$$\begin{aligned}\eta_k &= \mathrm{Var}(h_1(X_1, \ldots, X_k)) = EX_{1:k}^{2r} - E^2 X_{1:k}^r \\ &= \frac{\Gamma(2r+1) - \Gamma^2(r+1)}{k^{2r}}.\end{aligned}$$

Now suppose that $j < k$. Then

$$h_{1j}(x_1, \ldots, x_j) = E\left[U_j^r\right] \qquad \text{and} \qquad h_{2j}(x_1, \ldots, x_j) = E\left[U_j^{r+1}\right] \tag{6.8}$$

where

$$U_j = \min(x_1, \ldots, x_j, X_{j+1}, \ldots, X_k).$$

Now we write

$$z_j = \min(x_1, \ldots, x_j) \qquad \text{and} \qquad Y_j = \min(X_{j+1}, \ldots, X_k)$$

so that

$$U_j = \begin{cases} z_j & \text{if } Y_j > z_j \\ Y_j & \text{if } Y_j < z_j. \end{cases}$$

Then U_j has possible values in $[0, z_j]$ and its distribution consists of a lump of probability at $U_j = z_j$, viz.

$$p_j = P[U_j = z_j] = P[Y_j > z_j],$$

and for $y \in [0, z_j)$ it has a density $f_j(y)$ equal to the density of Y_j. But since $Y_j \sim \mathrm{Exp}(k-j)$ then

$$p_j = e^{-(k-j)z_j} \qquad \text{and} \qquad f_j(y) = (k-j)e^{-(k-j)y}$$

and it follows that

$$\begin{aligned} EU_j^r &= z_j^r e^{-(k-j)z_j} + (k-j)\int_0^{z_j} y^r e^{-(k-j)y} dy \\ &= \phi_{rj}(z_j) \quad \text{say,} \end{aligned} \tag{6.9}$$

and the integral exists if $r > -1$.

Referring now to (6.8)–(6.9) it follows that

$$h_{1j}(X_1, \ldots, X_j) = \phi_{rj}(Z_j) \qquad \text{and} \qquad h_{2j}(X_1, \ldots, X_j) = \phi_{r+1j}(Z_j)$$

where

$$Z_j = \min(X_1, \ldots, X_j) \sim \mathrm{Exp}(j)$$

and then in (6.7)

$$\begin{aligned} \zeta_j &= \mathrm{Cov}(\phi_{rj}(Z_j), \phi_{r+1j}(Z_j)) \\ &= E\left[\phi_{rj}(Z_j)\phi_{r+1j}(Z_j)\right] - E\left[\phi_{rj}(Z_j)\right]E\left[\phi_{r+1j}(Z_j)\right]. \end{aligned} \tag{6.10}$$

and

$$\begin{aligned} \eta_j &= \mathrm{Var}\left(h_{1j}(X_1, \ldots, X_j)\right) = \mathrm{Var}\left(\phi_{rj}(Z_j)\right) \\ &= E\left[\phi_{rj}^2(Z_j)\right] - E^2\left[\phi_{rj}(Z_j)\right]. \end{aligned} \tag{6.11}$$

To investigate (6.10) and (6.11) for given j we write

$$\phi_{rj}(z) = H_r(z) + (k-j)G_r(z)$$

where

$$H_r(z) = z^r e^{-(k-j)z} \qquad \text{and} \qquad G_r(z) = \int_0^z y^r e^{-(k-j)y} dy.$$

Then

$$\begin{aligned} \eta_j = E\left[H_r^2(Z_j)\right] + 2(k-j)E\left[H_r(Z_j)G_r(Z_j)\right] + (k-j)^2 E\left[G_r^2(Z_j)\right] \\ - \left(E\left[H_r(Z_j)\right] + (k-j)E\left[G_r(Z_j)\right]\right)^2 \end{aligned} \tag{6.12}$$

and

$$\begin{aligned}\zeta_j &= E\left[H_r(Z_j)H_{r+1}(Z_j)\right] + (k-j)E\left[H_r(Z_j)G_{r+1}(Z_j)\right] \\ &\quad + (k-j)E\left[H_{r+1}(Z_j)G_r(Z_j)\right] + (k-j)^2E\left[G_r(Z_j)G_{r+1}(Z_j)\right] \\ &\quad - E\left[H_r(Z_j) + (k-j)G_r(Z_j)\right]E\left[H_{r+1}(Z_j) + (k-j)G_{r+1}(Z_j)\right].\end{aligned} \tag{6.13}$$

Referring to Section 6.1

$$G_r(z) = \int_0^z y^r e^{-(k-j)y}dy = \frac{1}{(k-j)^{r+1}}\gamma(r+1;(k-j)z)$$

and by (6.3) we have

$$G_r(z) = \frac{1}{r+1}e^{-(k-j)z}z^{r+1}{}_1F_1(1;r+2;(k-j)z). \tag{6.14}$$

Next since Z_j has pdf je^{-jz} then

$$E\left[H_r(Z_j)\right] = j\int_0^\infty z^r e^{-kz}dz = j\frac{\Gamma(r+1)}{k^{r+1}}$$

and by (6.14) and (6.5)

$$\begin{aligned}E\left[G_r(Z_j)\right] &= j\int_0^\infty e^{-jz}G_r(z)dz = \frac{j}{r+1}\int_0^\infty e^{-kz}z^{r+1}{}_1F_1(1;r+2;(k-j)z)dz \\ &= \frac{j}{r+1}\frac{\Gamma(r+2)}{k^{r+2}}{}_2F_1\left(1;r+2;r+2;\frac{k-j}{k}\right) = \frac{\Gamma(r+1)}{k^{r+1}}.\end{aligned}$$

Hence

$$E\phi_{rj}(Z_j) = EH_r(Z_j) + (k-j)EG_r(Z_j) = \frac{\Gamma(r+1)}{k^r}.$$

Now

$$E\left[H_r^2(Z_j)\right] = j\int_0^\infty z^{2r}e^{-(2k-j)z}dz = j\frac{\Gamma(2r+1)}{(2k-j)^{2r+1}}$$

and using (6.5) and (6.4) we have

$$\begin{aligned}E\left[H_r(Z_j)G_r(Z_j)\right] &= \frac{j}{r+1}\int_0^\infty e^{-(2k-j)z}z^{2r+1}{}_1F_1(1;r+2;(k-j)z)dz \\ &= \frac{j}{r+1}\frac{\Gamma(2r+2)}{(2k-j)^{2r+2}}{}_2F_1\left(1,2r+2;r+2;\frac{k-j}{2k-j}z\right) \\ &= \frac{j}{r+1}\frac{\Gamma(2r+2)}{(2k-j)^{2r+2}}{}_2F_1\left(2r+2,1;r+2;\frac{k-j}{2k-j}z\right) \\ &= \frac{j}{r+1}\frac{\Gamma(2r+2)}{(2k-j)^{2r+2}}(r+1)\left(\frac{k-j}{2k-j}\right)^{-(r+1)} \\ &\quad \times\left(1-\frac{k-j}{2k-j}\right)^{r+1}B_{\frac{k-j}{2k-j}}(r+1,r+1)\end{aligned}$$

$$= j\frac{\Gamma(2r+2)}{k^{r+1}(k-j)^{r+1}}B_{\frac{k-j}{2k-j}}(r+1,r+1) \qquad \text{if } r>-1/2.$$

Now using (6.2) we get

$$\begin{aligned}\frac{d}{dz}G_r(z) &= \frac{1}{r+1}\frac{d}{dz}\left[e^{-(k-j)z}z^{r+1}{}_1F_1(1;r+2;(k-j)z)\right]\\ &= \frac{1}{r+1}\left[(r+1)z^r{}_1F_1(r+1;r+1;-(k-j)z)\right]\\ &= z^r{}_1F_1\left[r+1;r+1;-(k-j)z\right]\\ &= z^re^{-(k-j)z}.\end{aligned}$$

Hence integrating by parts and using (6.5) and (6.4) we obtain

$$\begin{aligned}E\left[G_r^2(Z_j)\right] &= \int_0^\infty \left(je^{-jz}\right)G_r^2(z)dz\\ &= -\left[e^{-jz}G_r^2(z)\right]_0^\infty + 2\int_0^\infty e^{-jz}G_r(z)G_r'(z)dz\\ &= \frac{2}{r+1}\int_0^\infty e^{-(2k-j)z}z^{2r+1}{}_1F_1(1;r+2;(k-j)z)dz\\ &= \frac{2}{r+1}\cdot\frac{\Gamma(2r+2)}{(2k-j)^{2r+2}}\,{}_2F_1(1,2r+2;r+2;(k-j)z)\\ &= 2\frac{\Gamma(2r+2)}{k^{r+1}(k-j)^{r+1}}B_{\frac{k-j}{2k-j}}(r+1,r+1).\end{aligned}$$

Hence by (6.12)

$$\begin{aligned}\eta_j = &\frac{j\Gamma(2r+1)}{(2k-j)^{2r+1}} + 2(k-j)\frac{j\Gamma(2r+2)}{k^{r+1}(k-j)^{r+1}}B_{\frac{k-j}{2k-j}}(r+1,r+1)\\ &+(k-j)^2\frac{2\Gamma(2r+2)}{k^{r+1}(k-j)^{r+1}}B_{\frac{k-j}{2k-j}}(r+1,r+1) - \frac{\Gamma^2(r+1)}{k^{2r}}\\ &= 2\frac{\Gamma(2r+2)}{k^r(k-j)^r}B_{\frac{k-j}{2k-j}}(r+1,r+1) + j\frac{\Gamma(2r+1)}{(2k-j)^{2r+1}} - \frac{\Gamma^2(r+1)}{k^{2r}}.\end{aligned}$$

The corresponding derivation of ζ_j in (6.13) is similar. We have

$$E\left[H_r(Z_j)H_{r+1}(Z_j)\right] = j\int_0^\infty z^{2r+1}e^{-(2k-j)z}dz = j\frac{\Gamma(2r+2)}{(2k-j)^{2r+2}} \text{ if } r>-1$$

$$\begin{aligned}E\left[H_r(Z)G_{r+1}(Z_j)\right] &= \frac{j}{r+2}\int_0^\infty e^{-(2k-j)z}z^{2r+2}\;{}_1F_1(1;r+3;(k-j)z)dz\\ &= \frac{j}{r+2}\frac{\Gamma(2r+3)}{(2k-j)^{2r+3}}\;{}_2F_1(2r+3;1;r+3;\frac{k-j}{2k-j})\end{aligned}$$

$$= j\frac{\Gamma(2r+3)}{k^{r+1}(k-j)^{r+2}}B_{\frac{k-j}{2k-j}}(r+2,r+1)$$

$$\begin{aligned}
&E\left[H_{r+1}(Z_j)G_r(Z_j)\right] \\
&\qquad = j\frac{\Gamma(2r+3)}{(r+1)(2k-j)^{2r+3}}\ {}_2F_1\left(2r+3,1;r+2;\frac{k-j}{2k-j}\right) \\
&\qquad = j\frac{\Gamma(2r+3)}{k^{r+2}(k-j)^{r+1}}B_{\frac{k-j}{2k-j}}(r+1,r+2).
\end{aligned}$$

Now we see that

$$E\left[G_r(Z_j)G_{r+1}(Z_j)\right] = \int_0^\infty \left(je^{-jz}\right)G_r(z)G_{r+1}(z)dz.$$

Here we integrate by parts, giving

$$\begin{aligned}
E\left[G_r(Z_j)G_{r+1}(Z_j)\right] &= -\left[e^{-jz}G_r(z)G_{r+1}(z)\right]_0^\infty \\
&\quad + \int_0^\infty e^{-jz}\left(G_r'(z)G_{r+1}(z)+G_{r+1}'(z)G_r(z)\right)dz \\
&= \frac{1}{r+2}\int_0^\infty e^{-(2k-j)z}z^{2r+2}\ {}_1F_1(1;r+3;(k-j)z)dz \\
&\quad + \frac{1}{r+2}\int_0^\infty e^{-(2k-j)z}z^{2r+2}\ {}_1F_1(1;r+2;(k-j)z)dz \\
&= \frac{\Gamma(2r+3)}{k^{r+1}(k-j)^{r+2}}B_{\frac{k-j}{2k-j}}(r+2,r+1) + \frac{\Gamma(2r+3)}{k^{r+2}(k-j)^{r+1}}B_{\frac{k-j}{2k-j}}(r+1,r+2).
\end{aligned}$$

Therefore

$$\begin{aligned}
&E\left[\phi_{rj}(Z_j)\phi_{(r+1)j}(Z_j)\right] \\
&\quad = j\frac{\Gamma(2r+2)}{(2k-j)^{2r+2}} + (k-j)j\frac{\Gamma(2r+3)}{k^{r+1}(k-j)^{r+2}}B_{\frac{k-j}{2k-j}}(r+2,r+1) \\
&\qquad + (k-j)j\frac{\Gamma(2r+3)}{k^{r+2}(k-j)^{r+1}}B_{\frac{k-j}{2k-j}}(r+1,r+2) \\
&\qquad + (k-j)^2\left(\frac{\Gamma(2r+3)}{k^{r+1}(k-j)^{r+2}}B_{\frac{k-j}{2k-j}}(r+2,r+1)\right. \\
&\qquad\qquad \left.+\frac{\Gamma(2r+3)}{k^{r+2}(k-j)^{r+1}}B_{\frac{k-j}{2k-j}}(r+1,r+2)\right) \\
&\quad = j\frac{\Gamma(2r+2)}{(2k-j)^{2r+2}} + \frac{\Gamma(2r+3)}{k^{r+1}(k-j)^{r}}\left[\frac{k}{k-j}B_{\frac{k-j}{2k-j}}(r+2,r+1)\right. \\
&\qquad\qquad \left.+B_{\frac{k-j}{2k-j}}(r+1,r+2)\right]
\end{aligned}$$

$$= j\frac{\Gamma(2r+2)}{(2k-j)^{2r+2}} + (2k-j)j\frac{\Gamma(2r+3)}{k^{r+1}(k-j)^{r+1}}\left[\frac{k}{2k-j}B_{\frac{k-j}{2k-j}}(r+2,r+1) + \frac{k-j}{2k-j}B_{\frac{k-j}{2k-j}}(r+1,r+2)\right]$$

and after using (6.6) we have

$$E\left[\phi_{rj}(Z_j)\phi_{(r+1)j}(Z_j)\right] = j\frac{\Gamma(2r+2)}{(2k-j)^{2r+2}} + 2(2k-j)\frac{(2r+3)\Gamma(2r+2)}{k^{r+1}(k-j)^{r+1}}B_{\frac{k-j}{2k-j}}(r+2,r+2).$$

Finally we have

$$\zeta_j = 2(2k-j)\frac{(2r+3)\Gamma(2r+2)}{k^{r+1}(k-j)^{r+1}}B_{\frac{k-j}{2k-j}}(r+2,r+2) + j\frac{\Gamma(2r+2)}{(2k-j)^{2r+2}} - \frac{\Gamma(r+1)\Gamma(r+2)}{k^{2r+1}}$$
$$= 2(2k-j)\frac{\Gamma(2r+2)+\Gamma(2r+3)}{k^{r+1}(k-j)^{r+1}}B_{\frac{k-j}{2k-j}}(r+2,r+2) + j\frac{\Gamma(2r+2)}{(2k-j)^{2r+2}} - \frac{\Gamma(r+1)\Gamma(r+2)}{k^{2r+1}}.$$

Therefore

$$\mathrm{Var}\left(\overline{V}_{n1}^{(r,k)}\right) = \frac{1}{\binom{n}{k}}\sum_{j=1}^{k}\binom{k}{j}\binom{n-k}{k-j}\eta_j^{(r)}$$
$$= \frac{1}{\binom{n}{k}}\sum_{j=1}^{k-1}\binom{k}{j}\binom{n-k}{k-j}\left[2\frac{\Gamma(2r+2)}{k^r(k-j)^r}B_{\frac{k-j}{2k-j}}(r+1,r+1) + j\frac{\Gamma(2r+1)}{(2k-j)^{2r+1}} - \frac{\Gamma^2(r+1)}{k^{2r}}\right] + \frac{1}{\binom{n}{k}}\cdot\frac{\Gamma(2r+1)-\Gamma^2(r+1)}{k^{2r}}$$

$$\mathrm{Cov}\left(\overline{V}_{n1}^{(r,k)},\overline{V}_{n2}^{(r,k)}\right) = \frac{1}{\binom{n}{k}}\sum_{j=1}^{k}\binom{k}{j}\binom{n-k}{k-j}\zeta_j$$
$$= \frac{1}{\binom{n}{k}}\sum_{j=1}^{k-1}\binom{k}{j}\binom{n-k}{k-j}$$
$$\times\left[2(2k-j)\frac{\Gamma(2r+2)+\Gamma(2r+3)}{k^{r+1}(k-j)^{r+1}}B_{\frac{k-j}{2k-j}}(r+2,r+2)\right.$$

$$+j\frac{\Gamma(2r+2)}{(2k-j)^{2r+2}}-\frac{\Gamma(r+1)\Gamma(r+2)}{k^{2r+1}}\Bigg] + \frac{1}{\binom{n}{k}}\cdot\frac{\Gamma(2r+2)-\Gamma(r+1)\Gamma(r+2)}{k^{2r+1}}$$

$$\mathrm{Var}\left(\overline{V}_{n2}^{(r,k)}\right)=\frac{1}{\binom{n}{k}}\sum_{j=1}^{k}\binom{k}{j}\binom{n-k}{k-j}\eta_j^{(r+1)}$$

$$=\frac{1}{\binom{n}{k}}\sum_{j=1}^{k-1}\binom{k}{j}\binom{n-k}{k-j}\left[2\frac{\Gamma(2r+4)}{k^{r+1}(k-j)^{r+1}}B_{\frac{k-j}{2k-j}}(r+2,r+2)\right.$$
$$\left.+j\frac{\Gamma(2r+3)}{(2k-j)^{2r+3}}-\frac{\Gamma^2(r+2)}{k^{2r+2}}\right]+\frac{1}{\binom{n}{k}}\cdot\frac{\Gamma(2r+3)-\Gamma^2(r+2)}{k^{2r+2}}.$$

7. Simulations

7.1. Exponential Distribution

7.1.1.

We have selected tests and alternatives in Table 1 from Cabaña and Cabaña [3] as standards of comparison with our tests. When $n=20$ the test–statistics $\hat{T}_n^{(r,k)}$, $\hat{T}_n^{*(r,k)}$ and their components were investigated for $r=-0.499$, -0.45, -0.4, -0.1, 0.1, 0.3, 0.5, 0.7, 0.9, 1.0, 1.1, 1.3, 1.5, 1.7, 2.0 with $k=3$ and $k=4$. (The case $k=2$ was discussed in Morris and Szynal [19] and Morris et al [12]). Though $\hat{T}_{n;c1}^{(r,k)}$ and $\hat{T}_{n;c3}^{(r+1,k)}$ are identical, and similarly for duals, we do simulations for all tests. Critical values were simulated using 100,000 samples of size 20, and the associated powers were obtained using 50,000 samples, but only some results are presented here. We include here (Table 2 and Table 3) simulations for some of our favorable omnibus tests with Av. powers ≥ 46. For samples of size 50 we include (Table 4, Table 5 and Table 6) simulations for tests with Av. power ≥ 74.5. Moreover, many tests have powers greater than those in Table 1. These are shown in boldface, while bold face with star denotes the maximum power.

The meaning of the headings and the test–statistics can be found in Cabaña and Cabaña [3] and Henze and Meintanis [6] and their references:

$TEEP$: The transformed estimated empirical process (TEEP) test

	Alt.	$TEEP$	EP	$BHKS$	$BHCM$	S	CO	BH	T
$n = 20$	$W(0.8)$	26	24	17	22	24	28	24	1
	$W(1.4)$	35	36	28	35	35	37	37	**45***
	$\Gamma(0.4)$	85	76	71	75	76	91	80	11
	$\Gamma(2)$	54	48	40	47	46	54	51	56
	$LN(0.8)$	37	25	30	27	24	33	29	34
	$LN(1.5)$	64	67	58	66	67	60	66	2
	HN	17	21	18	22	21	19	21	**31***
	U	47	66	52	70	70	50	61	**82***
	$CH(0.5)$	72	63	56	61	63	80	67	6
	$CH(1)$	12	15	13	16	15	13	15	**23***
	$CH(1.5)$	77	84	67	83	84	81	83	**89***
	$LF(2)$	24	28	24	30	29	25	25	**39***
	$LF(4)$	36	42	34	43	42	37	41	**54***
	$EV(0.5)$	11	13	18	16	15	13	15	**23***
	$EV(1.5)$	35	35	48	47	46	37	43	**58***
	$DL(1)$	25	20	20	21	19	25	23	28
	$DL(1.5)$	72	64	56	63	62	72	68	71
	Av	42.9	42.8	38.2	43.8	43.4	44.4	44.1	38.4
$n = 50$	$W(0.8)$	53	48	35	46	48	56	50	17
	$W(1.4)$	82	80	71	77	79	82	81	81
	$\Gamma(0.4)$	**100***	99	97	99	98	**100***	99	90
	$\Gamma(2)$	94	91	86	90	90	**96***	93	92
	$LN(0.8)$	73	45	62	60	47	66	58	66
	$LN(1.5)$	95	95	92	95	95	92	95	54
	HN	45	54	50	53	54	45	52	**60***
	U	93	98	99	99	99	91	97	**100***
	$CH(0.5)$	98	94	90	94	94	99	96	79
	$CH(1)$	31	38	36	37	38	30	35	**44***
	$CH(1.5)$	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***
	$LF(2)$	61	69	65	69	69	60	68	**74***
	$LF(4)$	81	87	82	87	87	80	86	**90***
	$EV(0.5)$	30	38	36	37	38	30	35	**44***
	$EV(1.5)$	80	90	88	90	90	78	87	**93***
	$DL(1)$	57	39	43	44	39	55	47	54
	$DL(1.5)$	99	97	96	97	97	99	98	98
	Av	74.8	74.2	72.2	74.9	74.2	74.1	75.1	72.7

Table 1: (Source: Cabaña and Cabaña [3]) Empirical comparision of the performances of the test $TEEP$ based on 100 000 replications and of seven other tests, under several alternatives. The entries are simulated powers of 5% tests.

EP: The Epps and Pulley test

$BHKS$: The Baringhaus and Henze test suggested by the Kolmogorov–Smirnov statistic

$BHCM$: The Baringhaus and Henze test suggested by the Cramér–von Mises statistic

S: The test based on spacing

CO: The Cox and Oakes test

BH: The Baringhaus and Henze test based on the empirical Laplace transform

T: The Henze and Meintanis test based on the empirical characteristic function (ECF).

The alternatives considered are:

$W(\theta)$ – Weibull distribution with parameters $(1, \theta)$

$\Gamma(\theta)$ – Gamma distribution with parameters $(1, \theta)$

$LN(\theta)$ – Lognormal distribution with parameters $(0, \theta)$

HN – Half–normal distribution: the law of $|Z|$, Z standard normal

U – Uniform distribution on $[0, 1]$

$EV(\theta)$ – Modified extreme value distribution: the law of $\log(1 - \theta \log U)$, U uniform on $[0, 1]$

$LF(\theta)$ – Linear increasing failure rate: the law of $\theta^{-1}(\sqrt{1 + 2Y\theta} - 1)$, $Y \sim \mathrm{Exp}(1)$

$DL(\theta)$ – Dhillon's Distribution: the law of $e^{(-\log U)^{1/(\theta+1)}} - 1$, U uniform on $[0, 1]$

$CH(\theta)$ – Chen Distribution: the law of $\left(\log\left(1 - \frac{1}{2}\log U\right)\right)^{1/\theta}$, U uniform on $[0, 1]$.

r	Alt.	$\hat{T}_n^{(r,3)}$	$\hat{T}_{n;c_1}^{(r,3)}$	$\hat{T}_{n;c_2}^{(r,3)}$	$\hat{T}_{n;c_3}^{(r,3)}$	$\hat{T}_{n;c_4}^{(r,3)}$	$\hat{T}_n^{*(r,3)}$	$\hat{T}_{n;c_1}^{*(r,3)}$	$\hat{T}_{n;c_2}^{*(r,3)}$	$\hat{T}_{n;c_3}^{*(r,3)}$	$\hat{T}_{n;c_4}^{*(r,3)}$
	$W(0.8)$	24	28	10	21	18	24	24	12	21	19
	$W(1.4)$	26	18	23	42	1	25	26	5	1	32
	$\Gamma(0.4)$	91	**93***	56	82	84	67	67	43	63	35
	$\Gamma(2.0)$	39	32	32	**59**	0	35	33	9	0	43
	$LN(0.8)$	24	26	16	**40**	1	21	16	10	6	25
	$LN(1.5)$	46	28	43	55	5	**68**	67	46	66	40
	HN	16	8	18	24	5	15	17	4	4	19
	U	51	21	56	60	9	67	70	19	6	66
-0.1	$CH(0.5)$	79	**83***	36	69	69	54	54	30	49	33
	$CH(1.0)$	11	6	13	16	5	10	11	5	5	12
	$CH(1.5)$	69	48	67	83	2	75	77	29	0	79
	$LF(2.0)$	22	10	23	31	5	21	23	5	3	25
	$LF(4.0)$	32	16	34	45	4	34	35	7	3	38
	$EV(0.5)$	11	6	13	16	5	10	11	5	5	12
	$EV(1.5)$	33	15	36	44	6	38	41	8	5	41
	$DL(1.0)$	16	14	12	**29**	0	15	13	5	3	18
	$DL(1.5)$	58	52	48	**76**	0	50	47	19	0	61
	Av.	38.2	29.6	31.6	46.7	12.9	37.0	37.1	15.4	14.0	35.2
	$W(0.8)$	24	27	8	20	18	24	21	18	22	17

	$W(1.4)$	29	30	15	42	1	30	34	1	2	33
	$\Gamma(0.4)$	91	**93***	54	78	85	77	71	61	69	38
	$\Gamma(2.0)$	42	49	19	**57**	0	39	44	0	1	44
	$LN(0.8)$	27	**41**	9	36	6	22	22	6	4	25
	$LN(1.5)$	49	36	28	58	8	67	35	63	**68**	23
	HN	18	13	15	24	4	20	22	3	6	20
	U	58	32	55	65	10	73	73	2	16	69
0.1	$CH(0.5)$	80	**83***	34	65	71	64	63	46	55	40
	$CH(1.0)$	13	9	11	17	5	13	15	4	6	13
	$CH(1.5)$	73	65	55	84	1	81	83	1	2	81
	$LF(2.0)$	24	18	19	32	4	26	29	3	6	27
	$LF(4.0)$	36	27	27	46	4	40	43	2	5	40
	$EV(0.5)$	13	9	11	17	5	13	15	4	6	13
	$EV(1.5)$	38	24	32	47	6	45	47	3	9	43
	$DL(1.0)$	18	25	7	27	2	16	18	3	2	19
	$DL(1.5)$	61	70	29	**74**	0	54	60	1	0	61
	Av.	40.8	38.3	25.2	46.4	13.5	41.5	40.9	13.0	16.4	35.7
	$W(0.8)$	19	22	5	20	11	28	18	24	26	18
	$W(1.4)$	37	42	3	36	13	9	0	17	8	6
	$\Gamma(0.4)$	84	86	22	66	74	84	54	81	80	63
	$\Gamma(2.0)$	54	**59**	3	46	25	14	0	25	6	11
	$LN(0.8)$	**55**	**43**	24	24	**54**	13	7	15	3	14
	$LN(1.5)$	51	52	6	64	12	**68**	62	48	**69**	31
	HN	21	22	6	24	6	5	3	9	10	3
	U	63	55	32	73	4	36	2	48	55	24
0.7	$CH(0.5)$	71	73	14	53	58	73	40	70	67	52
	$CH(1.0)$	14	15	5	17	5	3	3	6	8	3
	$CH(1.5)$	78	81	11	83	7	44	0	60	24	33
	$LF(2.0)$	27	29	6	32	6	7	2	13	13	5
	$LF(4.0)$	40	42	7	46	7	13	1	22	17	8
	$EV(0.5)$	14	15	5	17	5	3	4	6	8	3
	$EV(1.5)$	41	41	12	50	5	15	2	25	26	9
	$DL(1.0)$	**33**	**31**	12	19	28	8	4	11	3	8
	$DL(1.5)$	**74**	**78**	3	62	35	25	0	39	4	21
	Av.	45.7	46.3	10.5	42.9	21.0	26.3	11.9	30.6	25.2	18.4
	$W(0.8)$	17	21	4	20	9	**29**	21	24	27	18
	$W(1.4)$	39	42	4	33	23	3	1	7	10	2
	$\Gamma(0.4)$	81	82	16	62	67	85	63	81	83	66
	$\Gamma(2.0)$	**57**	**59**	6	41	41	5	0	12	8	4
	$LN(0.8)$	**56**	**40**	34	20	**62**	10	6	12	3	11
	$LN(1.5)$	52	55	4	65	14	**68**	66	43	**70***	31
	HN	22	24	4	23	9	1	4	3	11	1
	U	64	60	21	74	7	16	6	25	60	9
0.9	$CH(0.5)$	66	69	11	49	51	74	49	70	70	54
	$CH(1.0)$	15	16	4	16	7	1	5	3	8	2
	$CH(1.5)$	79	83	4	81	27	21	0	34	34	12
	$LF(2.0)$	28	31	4	30	11	2	3	5	14	1
	$LF(4.0)$	41	45	4	43	15	4	3	9	20	2
	$EV(0.5)$	15	16	5	16	7	1	5	3	8	2
	$EV(1.5)$	42	44	7	49	10	5	5	10	28	3
	$DL(1.0)$	**35**	**29**	18	16	**35**	5	3	7	3	6
	$DL(1.5)$	**77**	**76**	7	56	58	11	0	22	7	8
	Av.	46.3	46.7	9.3	40.7	26.7	20.1	14.0	21.8	27.3	13.7
	$W(0.8)$	17	21	4	20	8	**29**	22	24	28	18
	$W(1.4)$	40	42	5	31	27	1	1	4	10	1
	$\Gamma(0.4)$	79	80	14	60	64	86	66	81	84	68

	$\Gamma(2.0)$	**58**	**58**	9	38	47	3	0	7	9	2
	$LN(0.8)$	**56**	**38**	**40**	18	**63**	8	5	11	3	10
	$LN(1.5)$	53	57	3	65	14	67	67	42	**70***	32
	HN	22	24	4	22	11	1	5	2	10	1
	U	64	63	16	75	10	9	10	14	60	4
1.0	$CH(0.5)$	64	67	10	47	48	74	52	70	72	55
	$CH(1.0)$	15	17	4	15	9	1	5	2	8	2
	$CH(1.5)$	80	84	2	80	37	11	1	21	38	6
	$LF(2.0)$	29	32	4	29	14	1	4	2	13	1
	$LF(4.0)$	42	46	4	42	19	2	4	5	20	1
	$EV(0.5)$	15	16	4	15	9	1	5	2	8	2
	$EV(1.5)$	42	46	5	48	14	2	7	5	28	1
	$DL(1.0)$	**35**	**29**	21	15	**37**	4	3	6	3	5
	$DL(1.5)$	**77**	**75**	12	52	66	6	0	14	8	4
	Av.	46.4	46.7	9.5	39.5	29.2	18.0	15.2	18.3	27.7	12.6
	$W(0.8)$	15	20	3	20	7	**29**	24	23	**30**	19
	$W(1.4)$	42	41	13	24	37	0	4	0	10	0
	$\Gamma(0.4)$	72	74	12	53	54	86	74	82	86	72
	$\Gamma(2.0)$	**59**	54	25	29	**58**	0	2	1	10	0
	$LN(0.8)$	**51**	32	**53**	12	**59**	7	3	9	3	9
	$LN(1.5)$	54	60	2	66	17	67	**69**	40	**70***	34
	HN	24	25	6	18	17	1	8	1	9	1
	U	65	68	7	74	24	1	30	1	56	1
1.3	$CH(0.5)$	57	60	10	42	39	76	60	69	75	58
	$CH(1.0)$	16	17	5	13	12	1	7	1	6	2
	$CH(1.5)$	82	85	8	74	61	0	6	1	43	0
	$LF(2.0)$	31	33	6	24	22	0	8	0	12	1
	$LF(4.0)$	44	47	7	36	31	0	9	0	19	0
	$EV(0.5)$	16	17	5	13	12	1	7	1	6	2
	$EV(1.5)$	44	48	5	44	26	0	15	0	26	1
	$DL(1.0)$	**34**	25	**29**	11	**39**	3	2	4	3	5
	$DL(1.5)$	**78**	71	36	41	**77**	1	1	2	11	1
	Av.	46.3	45.7	13.6	35.0	34.9	16.1	19.3	14.0	27.9	12.1

Table 2: Powers of 5% tests based on 50,000 simulations using empirical critical values; $n = 20$ $(k = 3)$

r	Alt.	$\hat{T}_n^{(r,4)}$	$\hat{T}_{n;c_1}^{(r,4)}$	$\hat{T}_{n;c_2}^{(r,4)}$	$\hat{T}_{n;c_3}^{(r,4)}$	$\hat{T}_{n;c_4}^{(r,4)}$	$\hat{T}_n^{*(r,4)}$	$\hat{T}_{n;c_1}^{*(r,4)}$	$\hat{T}_{n;c_2}^{*(r,4)}$	$\hat{T}_{n;c_3}^{*(r,4)}$	$\hat{T}_{n;c_4}^{*(r,4)}$
	$W(0.8)$	20	25	7	16	18	25	24	15	16	24
	$W(1.4)$	30	26	23	44	2	26	29	1	0	29
	$\Gamma(0.4)$	90	**92**	59	78	85	75	74	41	46	75
	$\Gamma(2.0)$	46	46	32	**62**	1	33	37	1	0	38
	$LN(0.8)$	32	**42**	17	**47**	1	17	16	7	6	17
	$LN(1.5)$	27	27	16	45	6	**68**	55	54	58	46
	HN	18	11	18	23	6	17	19	2	3	19
	U	50	27	53	57	14	75	77	4	2	75
0.1	$CH(0.5)$	77	**82**	38	64	71	62	62	30	34	64
	$CH(1.0)$	13	8	13	17	5	12	14	3	3	14
	$CH(1.5)$	71	59	65	83	8	78	81	4	1	81
	$LF(2.0)$	24	16	23	32	6	24	27	2	2	27
	$LF(4.0)$	35	24	33	45	6	36	40	1	1	40

	$EV(0.5)$	13	8	13	17	6	12	14	3	4	14
	$EV(1.5)$	35	21	36	44	8	43	46	2	2	46
	$DL(1.0)$	21	24	12	**33**	1	13	13	3	3	14
	$DL(1.5)$	65	67	47	**79**	2	46	50	2	1	52
	Av.	39.2	35.6	29.8	46.3	14.5	38.8	39.9	10.3	10.6	39.6
	$W(0.8)$	20	24	6	16	17	24	20	19	17	21
	$W(1.4)$	34	35	18	**45***	2	27	27	2	1	29
	$\Gamma(0.4)$	89	91	54	75	85	75	56	59	53	66
	$\Gamma(2.0)$	50	55	22	**61**	1	33	35	1	0	37
	$LN(0.8)$	**38**	**50**	12	**44**	9	17	18	5	6	17
	$LN(1.5)$	31	33	11	49	9	66	30	65	61	34
	HN	20	15	16	25	5	18	17	6	4	19
	U	56	35	52	62	14	77	70	19	3	74
0.3	$CH(0.5)$	77	80	33	61	70	63	52	46	40	58
	$CH(1.0)$	14	11	12	18	5	13	11	7	5	13
	$CH(1.5)$	76	69	57	84	6	79	77	2	0	80
	$LF(2.0)$	27	21	21	34	5	25	23	6	3	26
	$LF(4.0)$	40	31	29	48	5	38	36	5	2	39
	$EV(0.5)$	14	11	12	18	5	13	11	7	5	13
	$EV(1.5)$	40	28	33	47	8	45	41	10	3	45
	$DL(1.0)$	25	**31**	9	**33**	4	13	14	3	3	14
	$DL(1.5)$	69	**76**	34	**79**	1	46	50	0	0	51
	Av.	42.4	40.8	25.4	**47.0***	14.7	39.5	34.5	15.5	12.0	37.5
	$W(0.8)$	17	21	5	15	14	24	17	21	19	19
	$W(1.4)$	37	39	13	**45***	3	24	14	24	1	25
	$\Gamma(0.4)$	87	89	43	71	82	75	37	68	59	62
	$\Gamma(2.0)$	53	**59**	14	**60**	3	31	21	26	0	33
	$LN(0.8)$	**48**	**52**	12	**41**	22	17	14	10	5	17
	$LN(1.5)$	31	37	7	51	10	65	36	64	64	31
	HN	21	18	13	26	4	16	8	20	5	16
	U	59	42	49	65	12	73	49	**83***	7	69
0.5	$CH(0.5)$	73	77	25	57	65	63	34	55	45	54
	$CH(1.0)$	15	13	10	19	4	11	5	14	6	11
	$CH(1.5)$	77	75	46	85	3	76	56	76	0	75
	$LF(2.0)$	28	25	16	35	4	22	11	25	4	22
	$LF(4.0)$	41	36	23	49	4	34	19	37	3	34
	$EV(0.5)$	15	13	10	19	4	11	6	15	6	11
	$EV(1.5)$	41	33	28	49	6	41	22	48	6	39
	$DL(1.0)$	**29**	**34**	8	**30**	11	12	9	10	3	13
	$DL(1.5)$	**73**	**79**	21	**77**	3	43	33	32	0	47
	Av.	43.8	43.6	20.2	46.7	14.9	37.7	23.1	36.9	13.6	33.8
	$W(0.8)$	11	17	3	15	7	26	15	23	22	19
	$W(1.4)$	41	44	7	42	18	10	0	18	3	9
	$\Gamma(0.4)$	77	81	13	63	65	78	42	76	70	65
	$\Gamma(2.0)$	**60**	**63***	8	55	32	15	1	25	1	15
	$LN(0.8)$	**61**	**49**	33	32	**60**	12	7	13	3	13
	$LN(1.5)$	33	45	2	56	11	66	56	54	67	34
	HN	22	23	7	27	8	6	2	11	8	5
	U	61	56	32	71	7	44	3	57	24	37
1.0	$CH(0.5)$	60	67	8	49	47	66	31	64	56	54
	$CH(1.0)$	16	17	7	19	7	4	3	7	7	4
	$CH(1.5)$	79	83	16	85	12	48	2	63	4	43
	$LF(2.0)$	30	32	8	36	9	9	1	15	8	7
	$LF(4.0)$	42	45	9	50	11	15	1	25	7	13
	$EV(0.5)$	16	17	7	19	7	4	3	7	7	3
	$EV(1.5)$	42	43	14	52	7	18	1	29	13	15

	$DL(1.0)$	**38**	**35**	17	25	**32**	8	4	10	2	8
	$DL(1.5)$	**79**	**80**	9	70	43	25	1	37	0	25
	Av.	45.2	46.9	11.7	45.0	22.5	26.6	10.2	31.4	17.9	21.7
	$W(0.8)$	7	15	2	16	3	27	19	23	25	19
	$W(1.4)$	44	**45***	16	37	33	1	1	3	6	1
	$\Gamma(0.4)$	61	72	4	58	37	81	59	78	77	69
	$\Gamma(2.0)$	**63***	**60**	29	46	54	2	0	5	3	2
	$LN(0.8)$	**57**	**41**	**56**	23	**63**	8	5	10	3	10
	$LN(1.5)$	33	51	1	60	8	66	63	48	**69**	38
	HN	24	26	8	25	15	1	5	2	9	1
	U	62	65	14	74	17	8	7	13	42	6
1.5	$CH(0.5)$	44	57	4	45	23	69	45	65	64	57
	$CH(1.0)$	17	18	7	18	11	1	5	1	7	1
	$CH(1.5)$	81	85	11	83	49	8	0	16	12	7
	$LF(2.0)$	32	35	9	34	19	1	4	2	10	1
	$LF(4.0)$	45	49	10	47	26	1	3	3	12	1
	$EV(0.5)$	17	19	7	18	11	1	5	1	7	2
	$EV(1.5)$	44	49	8	51	20	2	5	4	19	1
	$DL(1.0)$	**38**	**31**	**31**	19	**39**	5	2	6	2	6
	$DL(1.5)$	**81***	**77**	39	61	**73**	5	0	11	2	5
	Av.	44.1	46.8	15.2	42.1	29.4	16.7	13.5	17.2	21.7	13.4
	$W(0.8)$	6	15	2	17	2	27	20	22	26	19
	$W(1.4)$	**45***	43	23	34	38	0	2	1	7	0
	$\Gamma(0.4)$	56	68	4	56	27	82	64	78	79	70
	$\Gamma(2.0)$	**63***	**58**	40	43	**59**	1	0	2	4	1
	$LN(0.8)$	**55**	37	**60**	20	**61**	8	4	9	3	9
	$LN(1.5)$	34	53	1	61	6	66	65	47	**69**	39
	HN	25	27	10	24	18	1	6	1	9	1
	U	64	68	12	75	24	2	13	4	48	2
1.7	$CH(0.5)$	39	54	3	44	17	70	50	65	67	57
	$CH(1.0)$	18	19	9	18	13	1	7	1	8	1
	$CH(1.5)$	82	85	23	81	60	2	1	5	17	2
	$LF(2.0)$	33	35	12	32	23	0	5	1	11	1
	$LF(4.0)$	47	50	16	45	32	0	5	1	13	0
	$EV(0.5)$	18	19	8	18	13	1	6	1	7	1
	$EV(1.5)$	46	51	11	51	26	0	8	1	22	1
	$DL(1.0)$	**38**	**29**	**36**	17	**40***	4	2	5	2	5
	$DL(1.5)$	**81***	**75**	55	56	**77**	2	0	5	3	2
	Av.	44.1	46.2	19.1	40.7	31.6	15.7	15.3	14.7	23.3	12.5

Table 3: Powers of 5% tests based on 100,000 simulations using empirical critical values; $n = 20$ $(k = 4)$

r	Alt.	$\hat{T}_n^{(r,2)}$	$\hat{T}_{n;c_1}^{(r,2)}$	$\hat{T}_{n;c_2}^{(r,2)}$	$\hat{T}_{n;c_3}^{(r,2)}$	$\hat{T}_{n;c_4}^{(r,2)}$	$\hat{T}_n^{*(r,2)}$	$\hat{T}_{n;c_1}^{*(r,2)}$	$\hat{T}_{n;c_2}^{*(r,2)}$	$\hat{T}_{n;c_3}^{*(r,2)}$	$\hat{T}_{n;c_4}^{*(r,2)}$
	$W(0.8)$	53	39	45	53	21	49	35	47	48	40
	$W(1.4)$	76	9	81	**83***	0	39	18	45	45	0
	$\Gamma(0.4)$	**100***	**100***	73	**100***	97	99	81	98	99	90
	$\Gamma(2.0)$	92	29	94	95	0	28	27	31	35	0
	$LN(0.8)$	63	22	64	71	0	7	12	4	6	8
	$LN(1.5)$	91	27	94	92	27	**96***	91	89	**96***	83
	HN	42	3	49	49	5	41	9	48	46	0
	U	94	10	96	95	8	99	91	99	99	0

-0.45	$CH(0.5)$	99	98	73	98	87	95	67	94	94	86
	$CH(1.0)$	28	3	34	33	5	30	4	37	34	0
	$CH(1.5)$	**100***	49	**100***	**100***	1	87	94	83	92	0
	$LF(2.0)$	58	4	66	65	4	53	17	60	59	0
	$LF(4.0)$	79	8	85	84	4	70	35	75	75	0
	$EV(0.5)$	27	3	34	33	5	30	4	37	34	0
	$EV(1.5)$	79	6	85	84	6	88	49	91	91	0
	$DL(1.0)$	49	6	52	**58**	0	7	**7**	7	8	5
	$DL(1.5)$	99	62	99	99	0	21	43	21	28	0
	Av.	72.2	28.1	71.9	76.0	15.9	55.2	40.3	56.8	58.3	18.3
	$W(0.8)$	52	53	16	50	27	52	43	42	52	27
	$W(1.4)$	66	68	34	81	0	60	73	3	70	30
	$\Gamma(0.4)$	**100***	**100***	76	99	99	99	92	98	99	93
	$\Gamma(2.0)$	86	93	40	92	0	75	85	10	74	55
	$LN(0.8)$	64	**86**	9	53	22	32	39	9	18	33
	$LN(1.5)$	93	70	92	95	13	95	94	78	**96***	70
	HN	40	24	33	54	5	36	49	0	53	9
	U	97	63	97	99	14	98	99	19	99	78
-0.1	$CH(0.5)$	99	99	49	96	94	97	83	95	97	84
	$CH(1.0)$	27	16	24	39	5	23	34	0	38	4
	$CH(1.5)$	**100***	97	97	**100***	0	**100***	**100***	43	**100***	93
	$LF(2.0)$	56	37	44	71	4	53	66	1	68	17
	$LF(4.0)$	77	55	63	88	3	76	85	4	86	36
	$EV(0.5)$	27	15	24	38	5	23	34	0	38	4
	$EV(1.5)$	82	50	76	90	6	82	89	4	92	39
	$DL(1.0)$	38	**62**	7	46	8	23	33	4	21	17
	$DL(1.5)$	97	99	58	98	0	89	94	30	81	82
	Av.	70.6	64.0	49.4	75.8	17.9	65.5	70.1	25.9	69.4	45.5
	$W(0.8)$	48	53	9	41	27	55	47	33	56	20
	$W(1.4)$	71	82	1	64	34	37	35	19	73	1
	$\Gamma(0.4)$	**100***	**100***	47	89	99	**100***	98	98	**100***	75
	$\Gamma(2.0)$	91	95	1	72	70	59	22	44	85	3
	$LN(0.8)$	**93**	73	42	20	**96**	50	6	60	32	32
	$LN(1.5)$	93	92	56	95	24	95	**96***	36	94	75
	HN	41	47	8	51	7	14	42	4	45	1
	U	99	94	78	**100***	2	80	97	32	95	2
0.5	$CH(0.5)$	98	99	34	79	94	99	93	93	99	60
	$CH(1.0)$	28	32	8	37	5	7	32	2	30	2
	$CH(1.5)$	**100***	**100***	14	**100***	37	96	74	72	**100***	7
	$LF(2.0)$	57	63	8	66	10	25	53	8	62	0
	$LF(4.0)$	78	83	9	84	15	47	67	18	82	0
	$EV(0.5)$	28	32	8	37	5	7	32	2	30	2
	$EV(1.5)$	85	82	23	92	6	50	87	14	83	0
	$DL(1.0)$	**65**	**59**	20	20	**69**	28	6	32	32	15
	$DL(1.5)$	99	99	2	84	90	84	14	75	94	12
	Av.	75.0	75.5	21.5	66.4	40.4	54.7	53.1	37.7	70.0	18.1
	$W(0.8)$	46	51	9	36	28	55	50	31	**57**	20
	$W(1.4)$	72	82	1	52	54	27	60	2	72	0
	$\Gamma(0.4)$	99	**100***	43	77	98	**100***	99	98	**100***	87
	$\Gamma(2.0)$	91	94	4	56	86	45	58	8	86	0
	$LN(0.8)$	**90**	63	54	13	**96**	39	10	42	37	24
	$LN(1.5)$	94	93	44	94	41	95	**96***	35	94	72
	HN	41	51	4	44	15	11	50	1	41	2
	U	99	97	53	**100***	14	90	99	1	92	2
0.7	$CH(0.5)$	97	97	33	64	94	99	96	94	99	72
	$CH(1.0)$	27	35	4	32	9	6	36	1	27	3

	$CH(1.5)$	**100***	**100***	2	**100***	79	94	**100***	9	**100***	0
	$LF(2.0)$	57	67	3	58	23	20	65	0	58	1
	$LF(4.0)$	79	86	2	77	37	40	83	1	79	0
	$EV(0.5)$	27	35	4	32	9	6	37	1	27	3
	$EV(1.5)$	84	86	9	90	23	49	92	0	79	1
	$DL(1.0)$	**64**	52	28	12	**74***	19	14	19	34	11
	$DL(1.5)$	99	99	6	69	98	72	61	22	95	1
	Av.	74.5	75.8	17.8	59.1	51.7	50.9	65.0	21.5	69.1	17.6
0.9	$W(0.8)$	45	49	9	31	32	55	52	30	**58**	20
	$W(1.4)$	72	81	2	34	69	24	68	0	71	0
	$\Gamma(0.4)$	99	99	45	57	98	**100***	99	99	**100***	92
	$\Gamma(2.0)$	90	92	9	34	92	35	72	1	86	0
	$LN(0.8)$	**86**	53	62	11	**92**	27	17	30	40	18
	$LN(1.5)$	94	95	35	90	62	94	**96***	35	93	69
	HN	42	54	1	33	27	13	52	1	37	3
	U	99	99	25	**100***	46	94	99	1	89	6
	$CH(0.5)$	96	96	36	45	94	99	97	94	**100***	78
	$CH(1.0)$	28	38	2	24	17	8	37	2	24	5
	$CH(1.5)$	**100***	**100***	0	98	95	95	**100***	0	**100***	0
	$LF(2.0)$	58	70	1	44	40	22	67	1	54	2
	$LF(4.0)$	79	88	1	64	59	43	86	0	76	1
	$EV(0.5)$	28	38	2	24	17	8	37	2	24	5
	$EV(1.5)$	85	90	3	83	47	57	91	0	75	3
	$DL(1.0)$	**61**	45	34	8	**73**	12	20	12	36	7
	$DL(1.5)$	98	98	16	45	99	59	80	2	96	0
	Av.	74.1	75.5	16.7	48.6	62.2	49.7	68.7	18.2	68.1	18.2
1.0	$W(0.8)$	44	48	9	27	34	55	53	30	**58**	20
	$W(1.4)$	72	80	4	24	75	25	70	0	70	0
	$\Gamma(0.4)$	99	98	47	46	98	**100***	**100***	99	**100***	93
	$\Gamma(2.0)$	90	90	13	23	93	35	76	0	86	0
	$LN(0.8)$	**83**	46	65	11	**88**	23	19	26	41	16
	$LN(1.5)$	94	95	32	87	71	94	95	36	93	69
	HN	43	55	1	27	34	16	51	1	35	4
	U	99	99	14	**100***	64	95	98	1	87	11
	$CH(0.5)$	95	94	38	36	94	99	97	94	99	80
	$CH(1.0)$	28	39	2	19	21	10	36	2	22	6
	$CH(1.5)$	**100***	**100***	0	96	98	95	**100***	0	99	0
	$LF(2.0)$	59	70	1	35	49	25	67	1	51	3
	$LF(4.0)$	80	88	1	53	69	46	86	0	74	2
	$EV(0.5)$	28	38	2	19	21	10	35	3	22	6
	$EV(1.5)$	85	91	1	77	60	61	90	1	72	5
	$DL(1.0)$	**59**	41	37	7	**70**	10	22	10	36	6
	$DL(1.5)$	98	97	24	31	99	56	84	1	96	0
	Av.	73.8	74.7	17.1	42.1	67.0	50.2	69.4	18.0	67.2	19.0

Table 4: Powers of 5% tests based on 100,000 simulations using empirical critical values; $n = 50$ ($k = 2$)

r	Alt.	$\hat{T}_n^{(r,3)}$	$\hat{T}_{n;c_1}^{(r,3)}$	$\hat{T}_{n;c_2}^{(r,3)}$	$\hat{T}_{n;c_3}^{(r,3)}$	$\hat{T}_{n;c_4}^{(r,3)}$	$\hat{T}_n^{*(r,3)}$	$\hat{T}_{n;c_1}^{*(r,3)}$	$\hat{T}_{n;c_2}^{*(r,3)}$	$\hat{T}_{n;c_3}^{*(r,3)}$	$\hat{T}_{n;c_4}^{*(r,3)}$
	$W(0.8)$	48	39	36	50	21	31	37	19	30	24
	$W(1.4)$	72	12	78	82	1	7	32	5	0	60
	$\Gamma(0.4)$	**100***	**100***	66	**100***	97	83	84	72	83	50

	$\Gamma(2.0)$	92	38	93	**96***	0	20	40	16	1	84
	$LN(0.8)$	**74**	37	**75**	**84**	0	33	14	35	20	**75**
	$LN(1.5)$	86	23	92	88	17	92	92	79	92	73
	HN	35	3	44	43	6	3	19	3	3	25
	U	86	11	91	90	12	37	97	24	2	87
-0.4	$CH(0.5)$	99	97	57	99	86	67	70	54	67	37
	$CH(1.0)$	23	3	30	29	6	3	11	5	5	15
	$CH(1.5)$	99	53	**100***	**100***	7	61	98	47	4	98
	$LF(2.0)$	51	5	60	60	5	4	31	3	1	40
	$LF(4.0)$	73	10	80	80	6	10	54	6	1	63
	$EV(0.5)$	23	3	30	29	6	3	11	5	5	15
	$EV(1.5)$	70	8	78	77	7	11	69	6	1	62
	$DL(1.0)$	54	11	57	**67**	0	14	10	14	8	51
	$DL(1.5)$	99	71	99	**100***	1	47	56	40	4	97
	Av.	69.7	30.8	68.6	74.8	16.3	31.0	48.4	25.5	19.2	56.2
	$W(0.8)$	48	49	14	49	25	44	45	15	39	28
	$W(1.4)$	69	62	52	**83***	1	71	71	32	4	72
	$\Gamma(0.4)$	**100***	**100***	79	**100***	99	95	95	76	94	39
	$\Gamma(2.0)$	89	91	67	95	0	86	81	60	1	90
	$LN(0.8)$	68	**90**	25	**75**	4	62	30	61	8	72
	$LN(1.5)$	90	56	90	91	11	95	95	73	95	54
	HN	39	19	38	50	7	47	50	10	20	39
	U	93	51	94	95	26	**100***	99	69	34	96
-0.1	$CH(0.5)$	99	99	52	98	92	87	88	55	84	31
	$CH(1.0)$	26	12	26	34	7	32	35	7	20	25
	$CH(1.5)$	99	94	98	**100***	9	**100***	**100***	90	3	**100***
	$LF(2.0)$	55	30	52	66	7	63	66	18	18	56
	$LF(4.0)$	76	47	72	85	8	83	85	36	15	77
	$EV(0.5)$	26	12	26	34	7	32	35	7	20	25
	$EV(1.5)$	77	40	76	84	12	89	91	35	31	78
	$DL(1.0)$	46	**63**	19	**60**	1	41	27	31	3	52
	$DL(1.5)$	98	99	86	99	0	96	91	86	0	98
	Av.	70.5	59.7	56.9	76.3	18.5	71.8	69.6	44.8	28.8	60.7
	$W(0.8)$	47	51	9	47	24	47	43	29	42	21
	$W(1.4)$	71	72	37	82	1	74	78	4	18	70
	$\Gamma(0.4)$	**100***	**100***	79	99	99	99	94	90	96	37
	$\Gamma(2.0)$	90	95	44	94	0	87	90	14	7	88
	$LN(0.8)$	**85**	**91**	11	67	23	63	53	31	5	73
	$LN(1.5)$	91	71	76	93	9	95	53	92	95	62
	HN	43	28	34	52	6	51	53	3	35	39
	U	96	67	94	97	30	99	99	25	84	96
0.1	$CH(0.5)$	99	99	51	96	93	95	93	75	89	29
	$CH(1.0)$	29	18	24	36	6	35	37	5	29	25
	$CH(1.5)$	**100***	98	95	**100***	5	**100***	**100***	48	32	**100***
	$LF(2.0)$	58	41	45	68	6	67	69	3	39	55
	$LF(4.0)$	79	60	63	86	6	85	87	6	44	77
	$EV(0.5)$	29	18	24	36	6	35	37	5	29	24
	$EV(1.5)$	81	54	73	87	12	90	90	6	71	79
	$DL(1.0)$	53	**69**	9	55	9	44	44	12	3	52
	$DL(1.5)$	98	**100***	64	99	0	97	97	38	2	98
	Av.	73.5	66.5	49.0	76.2	19.7	74.3	71.6	28.5	42.4	60.2
	$W(0.8)$	44	50	5	44	20	49	25	44	47	21
	$W(1.4)$	74	81	12	79	7	52	2	67	48	33
	$\Gamma(0.4)$	**100***	**100***	59	97	98	99	73	99	99	54
	$\Gamma(2.0)$	92	**96***	9	90	18	69	8	81	41	57
	$LN(0.8)$	**95**	**86**	25	50	**82**	48	29	41	6	53

	$LN(1.5)$	91	86	34	94	11	95	89	84	**96***	63
	HN	45	41	22	55	4	28	1	42	47	11
	U	98	87	91	99	20	95	17	97	99	75
0.5	$CH(0.5)$	98	99	36	92	91	96	55	96	94	49
	$CH(1.0)$	31	27	17	40	4	17	2	28	35	5
	$CH(1.5)$	**100***	**100***	74	**100***	1	99	35	**100***	98	92
	$LF(2.0)$	61	57	27	71	3	44	1	59	61	20
	$LF(4.0)$	81	77	38	88	3	67	3	79	77	40
	$EV(0.5)$	31	27	17	39	4	17	2	27	35	5
	$EV(1.5)$	85	73	56	91	5	73	3	83	91	41
	$DL(1.0)$	**67**	**68**	11	42	45	32	13	34	8	32
	$DL(1.5)$	99	**100***	13	97	24	88	25	93	39	82
	Av.	76.1	73.7	32.0	74.7	25.9	62.7	22.7	67.8	60.0	43.1
	$W(0.8)$	43	49	5	42	18	50	33	40	49	20
	$W(1.4)$	75	82	6	76	17	38	0	51	55	14
	$\Gamma(0.4)$	**100***	**100***	46	95	97	99	88	98	99	72
	$\Gamma(2.0)$	93	**96***	4	87	43	56	0	72	52	32
	$LN(0.8)$	**96**	**81**	42	41	**93**	43	14	52	8	44
	$LN(1.5)$	92	89	20	95	15	95	93	66	**96***	57
	HN	46	45	16	55	4	17	6	23	48	3
	U	98	92	86	99	12	89	3	86	99	45
0.7	$CH(0.5)$	97	98	29	89	89	97	75	95	96	61
	$CH(1.0)$	32	31	13	40	4	9	8	13	35	2
	$CH(1.5)$	**100***	**100***	49	**100***	4	96	0	97	99	71
	$LF(2.0)$	62	62	18	71	5	30	3	37	62	6
	$LF(4.0)$	81	82	24	88	5	52	2	59	81	16
	$EV(0.5)$	32	31	13	40	4	9	8	13	35	2
	$EV(1.5)$	85	79	43	91	4	58	5	60	90	16
	$DL(1.0)$	**70**	**65**	19	36	**59**	26	6	37	11	23
	$DL(1.5)$	99	**100***	4	96	60	80	1	90	56	61
	Av.	76.5	75.3	25.8	73.0	31.5	55.4	20.4	58.2	63.0	32.1
	$W(0.8)$	41	48	4	41	17	50	39	37	51	20
	$W(1.4)$	76	82	3	73	31	23	4	27	59	3
	$\Gamma(0.4)$	99	**100***	36	93	96	99	94	98	99	84
	$\Gamma(2.0)$	93	95	5	83	64	39	1	50	60	12
	$LN(0.8)$	**95**	**74**	59	33	**96**	36	8	50	10	36
	$LN(1.5)$	92	91	12	95	19	95	95	59	**96***	50
	HN	46	49	10	55	7	9	19	8	48	1
	U	97	95	77	99	6	78	33	58	99	14
0.9	$CH(0.5)$	97	98	25	85	87	97	84	95	97	71
	$CH(1.0)$	32	34	10	40	6	4	19	4	33	1
	$CH(1.5)$	**100***	**100***	23	**100***	24	89	2	85	**100***	33
	$LF(2.0)$	62	66	11	70	9	16	17	15	62	1
	$LF(4.0)$	82	84	13	87	12	34	14	30	81	3
	$EV(0.5)$	32	34	10	40	6	4	19	4	33	1
	$EV(1.5)$	86	84	29	92	6	40	30	28	89	3
	$DL(1.0)$	**71**	**60**	29	30	**68**	19	3	30	14	17
	$DL(1.5)$	99	99	6	93	85	65	0	77	66	31
	Av.	76.6	76.2	21.3	71.1	37.6	47.0	28.4	44.3	64.6	22.6
	$W(0.8)$	40	48	4	40	17	51	41	36	52	21
	$W(1.4)$	77	**83***	4	71	38	17	10	16	60	1
	$\Gamma(0.4)$	99	99	33	91	95	99	95	98	99	88
	$\Gamma(2.0)$	93	95	8	80	73	31	2	36	62	6
	$LN(0.8)$	**95**	70	67	28	**96**	34	6	47	11	34
	$LN(1.5)$	92	92	8	95	22	95	95	57	**96***	47
	HN	47	51	8	54	9	6	28	4	46	1

	U	99	96	70	**100***	6	74	62	38	98	5
1.0	$CH(0.5)$	96	97	24	82	86	97	87	94	97	75
	$CH(1.0)$	32	35	8	39	7	3	26	2	32	1
	$CH(1.5)$	**100***	**100***	14	**100***	40	84	11	69	**100***	16
	$LF(2.0)$	62	67	8	69	13	12	29	7	61	0
	$LF(4.0)$	82	86	9	86	19	27	28	17	81	1
	$EV(0.5)$	32	35	8	39	7	3	25	2	32	2
	$EV(1.5)$	86	85	22	92	9	33	52	15	88	1
	$DL(1.0)$	**70**	**58**	34	27	**71**	17	3	26	15	16
	$DL(1.5)$	99	99	11	91	91	57	1	64	70	19
	Av.	76.5	76.2	20.1	69.7	41.0	43.5	35.3	37.1	64.7	19.6
	$W(0.8)$	39	46	5	38	17	52	45	35	55	22
	$W(1.4)$	77	81	9	62	57	3	37	1	62	0
	$\Gamma(0.4)$	98	98	31	83	93	**100***	98	99	**100***	94
	$\Gamma(2.0)$	93	93	25	69	86	8	25	6	68	1
	$LN(0.8)$	**92**	59	**81**	19	**95**	23	5	36	15	25
	$LN(1.5)$	92	94	3	95	32	95	**96***	54	95	46
	HN	48	54	5	51	18	2	45	0	43	1
	U	99	98	41	**100***	16	75	99	2	97	1
1.3	$CH(0.5)$	94	95	25	72	83	98	92	95	98	83
	$CH(1.0)$	33	38	5	37	12	2	34	1	29	2
	$CH(1.5)$	**100***	**100***	2	**100***	78	56	84	8	**100***	0
	$LF(2.0)$	64	70	4	64	27	3	54	0	59	1
	$LF(4.0)$	83	88	4	83	40	8	68	1	80	0
	$EV(0.5)$	33	38	5	37	12	2	34	1	28	2
	$EV(1.5)$	87	89	8	91	26	16	88	0	84	1
	$DL(1.0)$	**68**	49	48	18	**74***	9	6	17	18	11
	$DL(1.5)$	99	98	40	82	98	21	18	18	78	3
	Av.	76.5	75.8	20.0	64.6	50.7	33.7	54.6	22.0	65.1	17.3
	$W(0.8)$	38	44	5	36	18	53	48	35	56	24
	$W(1.4)$	77	79	17	55	66	1	49	0	62	0
	$\Gamma(0.4)$	98	97	33	76	92	**100***	99	99	**100***	95
	$\Gamma(2.0)$	93	90	41	59	90	2	42	1	70	0
	$LN(0.8)$	**89**	50	**85**	14	**92**	17	6	28	17	21
	$LN(1.5)$	93	94	2	94	41	95	**96***	54	95	48
	HN	49	56	4	47	26	3	49	1	40	2
	U	99	99	22	**100***	33	84	99	0	95	2
1.5	$CH(0.5)$	93	92	27	64	82	98	94	95	98	85
	$CH(1.0)$	34	40	4	34	16	3	36	1	26	3
	$CH(1.5)$	**100***	**100***	6	99	90	40	98	0	99	0
	$LF(2.0)$	65	71	5	60	37	4	61	0	57	1
	$LF(4.0)$	84	88	6	79	55	8	78	0	78	0
	$EV(0.5)$	34	40	4	34	16	3	36	1	26	3
	$EV(1.5)$	87	91	5	89	41	22	91	0	82	1
	$DL(1.0)$	**66**	43	54	13	**72**	6	9	12	19	9
	$DL(1.5)$	99	97	62	73	99	5	40	4	82	1
	Av.	76.3	74.8	22.5	60.4	56.9	32.1	60.6	19.5	64.9	17.3

Table 5: Powers of 5% tests based on 100,000 simulations using empirical critical values; $n = 50$ $(k = 3)$

r	Alt.	$\hat{T}_n^{(r,4)}$	$\hat{T}_{n;c_1}^{(r,4)}$	$\hat{T}_{n;c_2}^{(r,4)}$	$\hat{T}_{n;c_3}^{(r,4)}$	$\hat{T}_{n;c_4}^{(r,4)}$	$\hat{T}_n^{*(r,4)}$	$\hat{T}_{n;c_1}^{*(r,4)}$	$\hat{T}_{n;c_2}^{*(r,4)}$	$\hat{T}_{n;c_3}^{*(r,4)}$	$\hat{T}_{n;c_4}^{*(r,4)}$
	$W(0.8)$	43	47	13	46	24	48	44	14	24	52
	$W(1.4)$	70	56	60	**83***	3	66	66	33	0	75
	$\Gamma(0.4)$	**100***	**100***	81	**100***	98	99	94	38	71	97
	$\Gamma(2.0)$	90	88	79	**96***	3	82	75	57	2	88
	$LN(0.8)$	**76**	**92**	47	**86**	1	56	24	54	21	50
	$LN(1.5)$	84	44	84	86	9	94	95	64	89	90
	HN	36	16	38	45	8	42	49	11	2	48
	U	87	42	89	90	33	99	**100***	77	7	99
-0.1	$CH(0.5)$	99	99	54	98	91	95	86	27	53	96
	$CH(1.0)$	24	11	26	31	7	28	35	6	4	33
	$CH(1.5)$	99	91	98	**100***	32	**100***	**100***	92	12	**100***
	$LF(2.0)$	52	25	52	62	10	59	64	21	1	65
	$LF(4.0)$	73	41	72	81	13	80	84	40	1	84
	$EV(0.5)$	24	10	26	31	8	28	35	6	4	33
	$EV(1.5)$	71	34	73	78	16	87	91	41	1	87
	$DL(1.0)$	53	**62**	32	**69**	0	38	22	30	9	40
	$DL(1.5)$	99	99	94	**100***	10	94	86	83	8	97
	Av.	69.4	56.2	60.1	75.4	21.5	70.4	67.5	40.9	18.2	72.5
	$W(0.8)$	43	48	8	44	23	47	46	20	29	40
	$W(1.4)$	72	68	50	**83***	3	70	75	8	0	74
	$\Gamma(0.4)$	**100***	**100***	83	99	99	99	98	66	83	77
	$\Gamma(2.0)$	91	94	64	**96***	2	82	85	20	0	87
	$LN(0.8)$	**84**	**94**	25	**81**	6	53	36	35	15	54
	$LN(1.5)$	86	59	62	88	9	95	77	86	92	42
	HN	41	24	37	48	9	50	55	3	6	48
	U	92	57	91	93	42	**100***	**100***	41	3	99
0.1	$CH(0.5)$	99	99	55	97	92	95	94	47	67	85
	$CH(1.0)$	28	15	26	33	7	35	39	3	8	33
	$CH(1.5)$	99	96	97	**100***	31	**100***	**100***	61	0	**100***
	$LF(2.0)$	56	36	50	64	10	65	70	4	3	65
	$LF(4.0)$	77	54	69	83	14	84	88	10	1	84
	$EV(0.5)$	27	15	25	33	7	35	39	3	8	33
	$EV(1.5)$	77	46	73	81	20	91	92	11	4	88
	$DL(1.0)$	56	**70**	19	**65**	2	38	33	15	6	42
	$DL(1.5)$	99	99	85	**100***	6	94	94	46	1	97
	Av.	72.2	63.3	54.0	75.9	22.5	72.6	71.8	28.1	19.2	67.5
	$W(0.8)$	42	49	6	44	21	45	32	32	33	31
	$W(1.4)$	75	75	38	**83***	2	67	69	1	2	68
	$\Gamma(0.4)$	**100***	**100***	79	99	99	98	67	88	89	62
	$\Gamma(2.0)$	93	**96***	47	95	2	79	84	0	0	84
	$LN(0.8)$	**94**	**94**	14	**76**	26	49	53	11	10	54
	$LN(1.5)$	87	70	38	90	9	94	47	93	93	49
	HN	44	31	34	51	8	48	43	10	13	42
	U	94	70	92	95	45	**100***	98	8	15	98

0.3	$CH(0.5)$	98	99	51	96	92	93	72	75	76	68
	$CH(1.0)$	30	20	24	35	7	33	28	13	16	27
	$CH(1.5)$	99	98	94	**100***	25	**100***	**100***	0	0	**100***
	$LF(2.0)$	60	45	45	68	9	63	60	7	10	58
	$LF(4.0)$	80	64	63	85	12	82	80	4	7	79
	$EV(0.5)$	30	20	24	35	7	33	28	13	16	27
	$EV(1.5)$	81	58	70	85	19	90	84	10	16	83
	$DL(1.0)$	**64**	**74***	11	**62**	11	35	41	4	4	41
	$DL(1.5)$	99	**100***	69	99	3	92	95	0	0	95
	Av.	74.8	68.3	47.1	76.4	23.5	70.6	63.6	21.8	23.5	62.7
	$W(0.8)$	39	48	4	42	18	43	20	40	37	26
	$W(1.4)$	75	79	28	82	3	59	40	53	6	57
	$\Gamma(0.4)$	**100***	**100***	71	99	98	97	37	96	93	62
	$\Gamma(2.0)$	93	**96***	30	94	4	73	62	52	1	75
	$LN(0.8)$	**96**	**92**	15	70	53	45	50	9	7	52
	$LN(1.5)$	87	77	20	92	10	94	69	94	94	49
	HN	45	36	30	53	7	40	17	50	22	31
	U	96	79	91	96	43	99	88	**100***	43	96
0.5	$CH(0.5)$	98	99	42	95	91	92	33	89	82	62
	$CH(1.0)$	31	24	22	37	6	26	9	38	22	18
	$CH(1.5)$	**100***	99	90	**100***	15	99	96	**100***	4	99
	$LF(2.0)$	61	52	39	70	7	55	29	63	21	47
	$LF(4.0)$	81	71	56	87	9	76	51	80	18	69
	$EV(0.5)$	31	24	22	37	6	26	9	38	22	18
	$EV(1.5)$	82	66	66	87	16	85	54	92	37	74
	$DL(1.0)$	**68**	**73**	9	57	25	31	31	11	3	37
	$DL(1.5)$	99	**100***	48	99	3	88	85	57	0	91
	Av.	75.4	71.5	40.1	76.3	24.4	66.5	45.8	62.4	30.2	56.7
	$W(0.8)$	36	46	3	41	14	45	24	40	42	23
	$W(1.4)$	78	82	12	80	13	40	0	57	24	29
	$\Gamma(0.4)$	99	**100***	44	97	96	98	71	98	97	79
	$\Gamma(2.0)$	94	**96***	10	92	32	55	2	72	12	49
	$LN(0.8)$	**97***	**86**	43	57	**89**	39	22	37	4	45
	$LN(1.5)$	87	86	5	93	14	94	89	84	95	40
	HN	47	45	20	56	6	21	2	33	38	10
	U	97	90	87	98	30	94	7	96	94	76
0.9	$CH(0.5)$	97	98	25	91	86	94	53	93	90	70
	$CH(1.0)$	33	31	16	40	6	12	4	21	31	5
	$CH(1.5)$	**100***	**100***	68	**100***	4	97	11	99	53	90
	$LF(2.0)$	63	61	25	71	6	34	1	49	44	18
	$LF(4.0)$	82	81	36	88	6	56	1	71	52	37
	$EV(0.5)$	33	31	16	40	6	12	4	21	31	5
	$EV(1.5)$	85	78	52	90	9	64	1	76	78	39
	$DL(1.0)$	**73**	**69**	20	48	54	24	9	30	4	27
	$DL(1.5)$	99	**100***	13	98	40	77	7	88	6	75
	Av.	76.5	75.1	29.2	75.3	30.1	56.2	18.1	62.6	46.7	42.0

	$W(0.8)$	35	45	3	40	13	46	26	40	43	23
	$W(1.4)$	78	**83***	10	80	18	34	0	51	29	22
	$\Gamma(0.4)$	**100***	**100***	36	96	95	98	78	98	97	83
	$\Gamma(2.0)$	94	**96***	8	91	41	50	1	68	17	41
	$LN(0.8)$	**97***	**83**	52	53	**93**	38	18	41	4	43
	$LN(1.5)$	87	87	3	93	16	94	91	81	95	39
	HN	48	47	18	56	6	17	4	26	40	6
	U	98	91	85	98	26	91	2	92	97	66
1.0	$CH(0.5)$	96	98	21	90	84	94	61	93	91	72
	$CH(1.0)$	33	31	15	40	6	9	6	15	32	3
	$CH(1.5)$	**100***	**100***	59	**100***	6	95	2	98	69	83
	$LF(2.0)$	63	63	22	71	7	28	2	40	48	12
	$LF(4.0)$	82	82	30	88	7	49	1	63	59	27
	$EV(0.5)$	33	31	15	40	6	9	6	15	32	3
	$EV(1.5)$	85	80	47	90	8	57	2	67	82	28
	$DL(1.0)$	**73**	**67**	24	45	**59**	23	7	31	4	25
	$DL(1.5)$	99	**100***	10	98	54	73	2	86	11	68
	Av.	76.5	75.5	27.0	74.6	32.0	53.2	18.1	59.2	50.1	37.9
	$W(0.8)$	33	44	3	39	11	47	33	38	47	23
	$W(1.4)$	79	**83***	8	77	34	15	2	25	40	6
	$\Gamma(0.4)$	99	99	22	94	92	99	89	98	98	90
	$\Gamma(2.0)$	95	95	14	87	67	29	0	44	31	17
	$LN(0.8)$	**96**	**76**	**74**	42	**96**	32	10	42	4	36
	$LN(1.5)$	88	90	1	94	18	95	94	75	**96***	43
	HN	49	51	12	56	9	5	12	8	43	1
	U	98	95	74	99	15	72	14	63	99	26
1.3	$CH(0.5)$	95	96	15	86	77	95	76	94	94	79
	$CH(1.0)$	34	35	11	41	8	3	15	4	32	1
	$CH(1.5)$	**100***	**100***	30	**100***	25	81	0	84	94	46
	$LF(2.0)$	64	67	13	71	12	11	9	14	54	2
	$LF(4.0)$	83	85	16	88	16	24	6	30	71	7
	$EV(0.5)$	34	35	11	41	8	3	15	4	32	1
	$EV(1.5)$	86	85	31	91	9	29	14	30	87	6
	$DL(1.0)$	**74***	**61**	40	38	**69**	17	4	26	6	19
	$DL(1.5)$	99	99	18	96	85	53	0	71	28	39
	Av.	**76.8***	76.3	23.1	72.9	38.4	41.7	23.1	44.0	56.4	26.1
	$W(0.8)$	32	42	3	38	10	48	37	37	49	24
	$W(1.4)$	79	82	11	74	45	6	6	10	45	2
	$\Gamma(0.4)$	98	99	18	92	88	99	93	98	99	92
	$\Gamma(2.0)$	94	94	26	84	77	14	1	25	39	7
	$LN(0.8)$	**95**	70	**83**	35	**96**	27	7	38	5	31
	$LN(1.5)$	88	91	1	94	20	95	94	72	**96***	47
	HN	50	53	9	56	13	2	23	2	44	1
	U	98	97	63	99	12	47	43	31	99	7
1.5	$CH(0.5)$	93	95	14	82	73	96	82	93	95	82

	$CH(1.0)$	35	37	9	41	9	1	23	1	32	1
	$CH(1.5)$	**100***	**100***	15	**100***	48	58	5	56	98	17
	$LF(2.0)$	65	69	9	71	18	3	21	4	57	1
	$LF(4.0)$	84	87	10	87	25	10	18	11	75	1
	$EV(0.5)$	35	38	9	41	10	1	22	1	32	1
	$EV(1.5)$	86	87	21	91	14	12	37	9	88	1
	$DL(1.0)$	**73**	57	48	33	**72**	13	3	21	8	15
	$DL(1.5)$	99	99	36	94	93	34	0	49	39	19
	Av.	**76.8***	76.3	22.7	71.3	42.6	33.3	30.3	32.9	58.7	20.5
	$W(0.8)$	31	42	3	37	10	49	39	37	50	25
	$W(1.4)$	80	82	18	71	55	1	15	3	48	1
	$\Gamma(0.4)$	98	98	17	89	85	99	95	98	99	94
	$\Gamma(2.0)$	94	93	40	80	84	5	5	10	45	2
	$LN(0.8)$	**93**	64	**88**	29	**95**	22	5	33	6	27
	$LN(1.5)$	89	93	0	94	23	95	95	70	**96***	50
	HN	51	55	8	55	18	1	33	1	43	1
	U	99	98	49	99	15	20	77	8	99	1
1.7	$CH(0.5)$	92	93	14	79	70	96	87	94	96	85
	$CH(1.0)$	36	39	8	40	12	1	29	1	30	1
	$CH(1.5)$	**100***	**100***	10	**100***	69	26	22	21	99	3
	$LF(2.0)$	66	71	9	69	26	1	35	1	57	0
	$LF(4.0)$	84	88	9	86	37	2	37	2	76	0
	$EV(0.5)$	36	39	8	40	13	1	29	1	30	1
	$EV(1.5)$	87	89	14	91	23	3	63	1	87	0
	$DL(1.0)$	**71**	53	56	28	**74***	10	3	16	9	13
	$DL(1.5)$	99	99	57	91	96	15	2	25	47	8
	Av.	**76.8***	76.1	24.0	69.4	47.3	26.3	39.5	24.7	59.8	18.4

Table 6: Powers of 5% tests based on 100,000 simulations using empirical critical values; $n = 50$ $(k = 4)$

Our simplest omnibus tests when $n = 20$ are

$$\hat{T}^{(1,3)}_{n;c_1} = \frac{45n(n-1)(n-2)}{4n^2-3n-7} \cdot \left[\frac{3}{n(n-1)(n-2)}\sum_{i=1}^{n-2}(n-i-1)(n-i)\frac{X_{i:n}}{\overline{X}_n} - \frac{1}{3}\right]^2$$

$$\hat{T}^{(1,4)}_{n;c_1} = \frac{560n(n-1)(n-2)(n-3)}{3(15n^3-50n^2-3n+62)} \cdot \left[\frac{4}{n(n-1)(n-2)(n-3)}\sum_{i=1}^{n-3}(n-i-2)(n-i-1)(n-i)\frac{X_{i:n}}{\overline{X}_n} - \frac{1}{4}\right]^2$$

with Av. power 46.7 and 46.9, respectively.

For $k = 2$ we have the test

$$\hat{T}_{n;c_1}^{(1,2)} = \frac{12n(n-1)}{n+1}\left[\frac{2}{n(n-1)}\sum_{i=1}^{n}(n-i)\frac{X_{i:n}}{\overline{X}_n} - \frac{1}{2}\right]^2$$

with Av. power 44.0 (cf. Morris and Szynal [19]).

Our most powerful test is $\hat{T}_{n;c_1}^{(0.3,4)}$ with Av. power 47.0.

When $n = 50$ our recommended test is also $\hat{T}_{n;c_1}^{(1,3)}$ with Av. power 76.2 and $\hat{T}_{n;c_1}^{(1.5,4)}$ with Av. power 76.3, respectively.

For $k = 2$ the test

$$\hat{T}_{n;c_1}^{(0.5,2)} = \frac{32n(n-1)}{\left((32\sqrt{2}-17)\pi - 64\sqrt{2}\arctan\sqrt{2}\right)(n-2) + 32 - 9\pi} \cdot \left(\frac{1}{\sqrt{\overline{X}_n}}\frac{1}{\binom{n}{2}}\sum_{i=1}^{n}(n-i)\sqrt{X_{i:n}} - \frac{1}{2}\sqrt{\frac{\pi}{2}}\right)^2$$

(cf. Morris and Szynal [19]) has Av. power 75.5.

Our most powerful test is $\hat{T}_{n}^{(1.5,4)}$ with Av. power 76.8.

The role of k (=2,3,4) in powers of these tests is illustrated in the following tables.

r	-0.1			0.3			0.5		
Alt.↓ Tests→	$\hat{T}_{n;c_3}^{(r,2)}$	$\hat{T}_{n;c_3}^{(r,3)}$	$\hat{T}_{n;c_3}^{(r,4)}$	$\hat{T}_{n;c_1}^{(r,2)}$	$\hat{T}_{n;c_3}^{(r,3)}$	$\hat{T}_{n;c_3}^{(r,4)}$	$\hat{T}_{n;c_1}^{(r,2)}$	$\hat{T}_{n;c_1}^{(r,3)}$	$\hat{T}_{n;c_3}^{(r,4)}$
$W(0.8)$	24	21	18	27	20	16	25	24	15
$W(1.4)$	37	42	44	37	41	**45***	38	40	**45***
$\Gamma(0.4)$	78	82	83	91	74	75	88	89	71
$\Gamma(2.0)$	49	**59**	**63***	55	54	**61**	55	**59**	**60**
$LN(0.8)$	27	**40**	**50**	**38**	32	**44**	36	**46**	**41**
$LN(1.5)$	65	55	44	54	60	49	59	48	51
HN	22	24	22	18	25	25	20	20	26
U	69	60	53	49	68	62	57	49	65
$CH(0.5)$	65	69	69	**81**	60	61	76	78	57
$CH(1.0)$	15	16	16	12	17	18	14	13	19
$CH(1.5)$	84	83	81	78	85	84	82	78	85
$LF(2.0)$	30	31	30	25	33	34	27	26	35
$LF(4.0)$	44	45	44	37	47	48	40	38	49
$EV(0.5)$	15	16	16	12	17	18	14	13	19
$EV(1.5)$	46	44	41	36	48	47	40	37	49
$DL(1.0)$	21	**29**	**35**	27	25	**33**	26	**31**	**30**
$DL(1.5)$	66	**76**	**80**	**74**	71	**79**	**73**	**78**	**77**
Av.	44.5	46.7	46.3	44.2	45.7	47.0	45.3	45.1	46.7

r	0.7			1.0			1.5		
Alt.↓ Tests→	$\hat{T}^{(r,2)}_{n;c_1}$	$\hat{T}^{(r,3)}_{n;c_1}$	$\hat{T}^{(r,4)}_{n;c_3}$	$\hat{T}^{(r,2)}_{n;c_1}$	$\hat{T}^{(r,3)}_{n;c_1}$	$\hat{T}^{(r,4)}_{n;c_1}$	$\hat{T}^{(r,2)}_{n;c_4}$	$\hat{T}^{(r,3)}_{n}$	$\hat{T}^{(r,4)}_{n;c_1}$
$W(0.8)$	24	22	15	23	21	17	17	15	15
$W(1.4)$	39	42	44	35	42	44	39	43	**45***
$\Gamma(0.4)$	84	86	68	75	80	81	65	68	72
$\Gamma(2.0)$	53	**59**	**58**	47	**58**	**63***	50	**59**	**60**
$LN(0.8)$	32	**43**	37	24	**38**	**49**	26	**47**	**41**
$LN(1.5)$	62	52	53	66	57	45	59	55	51
HN	22	22	26	22	24	23	25	25	26
U	64	55	68	71	63	56	72	67	65
$CH(0.5)$	71	73	53	62	67	67	51	53	57
$CH(1.0)$	15	15	19	15	17	17	18	17	18
$CH(1.5)$	84	81	85	84	84	83	85	82	85
$LF(2.0)$	29	29	35	29	32	32	33	32	35
$LF(4.0)$	43	42	50	43	46	45	47	46	49
$EV(0.5)$	14	15	19	15	16	17	17	17	19
$EV(1.5)$	44	41	50	47	46	43	51	46	49
$DL(1.0)$	24	**31**	28	19	**29**	**35**	21	**33**	**31**
$DL(1.5)$	71	**78**	**75**	63	**75**	**80**	65	**77**	**77**
Av.	45.6	46.3	46.1	43.5	46.7	46.9	43.6	46.0	46.8

Table 7: Powers of 5% selected tests using empirical values; $n = 20$

r	-0.1			0.3			0.5		
Alt.↓ Tests→	$\hat{T}^{(r,2)}_{n}$	$\hat{T}^{(r,3)}_{n;c_3}$	$\hat{T}^{(r,4)}_{n;c_3}$	$\hat{T}^{(r,2)}_{n}$	$\hat{T}^{(r,3)}_{n}$	$\hat{T}^{(r,4)}_{n;c_3}$	$\hat{T}^{(r,2)}_{n;c_1}$	$\hat{T}^{(r,3)}_{n}$	$\hat{T}^{(r,4)}_{n;c_3}$
$W(0.8)$	52	49	46	50	47	44	53	44	42
$W(1.4)$	66	**83***	**83***	71	74	**83***	82	74	82
$\Gamma(0.4)$	**100***	**100***	**100***	**100***	**100***	99	**100***	**100***	99
$\Gamma(2.0)$	86	95	**96***	90	92	95	95	92	94
$LN(0.8)$	64	**75**	**86**	**93**	**94**	**76**	73	**95**	70
$LN(1.5)$	93	91	86	93	91	90	92	91	92
HN	40	50	45	42	45	51	47	45	53
U	97	95	90	99	97	95	94	98	96
$CH(0.5)$	99	98	98	99	99	96	99	98	95
$CH(1.0)$	27	34	31	29	31	35	32	31	37
$CH(1.5)$	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***
$LF(2.0)$	56	66	62	58	61	68	63	61	70
$LF(4.0)$	77	85	81	78	81	85	83	81	87
$EV(0.5)$	27	34	31	28	31	35	32	31	37
$EV(1.5)$	82	84	78	85	84	85	82	85	87
$DL(1.0)$	38	**60**	**69**	**63**	**63**	**62**	**59**	**67**	57
$DL(1.5)$	97	99	**100***	99	99	99	99	99	99
Av.	70.6	76.3	75.4	75.2	75.8	76.4	75.5	76.1	76.3

r	0.7			1.0			1.5		
Alt.↓ Tests→	$\hat{T}^{(r,2)}_{n;c_1}$	$\hat{T}^{(r,3)}_{n}$	$\hat{T}^{(r,4)}_{n}$	$\hat{T}^{(r,2)}_{n;c_1}$	$\hat{T}^{(r,3)}_{n;c_3}$	$\hat{T}^{(r,4)}_{n}$	$\hat{T}^{(r,2)}_{n;c_4}$	$\hat{T}^{(r,3)}_{n}$	$\hat{T}^{(r,4)}_{n}$
$W(0.8)$	51	43	37	48	40	35	40	38	32
$W(1.4)$	82	75	76	80	77	78	79	77	79
$\Gamma(0.4)$	**100***	**100***	**100***	98	99	**100***	97	98	98
$\Gamma(2.0)$	94	93	94	90	93	94	90	93	94
$LN(0.8)$	63	**96**	**97***	46	**95**	**97***	47	**89**	**95**
$LN(1.5)$	93	92	86	95	92	87	93	93	88
HN	51	46	46	55	47	48	55	49	50
U	97	98	97	99	99	98	98	99	98
$CH(0.5)$	97	97	97	94	96	96	90	93	93
$CH(1.0)$	35	32	32	39	32	33	39	34	35
$CH(1.5)$	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***
$LF(2.0)$	67	62	62	70	62	63	71	65	65
$LF(4.0)$	86	81	82	88	82	82	88	84	84
$EV(0.5)$	35	32	32	38	32	33	39	34	35
$EV(1.5)$	86	85	83	91	86	85	90	87	86
$DL(1.0)$	52	**70**	**71**	41	**70**	**73**	42	**66**	**73**
$DL(1.5)$	99	99	99	97	99	99	97	99	99
Av.	75.8	76.5	76.1	74.7	76.5	76.5	73.8	76.3	76.8

Table 8: Powers of 5% selected tests using empirical values; $n = 50$

7.1.2.

Here we consider tests and alternatives from Meintanis et al [11] (Table 9). The case with $k = 2$ for sample sizes $n = 20$ and $n = 50$ was discussed in Morris et al [12]. Now we simulate powers of tests with $k = 3$ and $k = 4$ taking values of r as in Section 7.1.1. For $n = 20$ we include simulations for tests with Av. powers ≥ 51.5 (Table 10, Table 11). For $n = 50$ we include simulations for tests with Av. powers ≥ 70.0 (Table 12, Table 13).

		$T_{n,a}$				$H_{n,a}$			
	↓ Distributions a →	0.05	0.10	0.20	0.40	0.05	0.10	0.20	0.40
	$\Gamma(0.2)$	20	20	20	20	14	15	15	15
	$\Gamma(0.4)$	33	33	32	32	26	27	27	25
	$\Gamma(0.8)$	**59***	**59***	58	57	56	57	57	53
	$W(0.25)$	40	40	40	39	27	31	32	31
	$W(0.50)$	76	75	75	74	62	67	69	68
	$W(0.75)$	**94***	**94***	**94***	93	86	90	92	91
$n = 20$	$LF(0.50)$	**31***	**31***	**31***	**31***	15	18	20	20
	$LF(0.75)$	38	**39***	38	38	19	22	25	26
	$LF(1.0)$	**44***	**44***	**44***	**44***	22	27	30	31
	$LF(1.5)$	**53***	**53***	**53***	**53***	27	33	38	39
	HN	**50***	**50***	**50***	**50***	24	30	34	35

	$CH(1.0)$	**41***	**41***	**41***	**41***	18	22	25	26
	$CH(1.2)$	**75***	**75***	**75***	**75***	45	53	59	61
	Av.	50.3	50.3	50.1	49.8	33.2	37.8	40.2	40.1
$n = 50$	$\Gamma(0.2)$	28	28	28	27	23	23	23	22
	$\Gamma(0.4)$	51	51	51	49	53	53	51	48
	$\Gamma(0.8)$	85	85	84	83	**93***	90	91	88
	$W(0.25)$	66	66	65	64	55	58	60	59
	$W(0.50)$	98	98	97	97	95	96	97	97
	$W(0.75)$	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***
	$LF(0.50)$	**54***	**54***	**54***	**54***	25	31	35	38
	$LF(0.75)$	**67***	**67***	**67***	**67***	33	41	47	51
	$LF(1.0)$	**76***	**76***	**76***	**76***	41	50	57	61
	$LF(1.5)$	**86***	**86***	**86***	**86***	52	65	70	75
	HN	**84***	**84***	83	83	46	58	64	69
	$CH(1.0)$	**73***	**73***	**73***	**73***	32	41	48	54
	$CH(1.2)$	**99***	**99***	**99***	**99***	82	85	93	95
	Av.	**74.4***	**74.4***	74.1	73.7	56.2	60.8	64.3	65.9

Table 9: (Source, Table 3.2 in Meintanis et al [11]). Observed power at 10% nominal level, with sample size $n = 20$ and $n = 50$

r	Alt.	$\hat{T}_n^{(r,3)}$	$\hat{T}_{n;c_1}^{(r,3)}$	$\hat{T}_{n;c_2}^{(r,3)}$	$\hat{T}_{n;c_3}^{(r,3)}$	$\hat{T}_{n;c_4}^{(r,3)}$	$\hat{T}_n^{*(r,3)}$	$\hat{T}_{n;c_1}^{*(r,3)}$	$\hat{T}_{n;c_2}^{*(r,3)}$	$\hat{T}_{n;c_3}^{*(r,3)}$	$\hat{T}_{n;c_4}^{*(r,3)}$
-0.4	$\Gamma(0.2)$	**100***	**100***	95	**100***	**100***	95	94	91	94	74
	$\Gamma(0.4)$	**95***	**94**	**60**	**93**	**84**	**54**	**62**	**42**	**51**	**51**
	$\Gamma(0.8)$	24	26	17	21	22	18	15	16	16	20
	$W(0.25)$	**100***	**100***	98	**100***	**100***	**100***	**100***	97	**100***	94
	$W(0.5)$	**98***	**95**	65	**97**	**83**	**77**	**83**	64	76	63
	$W(0.75)$	49	45	37	47	34	31	36	22	28	36
	$LF(0.5)$	17	10	19	19	8	7	16	8	8	13
	$LF(0.75)$	20	11	23	23	8	7	21	7	7	16
	$LF(1.0)$	24	13	28	27	8	7	26	7	6	20
	$LF(1.5)$	30	15	34	34	8	6	33	6	5	25
	HN	27	13	31	30	8	7	30	6	6	22
	$CH(1.0)$	20	11	23	23	8	8	22	8	8	16
	$CH(1.2)$	47	23	53	54	7	8	54	5	3	43
	Av.	50.0	42.8	44.9	51.4	36.9	32.6	45.5	29.1	31.4	37.9
-0.1	$\Gamma(0.2)$	**100***	**100***	99	**100***	**100***	99	98	95	98	63
	$\Gamma(0.4)$	**95***	**95***	**62**	**89**	**88**	**77**	**76**	**46**	**66**	**51**
	$\Gamma(0.8)$	24	25	15	19	22	18	17	16	15	21
	$W(0.25)$	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***	88
	$W(0.5)$	**97**	**97**	54	**96**	**88**	**91**	**91**	71	**86**	54
	$W(0.75)$	48	48	22	46	34	43	42	21	33	42
	$LF(0.5)$	16	13	16	20	8	16	17	8	12	13
	$LF(0.75)$	20	15	19	24	8	20	21	8	13	17
	$LF(1.0)$	23	17	23	29	8	25	26	7	13	21
	$LF(1.5)$	29	21	28	36	8	32	33	7	13	27
	HN	26	18	26	33	8	29	30	7	13	24
	$CH(1.0)$	20	14	20	24	8	22	23	8	15	17
	$CH(1.2)$	45	34	41	57	5	52	55	9	10	46
	Av.	49.5	45.9	40.3	51.7	37.3	47.9	48.4	31.0	37.4	37.2
	$\Gamma(0.2)$	**100***	**100***	99	**100***	**100***	**100***	67	98	99	58
	$\Gamma(0.4)$	**94**	**95***	**61**	**87**	**88**	**85**	**82**	**63**	**71**	**54**

	$\Gamma(0.8)$	23	24	13	18	21	20	21	15	15	22
	$W(0.25)$	**100***	**100***	**100***	**100***	**100***	**100***	66	**100***	**100***	94
	$W(0.5)$	**97**	**97**	53	**95**	**89**	**95**	76	**84**	**89**	47
	$W(0.75)$	48	48	17	44	32	47	45	29	34	38
0.1	$LF(0.5)$	17	15	14	20	8	17	16	10	14	14
	$LF(0.75)$	20	18	16	25	7	21	21	10	16	17
	$LF(1.0)$	24	20	19	30	7	26	25	9	17	21
	$LF(1.5)$	30	25	23	37	7	33	32	8	18	27
	HN	27	23	22	33	7	30	29	9	19	25
	$CH(1.0)$	21	17	17	25	8	22	21	11	19	18
	$CH(1.2)$	46	42	33	58	4	54	53	5	19	46
	Av.	49.9	48.0	37.4	51.6	36.8	50.1	42.6	34.9	41.0	36.9
	$\Gamma(0.2)$	**100***	**100***	55	96	**100***	**100***	99	**100***	**100***	99
	$\Gamma(0.4)$	**86**	**88**	23	**72**	**76**	**90**	**69**	**86**	**86**	**74**
	$\Gamma(0.8)$	16	19	8	17	11	25	15	23	18	20
	$W(0.25)$	**100***	**100***	67	**100***	**100***	**100***	**100***	**100***	**100***	**100***
	$W(0.5)$	**95**	**96**	15	**89**	**84**	**96**	**87**	**91**	**95**	**77**
	$W(0.75)$	39	45	7	41	21	51	34	42	43	33
1.0	$LF(0.5)$	19	20	9	16	14	5	14	6	19	5
	$LF(0.75)$	23	25	9	21	17	5	15	6	24	4
	$LF(1.0)$	27	29	8	25	18	5	15	6	28	4
	$LF(1.5)$	34	37	8	33	21	6	16	8	35	3
	HN	30	33	8	30	19	5	17	7	33	3
	$CH(1.0)$	23	25	8	22	15	5	17	6	26	4
	$CH(1.2)$	52	57	7	52	29	10	14	13	54	3
	Av.	49.6	**51.8***	17.8	47.2	40.4	38.7	39.2	37.9	50.9	33.1
	$\Gamma(0.2)$	**100***	**100***	45	95	99	**100***	99	**100***	**100***	99
	$\Gamma(0.4)$	**85**	**87**	21	**70**	**74**	**90**	**71**	**86**	**86**	**75**
	$\Gamma(0.8)$	15	18	8	18	11	25	15	23	19	21
	$W(0.25)$	**100***	**100***	28	**100***	**100***	**100***	**100***	**100***	**100***	**100***
	$W(0.5)$	**94**	**95**	14	**88**	**83**	**96**	**89**	**91**	**95**	**79**
	$W(0.75)$	39	45	7	41	21	51	35	41	44	33
1.1	$LF(0.5)$	19	20	9	16	16	4	15	5	19	5
	$LF(0.75)$	24	25	9	20	18	4	17	4	24	4
	$LF(1.0)$	28	30	9	24	20	4	18	4	28	3
	$LF(1.5)$	35	38	9	32	24	4	19	5	36	3
	HN	31	34	9	29	21	4	20	4	33	3
	$CH(1.0)$	23	25	8	21	17	4	20	4	26	4
	$CH(1.2)$	53	58	8	50	34	6	20	7	54	2
	Av.	49.7	**51.8***	14.1	46.5	41.3	37.7	41.3	36.5	51.1	33.1

Table 10: Powers of 10% tests based on 100,000 simulations using empirical critical values; $n = 20$ $(k = 3)$

r	Alt.	$\hat{T}_n^{(r,4)}$	$\hat{T}_{n;c_1}^{(r,4)}$	$\hat{T}_{n;c_2}^{(r,4)}$	$\hat{T}_{n;c_3}^{(r,4)}$	$\hat{T}_{n;c_4}^{(r,4)}$	$\hat{T}_n^{*(r,4)}$	$\hat{T}_{n;c_1}^{*(r,4)}$	$\hat{T}_{n;c_2}^{*(r,4)}$	$\hat{T}_{n;c_3}^{*(r,4)}$	$\hat{T}_{n;c_4}^{*(r,4)}$
	$\Gamma(0.2)$	**100***	**100***	99	**100***	**100***	**100***	57	97	96	82
	$\Gamma(0.4)$	**93**	**94**	**61**	**86**	**88**	**84**	**67**	**62**	**55**	**75**
	$\Gamma(0.8)$	21	22	11	17	19	20	21	14	14	20
	$W(0.25)$	**100***	**100***	**100***	**100***	**100***	**100***	80	**100***	**100***	67
	$W(0.5)$	**96**	**97**	59	**95**	**89**	**94**	59	**84**	**80**	71
	$W(0.75)$	42	46	13	42	31	46	39	31	28	41
0.3	$LF(0.5)$	18	16	14	20	8	16	13	15	11	15
	$LF(0.75)$	22	19	16	25	8	20	16	17	10	19
	$LF(1.0)$	26	22	19	30	8	24	19	18	10	23
	$LF(1.5)$	33	28	24	38	8	31	26	19	9	30

	HN	29	24	22	34	8	29	23	20	10	27
	$CH(1.0)$	22	19	17	25	8	21	17	19	12	20
	$CH(1.2)$	49	44	33	57	6	52	44	20	6	50
	Av.	50.1	48.7	37.5	51.6	37.0	48.9	36.9	39.6	33.9	41.5
	$\Gamma(0.2)$	**100***	**100***	99	**100***	**100***	**100***	62	99	97	91
	$\Gamma(0.4)$	**92**	**93**	**54**	**85**	**86**	**83**	**45**	**74**	**61**	**72**
	$\Gamma(0.8)$	18	21	10	17	16	21	21	15	14	21
	$W(0.25)$	**100***	**100***	**100***	**100***	**100***	**100***	94	**100***	**100***	95
	$W(0.5)$	**96**	**97**	53	**94**	**89**	**93**	51	**89**	**83**	69
	$W(0.75)$	39	45	10	42	27	45	31	36	29	38
0.5	$LF(0.5)$	18	17	13	20	9	14	8	21	13	13
	$LF(0.75)$	23	21	15	26	9	19	9	26	13	17
	$LF(1.0)$	27	25	16	31	8	22	10	31	13	20
	$LF(1.5)$	33	30	20	38	8	29	13	39	13	26
	HN	30	27	19	35	8	26	11	36	14	23
	$CH(1.0)$	23	20	15	26	9	20	9	29	16	17
	$CH(1.2)$	49	49	26	58	6	48	24	59	11	44
	Av.	49.7	49.7	34.6	51.6	36.6	47.6	29.8	50.2	36.7	41.9
	$\Gamma(0.2)$	**100***	**100***	89	99	**100***	**100***	91	**100***	99	98
	$\Gamma(0.4)$	**86**	**89**	28	**79**	**77**	**84**	**45**	**82**	**72**	**72**
	$\Gamma(0.8)$	14	18	7	17	11	22	16	20	15	21
	$W(0.25)$	**100***	**100***	99	**100***	**100***	**100***	**100***	**100***	**100***	**100***
	$W(0.5)$	**94**	**96**	27	**92**	**84**	**93**	71	**91**	**89**	**77**
	$W(0.75)$	33	43	6	41	20	46	26	40	34	35
1.0	$LF(0.5)$	20	20	11	20	13	8	8	12	17	7
	$LF(0.75)$	25	25	11	26	14	10	7	16	19	8
	$LF(1.0)$	28	29	12	30	14	11	6	18	21	9
	$LF(1.5)$	35	36	13	38	15	15	5	24	23	11
	HN	32	32	12	34	14	13	6	21	23	10
	$CH(1.0)$	24	24	12	26	13	10	8	16	22	8
	$CH(1.2)$	53	55	14	57	19	28	3	40	26	20
	Av.	49.5	51.2	26.2	50.7	38.0	41.6	30.1	44.6	43.0	36.5
	$\Gamma(0.2)$	**100***	**100***	60	98	99	**100***	96	**100***	99	99
	$\Gamma(0.4)$	**82**	**86**	15	**75**	**69**	**85**	**55**	**82**	**76**	**74**
	$\Gamma(0.8)$	12	17	7	17	9	23	15	21	16	20
	$W(0.25)$	**100***	**100***	90	**100***	**100***	**100***	**100***	**100***	**100***	**100***
	$W(0.5)$	**92**	**95**	11	**91**	**81**	**94**	**80**	**91**	**91**	**81**
	$W(0.75)$	31	42	4	41	17	46	28	41	37	35
1.3	$LF(0.5)$	21	20	12	19	16	5	11	7	18	5
	$LF(0.75)$	25	26	12	24	18	5	11	8	22	4
	$LF(1.0)$	29	30	13	29	19	6	11	9	24	4
	$LF(1.5)$	36	38	13	37	22	7	10	12	28	4
	HN	32	33	12	33	19	6	11	10	28	4
	$CH(1.0)$	25	25	12	25	16	6	13	8	25	5
	$CH(1.2)$	55	58	14	56	30	12	6	21	37	7
	Av.	49.3	51.5	21.1	49.6	39.6	38.1	34.3	39.2	46.2	34.1
	$\Gamma(0.2)$	**100***	**100***	24	97	99	**100***	98	**100***	**100***	99
	$\Gamma(0.4)$	**80**	**84**	12	**73**	**65**	**86**	**61**	**83**	**79**	**75**
	$\Gamma(0.8)$	12	17	6	17	8	23	14	21	16	20
	$W(0.25)$	**100***	**100***	11	**100***	**100***	**100***	**100***	**100***	**100***	**100***
	$W(0.5)$	**91**	**94**	7	**89**	**78**	**94**	**83**	**90**	**92**	**82**
	$W(0.75)$	30	41	4	40	15	47	30	41	38	35
1.5	$LF(0.5)$	21	20	13	18	17	4	13	5	18	5
	$LF(0.75)$	26	26	14	23	20	4	13	5	22	4
	$LF(1.0)$	31	31	14	28	22	4	13	5	26	3

	$LF(1.5)$	38	38	15	35	26	4	13	6	30	3
	HN	34	34	14	32	23	4	14	5	29	3
	$CH(1.0)$	26	26	13	24	19	4	15	5	25	4
	$CH(1.2)$	56	58	18	54	37	5	10	10	42	3
	Av.	49.6	51.5	12.7	48.5	40.7	36.7	36.7	36.5	47.4	33.5

Table 11: Powers of 10% tests based on 100,000 simulations using empirical critical values; $n = 20$ ($k = 4$)

r	Alt.	$\hat{T}_n^{(r,3)}$	$\hat{T}_{n;c_1}^{(r,3)}$	$\hat{T}_{n;c_2}^{(r,3)}$	$\hat{T}_{n;c_3}^{(r,3)}$	$\hat{T}_{n;c_4}^{(r,3)}$	$\hat{T}_n^{*(r,3)}$	$\hat{T}_{n;c_1}^{*(r,3)}$	$\hat{T}_{n;c_2}^{*(r,3)}$	$\hat{T}_{n;c_3}^{*(r,3)}$	$\hat{T}_{n;c_4}^{*(r,3)}$
	$\Gamma(0.2)$	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**98**
	$\Gamma(0.4)$	**100***	**100***	**83**	**100***	**99**	**97**	**97**	**78**	**95**	46
	$\Gamma(0.8)$	37	39	16	32	28	26	26	18	19	32
	$W(0.25)$	**100***	**100***	**100***	**100***	**100***	**91**	**91**	**91**	**100***	**90**
	$W(0.5)$	**100***	**100***	72	**100***	**99**	**100***	**100***	94	**100***	71
	$W(0.75)$	79	76	27	79	44	70	71	28	60	48
-0.1	$LF(0.5)$	28	19	27	34	9	34	35	10	29	21
	$LF(0.75)$	38	24	36	46	10	46	49	10	35	30
	$LF(1.0)$	46	30	44	55	10	56	59	12	39	38
	$LF(1.5)$	59	38	56	68	11	69	72	17	42	52
	HN	53	33	51	62	11	65	68	14	44	45
	$CH(1.0)$	38	23	37	45	11	51	53	11	44	30
	$CH(1.2)$	85	69	78	91	8	93	94	38	44	82
	Av.	66.3	57.8	55.9	70.1	41.6	68.9	70.5	40.0	57.7	52.4
	$\Gamma(0.2)$	**100***	**100***	**100***	**100***	**100***	**100***	**97**	**100***	**100***	**88**
	$\Gamma(0.4)$	**100***	**100***	**78**	**99**	**99**	**99**	46	**99**	**98**	46
	$\Gamma(0.8)$	33	37	12	29	24	32	25	24	23	26
	$W(0.25)$	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***
	$W(0.5)$	**100***	**100***	68	**100***	**99**	**100***	81	**100***	**100***	72
	$W(0.75)$	76	80	13	76	43	75	34	70	68	36
0.3	$LF(0.5)$	30	27	21	36	8	27	11	39	38	14
	$LF(0.75)$	40	35	27	49	8	38	16	52	49	20
	$LF(1.0)$	49	43	32	58	8	47	21	61	58	26
	$LF(1.5)$	62	55	41	71	8	62	32	74	70	38
	HN	56	48	39	66	9	56	26	70	67	32
	$CH(1.0)$	42	34	30	50	9	40	16	57	56	20
	$CH(1.2)$	87	84	59	93	5	88	63	94	91	69
	Av.	67.3	64.8	47.7	71.3	40.1	66.5	43.7	72.3	70.8	45.3
	$\Gamma(0.2)$	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**88**
	$\Gamma(0.4)$	**100***	**100***	**69**	**99**	**99**	**99**	**77**	**99**	**99**	**61**
	$\Gamma(0.8)$	30	35	11	28	22	32	18	30	25	25
	$W(0.25)$	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**93**
	$W(0.5)$	**100***	**100***	56	**100***	**99**	**100***	97	**100***	**100***	60
	$W(0.75)$	75	80	11	75	42	76	41	71	71	36
0.5	$LF(0.5)$	30	29	17	37	8	21	7	30	39	8
	$LF(0.75)$	40	39	22	49	8	31	5	41	51	11
	$LF(1.0)$	49	48	26	59	8	39	4	51	61	14
	$LF(1.5)$	63	60	33	72	8	54	3	65	73	22
	HN	57	53	31	67	8	47	4	58	69	17
	$CH(1.0)$	42	38	25	51	9	33	7	42	57	10
	$CH(1.2)$	87	87	45	93	5	82	4	89	93	48
	Av.	67.2	66.9	41.9	71.5	39.6	62.6	36.0	67.4	72.2	37.8
	$\Gamma(0.2)$	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***
	$\Gamma(0.4)$	**100***	**100***	**58**	**98**	**99**	**100***	**90**	**99**	**99**	**78**

	$\Gamma(0.8)$	29	34	10	27	20	34	18	31	27	24
	$W(0.25)$	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***
	$W(0.5)$	**100***	**100***	40	**100***	**99**	**100***	**99**	**100***	**100***	68
	$W(0.75)$	74	80	9	73	41	76	53	65	74	35
0.7	$LF(0.5)$	30	32	14	36	10	18	17	18	39	5
	$LF(0.75)$	41	43	17	49	10	26	18	25	52	5
	$LF(1.0)$	50	52	19	59	10	34	17	33	62	6
	$LF(1.5)$	63	65	23	73	10	47	15	45	74	9
	HN	58	58	24	67	9	41	18	38	70	7
	$CH(1.0)$	43	42	20	52	9	28	24	24	57	5
	$CH(1.2)$	89	90	31	93	11	77	8	73	94	24
	Av.	67.5	68.9	35.8	71.4	40.6	60.0	44.5	57.7	72.9	35.8
	$\Gamma(0.2)$	**100***	**100***	89	**100***	**100***	**100***	**100***	**100***	**100***	**100***
	$\Gamma(0.4)$	**100***	**100***	45	**95**	**98**	**100***	**96**	**99**	**100***	**91**
	$\Gamma(0.8)$	27	31	10	25	18	34	20	31	29	24
	$W(0.25)$	**100***	**100***	**92**	**100***	**100***	**100***	**100***	**100***	**100***	**100***
	$W(0.5)$	**100***	**100***	25	**100***	**99**	**100***	**100***	**100***	**100***	90
	$W(0.75)$	72	78	9	69	40	77	63	60	76	37
1.0	$LF(0.5)$	31	35	10	35	13	15	33	6	38	4
	$LF(0.75)$	42	47	11	48	14	21	42	7	51	3
	$LF(1.0)$	51	56	12	58	16	28	48	9	61	3
	$LF(1.5)$	64	69	13	71	19	39	54	14	73	2
	HN	58	63	14	66	16	35	55	11	68	2
	$CH(1.0)$	43	47	14	51	13	25	50	6	54	4
	$CH(1.2)$	89	92	14	92	28	70	66	31	94	2
	Av.	67.3	70.5	27.5	70.0	44.1	57.2	63.6	44.1	72.5	35.5
	$\Gamma(0.2)$	**100***	**100***	**68**	**100***	**100***	**100***	**100***	**100***	**100***	**100***
	$\Gamma(0.4)$	**99**	**99**	43	**90**	**97**	**100***	**98**	**99**	**100***	**95**
	$\Gamma(0.8)$	25	29	9	24	17	36	23	31	32	24
	$W(0.25)$	**100***	**100***	24	**100***	**100***	**100***	**100***	**100***	**100***	**100***
	$W(0.5)$	**100***	**100***	23	**99**	**99**	**100***	**100***	**100***	**100***	96
	$W(0.75)$	70	76	10	65	41	78	68	59	78	40
1.3	$LF(0.5)$	31	36	9	32	19	15	38	3	35	6
	$LF(0.75)$	43	49	9	44	23	22	50	2	48	5
	$LF(1.0)$	52	58	9	53	26	28	59	2	58	4
	$LF(1.5)$	65	72	9	67	33	39	71	2	71	3
	HN	59	66	9	62	28	36	68	2	66	4
	$CH(1.0)$	44	50	9	49	20	27	57	3	50	6
	$CH(1.2)$	89	93	8	90	51	69	91	2	92	1
	Av.	67.6	71.4	18.3	67.2	50.2	57.7	71.1	38.8	71.6	37.1
	$\Gamma(0.2)$	**100***	**100***	**64**	**99**	**100***	**100***	**100***	**100***	**100***	**100***
	$\Gamma(0.4)$	**99**	**99**	43	**84**	**96**	**100***	**99**	**99**	**100***	**96**
	$\Gamma(0.8)$	24	28	9	22	17	36	25	31	33	24
	$W(0.25)$	**100***	**100***	22	**100***	**100***	**100***	**100***	**100***	**100***	**100***
	$W(0.5)$	**100***	**100***	25	98	**99**	**100***	**100***	**100***	**100***	98
	$W(0.75)$	69	74	10	61	43	78	71	58	79	41
1.5	$LF(0.5)$	32	36	9	29	23	16	39	3	34	6
	$LF(0.75)$	43	49	9	40	28	22	52	2	46	6
	$LF(1.0)$	52	59	9	49	34	29	61	2	56	5
	$LF(1.5)$	66	72	10	62	42	39	73	1	70	4
	HN	60	67	9	58	36	36	70	2	64	5
	$CH(1.0)$	45	51	9	45	25	28	57	3	48	8
	$CH(1.2)$	89	93	10	87	64	69	93	0	92	1
	Av.	67.5	71.4	18.2	64.1	54.4	57.9	72.3	38.5	70.8	38.0
	$\Gamma(0.2)$	**100***	**100***	**67**	**95**	**100***	**100***	**100***	**100***	**100***	**100***

	$\Gamma(0.4)$	**99**	**98**	47	**76**	**96**	**100***	**99**	**99**	**100***	**97**
	$\Gamma(0.8)$	23	26	9	21	16	36	27	30	34	25
	$W(0.25)$	**100***	**100***	25	**100***	**100***	**100***	**100***	**100***	**100***	**100***
	$W(0.5)$	**100***	**100***	31	96	**99**	**100***	**100***	**100***	**100***	98
	$W(0.75)$	68	73	12	57	45	79	74	58	80	44
1.7	$LF(0.5)$	33	36	11	25	27	16	38	4	32	8
	$LF(0.75)$	44	49	11	35	35	24	51	3	44	7
	$LF(1.0)$	53	59	12	43	41	30	61	2	54	7
	$LF(1.5)$	66	72	14	56	51	41	74	1	67	6
	HN	61	67	12	52	45	38	70	2	61	7
	$CH(1.0)$	45	51	10	41	32	29	56	4	45	10
	$CH(1.2)$	90	93	18	83	74	70	94	0	91	3
	Av.	67.8	71.0	21.4	60.0	58.6	58.7	72.6	38.8	69.9	39.4

Table 12: Powers of 10% tests based on 100,000 simulations using empirical critical values; $n = 50$ ($k = 3$)

r	Alt.	$\hat{T}_n^{(r,4)}$	$\hat{T}_{n;c_1}^{(r,4)}$	$\hat{T}_{n;c_2}^{(r,4)}$	$\hat{T}_{n;c_3}^{(r,4)}$	$\hat{T}_{n;c_4}^{(r,4)}$	$\hat{T}_n^{*(r,4)}$	$\hat{T}_{n;c_1}^{*(r,4)}$	$\hat{T}_{n;c_2}^{*(r,4)}$	$\hat{T}_{n;c_3}^{*(r,4)}$	$\hat{T}_{n;c_4}^{*(r,4)}$
	$\Gamma(0.2)$	**100***	**100***	**100***	**100***	**100***	**100***	**92**	**100***	**100***	**72**
	$\Gamma(0.4)$	**100***	**100***	**80**	**99**	**99**	**99**	44	**97**	**94**	**69**
	$\Gamma(0.8)$	29	35	10	29	21	29	24	21	18	28
	$W(0.25)$	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**76**
	$W(0.5)$	**100***	**100***	78	**100***	**99**	**100***	75	**100***	**100***	56
	$W(0.75)$	72	79	9	76	39	71	33	65	58	44
0.5	$LF(0.5)$	30	27	21	35	10	26	9	39	30	18
	$LF(0.75)$	40	35	28	47	10	37	13	51	36	26
	$LF(1.0)$	49	43	33	57	10	47	18	61	41	34
	$LF(1.5)$	62	55	42	70	12	61	28	73	45	47
	HN	56	48	39	64	12	55	23	69	47	40
	$CH(1.0)$	41	34	30	48	11	41	13	57	45	26
	$CH(1.2)$	86	83	59	92	10	87	57	93	50	77
	Av.	66.6	64.5	48.3	70.5	41.0	65.6	40.8	71.3	58.6	47.2
	$\Gamma(0.2)$	**100***	**100***	**100***	**100***	**100***	**100***	**99**	**100***	**100***	**98**
	$\Gamma(0.4)$	**100***	**100***	**72**	**99**	**99**	**99**	**57**	**99**	**96**	**76**
	$\Gamma(0.8)$	28	34	9	28	19	30	19	26	20	26
	$W(0.25)$	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***
	$W(0.5)$	**100***	**100***	70	**100***	**99**	**100***	91	**100***	**100***	64
	$W(0.75)$	71	79	7	75	37	70	33	67	61	40
0.7	$LF(0.5)$	31	29	19	36	10	22	5	34	34	12
	$LF(0.75)$	42	39	24	48	11	32	4	46	44	17
	$LF(1.0)$	51	47	29	58	10	40	4	55	50	23
	$LF(1.5)$	63	60	36	71	11	54	5	69	58	34
	HN	58	53	34	65	11	49	5	64	58	28
	$CH(1.0)$	42	38	27	49	11	34	5	48	52	17
	$CH(1.2)$	88	87	50	92	9	82	14	91	73	64
	Av.	67.1	66.5	44.3	70.9	40.5	62.4	34.1	69.1	65.0	46.1
	$\Gamma(0.2)$	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***
	$\Gamma(0.4)$	**100***	**100***	**54**	**98**	**98**	**99**	**81**	**99**	**98**	**87**
	$\Gamma(0.8)$	25	32	8	27	16	31	16	29	22	25
	$W(0.25)$	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***
	$W(0.5)$	**100***	**100***	48	**100***	**99**	**100***	98	**100***	**100***	86
	$W(0.75)$	69	78	6	73	34	72	44	65	66	39
1.0	$LF(0.5)$	32	32	16	37	11	16	13	19	38	6
	$LF(0.75)$	43	43	18	49	12	23	12	28	49	8
	$LF(1.0)$	52	52	21	59	12	30	10	36	57	10

	$LF(1.5)$	64	64	26	71	12	42	8	48	67	16
	HN	58	58	25	66	12	37	10	41	66	12
	$CH(1.0)$	44	42	21	50	11	25	16	27	56	7
	$CH(1.2)$	88	89	35	92	13	72	3	76	88	36
	Av.	67.3	68.4	36.9	71.0	40.8	57.4	39.2	58.9	69.7	40.9
	$\Gamma(0.2)$	**100***	**100***	**73**	**100***	**100***	**100***	**100***	**100***	**100***	**100***
	$\Gamma(0.4)$	**99**	**99**	32	**96**	**95**	**99**	**94**	**99**	**99**	**94**
	$\Gamma(0.8)$	22	28	7	25	13	32	18	29	25	24
	$W(0.25)$	**100***	**100***	**80**	**100***	**100***	**100***	**100***	**100***	**100***	**100***
	$W(0.5)$	**100***	**100***	16	**100***	98	**100***	**100***	**100***	**100***	97
	$W(0.75)$	67	75	6	69	31	74	57	61	71	43
1.5	$LF(0.5)$	33	35	12	35	16	10	30	5	37	4
	$LF(0.75)$	44	47	13	48	18	14	36	5	50	3
	$LF(1.0)$	53	57	14	57	21	18	40	7	59	2
	$LF(1.5)$	66	70	15	70	24	25	44	10	71	2
	HN	60	64	15	65	21	23	46	7	67	2
	$CH(1.0)$	45	47	14	50	16	17	45	4	54	3
	$CH(1.2)$	89	92	16	92	35	50	49	22	92	3
	Av.	67.6	70.4	24.0	69.7	45.3	50.9	58.5	42.2	71.1	36.7
	$\Gamma(0.2)$	**100***	**100***	**51**	**100***	**100***	**100***	**100***	**100***	**100***	**100***
	$\Gamma(0.4)$	**99**	**99**	30	**94**	**94**	**100***	**96**	**99**	**99**	**96**
	$\Gamma(0.8)$	21	27	7	24	13	33	20	30	27	25
	$W(0.25)$	**100***	**100***	10	**100***	**100***	**100***	**100***	**100***	**100***	**100***
	$W(0.5)$	**100***	**100***	13	**100***	98	**100***	**100***	**100***	**100***	98
	$W(0.75)$	66	74	6	67	31	75	62	61	73	44
1.7	$LF(0.5)$	34	36	12	34	19	10	34	3	36	4
	$LF(0.75)$	45	48	13	46	23	13	43	3	49	3
	$LF(1.0)$	54	58	13	56	26	17	49	3	59	2
	$LF(1.5)$	67	71	14	69	31	23	57	3	71	2
	HN	61	65	14	64	27	21	58	3	67	2
	$CH(1.0)$	46	49	13	49	20	18	51	2	53	4
	$CH(1.2)$	89	92	15	91	47	44	72	6	92	1
	Av.	67.9	70.7	16.4	68.7	48.3	50.2	64.7	39.4	71.3	37.0

Table 13: Powers of 10% tests based on 100,000 simulations using empirical critical values; $n = 50$ $(k = 4)$

Our simplest omnibus test when $n = 20$ is $\hat{T}_{n;c_1}^{(1,3)}$ with Av. power 51.8.

For $k = 2$ the test

$$\hat{T}_{n;c_1}^{*(1,2)} = \frac{3n(n-1)}{n+1}\left[\frac{4}{n(n-1)}\sum_{i=1}^{n}(n-i)\log\frac{1}{1-\exp\left(-X_{n-i+1}/\overline{X}_n\right)} - 1\right]^2$$

has Av. power 52.3 (cf. Morris et al [12]).

When $n = 50$ our recommended test is $\hat{T}_{n;c_3}^{(0.7,3)}$ with Av. power 72.9.

For $k = 2$ the test $\hat{T}_{n;c_1}^{*(1,2)}$ has Av. power 73.5.

The role of k (=2,3,4) in powers of these tests is illustrated in the following tables.

r	-0.1			0.3			0.5		
Alt.↓ Tests→	$\hat{T}^{*(r,2)}_{n;c_3}$	$\hat{T}^{(r,3)}_{n;c_3}$	$\hat{T}^{(r,4)}_{n;c_3}$	$\hat{T}^{*(r,2)}_{n;c_3}$	$\hat{T}^{(r,3)}_{n;c_3}$	$\hat{T}^{(r,4)}_{n;c_3}$	$\hat{T}^{*(r,2)}_{n;c_3}$	$\hat{T}^{(r,3)}_{n;c_1}$	$\hat{T}^{(r,4)}_{n;c_3}$
$\Gamma(0.2)$	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***
$\Gamma(0.4)$	**87**	**89**	**90**	**91**	**84**	**86**	**92**	**93**	**85**
$\Gamma(0.8)$	18	19	18	20	18	17	21	21	16
$W(0.25)$	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***
$W(0.5)$	**95**	**96**	**96**	**97**	**95**	**95**	**97**	**97**	**94**
$W(0.75)$	45	46	43	48	44	42	49	48	42
$LF(0.5)$	20	20	19	19	20	20	18	18	20
$LF(0.75)$	25	24	24	24	25	25	23	22	26
$LF(1.0)$	30	29	28	29	30	30	28	26	31
$LF(1.5)$	37	36	35	37	38	38	36	33	38
HN	35	33	31	34	34	34	33	29	35
$CH(1.0)$	26	24	23	25	25	25	24	22	26
$CH(1.2)$	58	57	54	58	58	57	57	52	58
Av.	52.0	51.7	51.0	52.4	51.5	51.6	52.1	50.8	51.6
r	0.7			1.0			1.5		
Alt.↓ Tests→	$\hat{T}^{(r,2)}_{n;c_1}$	$\hat{T}^{(r,3)}_{n;c_1}$	$\hat{T}^{(r,4)}_{n;c_3}$	$\hat{T}^{*(r,2)}_{n;c_1}$	$\hat{T}^{(r,3)}_{n;c_1}$	$\hat{T}^{(r,4)}_{n;c_1}$	$\hat{T}^{*(r,2)}_{n;c_1}$	$\hat{T}^{*(r,3)}_{n;c_3}$	$\hat{T}^{(r,4)}_{n;c_1}$
$\Gamma(0.2)$	**100***	**100***	**99**	**100***	**100***	**100***	**100***	**100***	**100***
$\Gamma(0.4)$	**89**	**91**	**82**	**88**	**88**	**89**	**92**	**89**	**84**
$\Gamma(0.8)$	19	19	17	18	19	18	21	21	17
$W(0.25)$	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***
$W(0.5)$	**96**	**97**	**94**	**96**	**96**	**96**	**97**	**96**	**94**
$W(0.75)$	47	46	41	45	45	43	49	47	41
$LF(0.5)$	19	19	21	20	20	20	18	18	20
$LF(0.75)$	24	24	26	25	25	25	23	23	26
$LF(1.0)$	29	28	31	30	29	29	28	28	31
$LF(1.5)$	37	35	39	38	37	36	36	35	38
HN	33	31	34	35	33	32	33	32	34
$CH(1.0)$	24	23	26	26	25	24	24	25	26
$CH(1.2)$	58	55	58	59	57	55	57	54	58
Av.	52.1	51.4	51.4	52.3	51.8	51.2	52.1	51.4	51.5

Table 14: Powers of 10% selected tests using empirical values; $n = 20$

r	-0.1			0.3			0.5		
Alt.↓ Tests→	$\hat{T}^{*(r,2)}_{n;c_3}$	$\hat{T}^{*(r,3)}_{n;c_1}$	$\hat{T}^{*(r,4)}_{n;c_1}$	$\hat{T}^{*(r,2)}_{n;c_3}$	$\hat{T}^{*(r,3)}_{n;c_2}$	$\hat{T}^{(r,4)}_{n;c_3}$	$\hat{T}^{*(r,2)}_{n;c_1}$	$\hat{T}^{*(r,3)}_{n;c_3}$	$\hat{T}^{*(r,4)}_{n;c_2}$
$\Gamma(0.2)$	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***
$\Gamma(0.4)$	**100***	**97**	**97**	**100***	**99**	**100***	**99**	**99**	**97**
$\Gamma(0.8)$	29	26	25	32	24	30	24	25	21
$W(0.25)$	**100***	**91**	**91**	**100***	**100***	**100***	**100***	**100***	**100***
$W(0.5)$	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***
$W(0.75)$	77	71	70	80	70	77	71	71	65
$LF(0.5)$	39	35	35	36	39	35	38	39	39
$LF(0.75)$	53	49	48	49	52	46	50	51	51
$LF(1.0)$	63	59	58	59	61	55	59	61	61
$LF(1.5)$	76	72	72	72	74	68	71	73	73
HN	71	68	67	67	70	62	69	69	69
$CH(1.0)$	56	53	53	51	57	46	58	57	57
$CH(1.2)$	95	94	94	94	94	91	93	93	93
Av.	73.7	70.5	70.1	72.4	72.3	69.9	71.6	72.2	71.3
r	0.7			1.0			1.5		

Alt.↓ Tests→	$\hat{T}_{n;c_1}^{*(r,2)}$	$\hat{T}_{n;c_3}^{*(r,3)}$	$\hat{T}_{n;c_3}^{(r,4)}$	$\hat{T}_{n;c_1}^{*(r,2)}$	$\hat{T}_{n;c_3}^{*(r,3)}$	$\hat{T}_{n;c_3}^{(r,4)}$	$\hat{T}_{n;c_4}^{(r,2)}$	$\hat{T}_{n;c_1}^{*(r,3)}$	$\hat{T}_{n;c_3}^{*(r,4)}$
$\Gamma(0.2)$	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***
$\Gamma(0.4)$	**99**	**99**	**99**	**100***	**100***	**98**	**98**	**99**	**99**
$\Gamma(0.8)$	27	27	28	30	29	27	24	25	25
$W(0.25)$	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***
$W(0.5)$	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***
$W(0.75)$	75	74	75	78	76	73	69	71	71
$LF(0.5)$	40	39	36	39	38	37	39	39	37
$LF(0.75)$	53	52	48	52	51	49	51	52	50
$LF(1.0)$	63	62	58	62	61	59	61	61	59
$LF(1.5)$	76	74	71	75	73	71	74	73	71
HN	72	70	65	71	68	66	69	70	67
$CH(1.0)$	58	57	49	55	54	50	52	57	54
$CH(1.2)$	96	94	92	95	94	92	94	93	92
Av.	73.6	72.9	70.9	73.5	72.5	71.0	71.6	72.3	71.1

Table 15: Powers of 10% selected tests using empirical values; $n = 50$

7.2. Rayleigh Distribution

We have selected tests and alternatives in Table 16 from Meintanis and Iliopoulos [10] as standards of comparison with our tests. When $n = 20$ the test–statistics $\hat{T}_n^{(r,k)}$, $\hat{T}_n^{*(r,k)}$ and their components were investigated for $r = -0.499$, -0.45, -0.4, -0.1, 0.1, 0.3, 0.5, 0.7, 0.9, 1.0, 1.1, 1.3, 1.5, 1.7, 2.0 with $k = 3$ and $k = 4$. (The case $k = 2$ was discussed in Morris et al [12]). Though $\hat{T}_{n;c_1}^{(r,k)}$ and $\hat{T}_{n;c_3}^{(r+1,k)}$ are identical, and similarly for duals, we do simulations for all tests. Critical values were simulated using 100,000 samples of size 20, and the associated powers were obtained using 50,000 samples, but only some results are presented here. We include here (Table 17 and Table 18) simulations for our favorable omnibus tests with Av. powers ≥ 63.0. Moreover, many tests have powers greater than those in Table 16. These are shown in boldface, while bold face with star denotes the maximum power. Such tests can be recommended for particular alternatives, but often they do not have good omnibus properties. Tests with $k = 4$ behave poorly so we give only tests with Av. power ≥ 60.0.

a	Alt.	$T_{n,a}^L$	BH^L	HE^L	HM_1^L	HM_2^L	$T_{n,a}^{MO}$	BH^{MO}	HE^{MO}	HM_1^{MO}	HM_2^{MO}
	$W(1.0)$	**97***	93	93	86	83	96	94	94	80	83
	$W(3.0)$	40	52	53	45	41	35	47	47	41	16
	$G(1.5)$	76	71	71	56	57	72	73	72	47	58
	$G(2.0)$	43	43	43	30	36	37	44	45	23	38
	$IG(0.5)$	98	98	98	96	96	96	98	98	93	96
	$IG(1.5)$	48	63	64	51	61	33	62	64	38	62
	$LN(0.8)$	66	75	75	62	70	55	74	75	50	69
1.0	$LN(1.5)$	**100***	**100***	**100***	99	99	**100***	**100***	**100***	99	99
	$GO(0.5)$	84	70	69	55	47	**84***	75	72	52	53

	$GO(1.5)$	57	32	30	24	15	60	40	37	25	22
	$PW(1.0)$	43	14	12	23	9	50	24	19	28	22
	$PW(2.0)$	**99***	91	90	86	64	**99***	96	94	87	82
	$LF(2.0)$	**70***	52	51	38	33	**70***	57	55	35	37
	$LF(4.0)$	57	38	36	26	23	**58***	43	41	24	27
	$EP(1.0)$	70	47	46	36	26	**71***	55	52	35	34
	$EP(2.0)$	16	28	29	26	35	13	24	25	25	20
	Av	66.5	60.4	60.0	52.4	49.7	64.3	62.9	61.9	48.9	51.1
	$W(1.0)$	96	92	91	85	83	96	95	94	84	86
	$W(3.0)$	47	51	50	47	27	43	44	41	35	0
	$G(1.5)$	75	69	68	59	61	75	75	73	57	63
	$G(2.0)$	44	43	43	36	41	43	48	47	34	44
	$IG(0.5)$	98	98	98	95	96	98	**99***	98	96	97
	$IG(1.5)$	57	65	65	60	65	50	67	69	57	69
	$LN(0.8)$	71	75	75	70	74	66	77	78	66	76
2.0	$LN(1.5)$	**100***	**100***	**100***	99	99	**100***	**100***	**100***	99	99
	$GO(0.5)$	80	65	63	52	49	83	75	71	54	56
	$GO(1.5)$	46	25	24	18	15	54	38	34	24	23
	$PW(1.0)$	30	8	7	13	6	37	21	19	26	24
	$PW(2.0)$	98	84	82	75	54	**99***	95	93	85	84
	$LF(2.0)$	63	47	46	35	36	68	58	54	38	41
	$LF(4.0)$	49	34	32	24	26	54	43	39	27	30
	$EP(1.0)$	61	40	40	30	27	67	54	49	36	35
	$EP(2.0)$	22	30	29	**35***	24	18	24	24	29	24
	Av	64.8	57.9	57.1	52.1	48.9	65.7	62.6	61.4	52.9	53.2
	$W(1.0)$	95	90	90	83	81	96	96	96	92	92
	$W(3.0)$	52	46	46	22	0	45	40	37	14	2
	$G(1.5)$	74	67	67	62	61	77	77	76	70	70
	$G(2.0)$	45	43	44	42	43	49	49	48	48	46
	$IG(0.5)$	98	98	97	96	95	**99***	**99***	98	98	97
	$IG(1.5)$	63	66	67	67	67	62	66	65	70	67
	$LN(0.8)$	75	76	76	74	74	75	77	76	78	76
5.0	$LN(1.5)$	**100***	**100***	99	99	99	**100***	**100***	**100***	**100***	**100***
	$GO(0.5)$	74	59	59	49	47	80	79	80	68	68
	$GO(1.5)$	34	20	20	14	14	46	44	50	32	34
	$PW(1.0)$	13	5	5	4	1	25	25	41	30	33
	$PW(2.0)$	93	73	72	51	41	97	97	98	94	96
	$LF(2.0)$	56	43	43	36	36	65	63	64	50	52
	$LF(4.0)$	41	30	31	26	26	50	48	50	38	39
	$EP(1.0)$	51	34	34	26	25	62	60	63	46	48
	$EP(2.0)$	28	29	29	19	0	21	19	17	17	17
	Av	62.0	54.9	54.9	48.1	44.4	65.6	64.9	66.2	59.1	58.6
	$W(1.0)$	93	88	82	79	78	96	96	96	96	95
	$W(3.0)$	51	41	41	0	0	44	40	41	37	32
	$G(1.5)$	72	66	69	59	58	77	77	**78***	77	75
	$G(2.0)$	45	43	**57***	41	41	47	48	54	49	47
	$IG(0.5)$	98	97	94	94	94	**99***	**99***	98	**99***	98
	$IG(1.5)$	65	67	**74***	65	64	60	62	68	65	64
	$LN(0.8)$	76	75	**79***	72	72	74	75	77	76	75
10.0	$LN(1.5)$	**100***	99	97	98	98	**100***	**100***	**100***	**100***	**100***
	$GO(0.5)$	69	56	58	44	43	81	80	81	80	78
	$GO(1.5)$	29	18	32	12	12	47	45	51	46	46
	$PW(1.0)$	9	3	5	1	0	26	25	33	29	33
	$PW(2.0)$	88	65	43	37	34	98	97	98	97	98
	$LF(2.0)$	52	41	51	34	33	65	64	67	64	62
	$LF(4.0)$	38	29	44	25	24	50	49	54	49	47

	$EP(1.0)$	45	31	42	23	22	62	61	65	61	60
	$EP(2.0)$	28	25	25	0	0	20	18	18	17	14
	Av	59.9	52.8	55.8	42.8	42.1	65.4	64.8	**67.4***	65.1	64.0

Table 16: (Source: Meintanis and Iliopoulos [10]) Percentage of rejection for 10,000 Monte Carlo samples of size $n = 20$ at significance level $\alpha = 0.05$

The meaning of the headings and the test–statistics can be found in Meintanis and Illiopoulos [10]:

T: Meintanis and Iliopoulos;

BH: Baringhaus and Henze;

HE: Henze;

HM: Henze and Meintanis.

The alternatives considered are:

— the Weibull distribution with density $\theta x^{\theta-1}\exp(-x^{\theta})$, denoted by $W(\theta)$;

— the gamma distribution with density $\Gamma(\theta)^{-1}x^{\theta-1}\exp(-x)$, denoted by $\Gamma(\theta)$;

— the inverse Gaussian law $IG(\theta)$ with density

$$(\theta/2\pi)^{1/2}x^{-3/2}\exp\left(-\theta(x-1)^2/2x\right);$$

— the lognormal law $LN(\theta)$ with density

$$(\theta x)^{-1}(2\pi)^{-1/2}\exp\left(-(\log x)^2/(2\theta^2)\right);$$

— the Gompertz law $GO(\theta)$ with distribution function $1-\exp\left(\theta^{-1}(1-e^x)\right)$;

— the power distribution $PW(\theta)$ with density $\theta^{-1}x^{(1-\theta)/\theta}$, $0 \leq x \leq 1$;

— the linear increasing failure rate law $LF(\theta)$ with density

$$(1+\theta x)\exp\left(-x-\theta x^2/2\right);$$

— the exponential–power $EP(\theta)$ law with distribution function

$$1-\exp\left(1-\exp\left(x^{\theta}\right)\right).$$

r	Alt.	$\hat{T}_n^{(r,3)}$	$\hat{T}_{n;c_1}^{(r,3)}$	$\hat{T}_{n;c_2}^{(r,3)}$	$\hat{T}_{n;c_3}^{(r,3)}$	$\hat{T}_{n;c_4}^{(r,3)}$	$\hat{T}_n^{*(r,3)}$	$\hat{T}_{n;c_1}^{*(r,3)}$	$\hat{T}_{n;c_2}^{*(r,3)}$	$\hat{T}_{n;c_3}^{*(r,3)}$	$\hat{T}_{n;c_4}^{*(r,3)}$
	$W(1.0)$	96	91	70	95	77	71	78	66	71	54
	$W(3.0)$	47	0	**54**	**54**	1	4	15	4	1	44
	$G(1.5)$	70	56	59	70	34	47	57	41	46	40
	$G(2.0)$	35	22	34	37	12	31	40	26	30	28
	$IG(0.5)$	97	84	98	97	46	92	94	88	92	77
	$IG(1.5)$	42	5	50	44	6	55	63	50	55	42
	$LN(0.8)$	59	21	66	62	6	64	71	57	63	51
-0.45	$LN(1.5)$	**100***	99	70	**100***	96	98	98	95	98	88
	$GO(0.5)$	80	77	51	77	62	34	43	29	33	30
	$GO(1.5)$	53	58	26	41	49	16	12	17	15	13
	$PW(1.0)$	42	**53**	16	24	49	33	1	36	34	3
	$PW(2.0)$	**99***	**99***	74	97	98	43	34	43	43	23
	$LF(2.0)$	65	64	39	58	51	25	32	22	24	26
	$LF(4.0)$	52	53	29	44	43	20	24	18	19	20
	$EP(1.0)$	65	66	36	57	54	20	22	18	19	20
	$EP(2.0)$	23	2	27	26	4	3	10	3	2	25
	Av.	64.1	53.2	50.0	61.5	42.9	41.0	43.4	38.2	40.3	36.6
	$W(1.0)$	95	96	48	93	84	87	87	68	85	43
	$W(3.0)$	38	27	34	**56**	1	38	38	9	1	45
	$G(1.5)$	66	68	19	68	42	65	65	38	60	41
	$G(2.0)$	31	32	16	37	14	43	43	21	39	30
	$IG(0.5)$	96	94	24	98	68	97	97	91	97	58
	$IG(1.5)$	43	19	52	52	3	66	66	44	65	36
	$LN(0.8)$	57	42	40	67	9	75	74	54	73	44
-0.1	$LN(1.5)$	**100***	**100***	83	**100***	99	**100***	99	98	99	74
	$GO(0.5)$	79	83	36	69	69	54	54	30	49	33
	$GO(1.5)$	56	**61***	30	30	54	17	17	18	15	15
	$PW(1.0)$	**52**	**53**	37	13	**56**	3	3	27	19	2
	$PW(2.0)$	**99***	**99***	92	91	**99***	66	56	59	68	20
	$LF(2.0)$	65	**70***	29	50	56	40	40	21	34	30
	$LF(4.0)$	53	**58***	25	36	47	28	28	16	24	24
	$EP(1.0)$	66	**71***	32	46	60	30	30	19	26	23
	$EP(2.0)$	22	10	25	31	5	24	26	5	5	27
	Av.	63.6	61.4	38.9	58.7	47.9	52.0	51.4	38.7	47.5	34.1
	$W(1.0)$	95	96	46	92	86	91	67	83	88	36
	$W(3.0)$	41	42	23	**56**	1	43	47	0	2	46
	$G(1.5)$	68	71	12	67	45	69	54	57	64	31
	$G(2.0)$	34	35	10	38	15	44	32	35	42	20
	$IG(0.5)$	97	96	24	98	76	98	54	96	97	61
	$IG(1.5)$	47	28	36	56	5	66	34	62	67	19
	$LN(0.8)$	60	50	22	69	14	75	42	70	75	28
0.1	$LN(1.5)$	**100***	**100***	87	**100***	99	**100***	62	99	**100***	84
	$GO(0.5)$	80	83	34	64	71	64	62	47	55	40
	$GO(1.5)$	56	58	28	26	55	25	29	16	18	30
	$PW(1.0)$	**54**	46	42	9	**59***	8	10	22	16	12
	$PW(2.0)$	**99***	**99***	93	86	**99***	82	79	71	77	32
	$LF(2.0)$	65	69	26	46	57	46	46	32	39	35
	$LF(4.0)$	52	56	21	32	47	33	34	22	28	29
	$EP(1.0)$	66	70	30	41	61	40	43	25	31	38
	$EP(2.0)$	25	17	21	33	5	29	32	4	9	29
	Av.	65.0	63.5	34.7	56.9	49.7	57.1	45.5	46.3	50.4	35.7
	$W(1.0)$	95	96	38	90	86	93	47	91	90	47
	$W(3.0)$	45	49	14	**55**	1	37	28	30	4	36
	$G(1.5)$	69	71	8	65	44	70	32	68	67	36

	$G(2.0)$	35	36	7	38	15	44	21	44	44	23
	$IG(0.5)$	97	97	26	98	81	98	77	98	98	60
	$IG(1.5)$	49	36	23	58	6	66	34	67	68	28
	$LN(0.8)$	62	56	12	70	17	75	43	75	76	36
0.3	$LN(1.5)$	**100***	**100***	86	**100***	99	**100***	91	**100***	**100***	74
	$GO(0.5)$	79	81	27	60	69	68	31	63	60	47
	$GO(1.5)$	52	51	23	22	51	32	30	25	22	35
	$PW(1.0)$	**51**	36	42	7	**56**	16	29	12	14	25
	$PW(2.0)$	**99***	**99***	90	78	**99***	92	42	86	83	84
	$LF(2.0)$	62	65	19	43	53	50	27	46	43	37
	$LF(4.0)$	49	51	16	29	43	37	24	33	31	30
	$EP(1.0)$	63	64	24	36	58	47	32	40	36	42
	$EP(2.0)$	27	21	17	34	4	25	15	28	12	21
	Av.	64.6	63.0	29.5	55.2	48.8	59.3	37.7	56.5	52.9	41.3
	$W(1.0)$	84	86	10	70	63	95	93	89	96	81
	$W(3.0)$	**56**	49	35	22	**56**	0	10	0	9	0
	$G(1.5)$	56	62	5	53	28	74	71	57	77	47
	$G(2.0)$	30	39	3	38	10	47	46	30	51	26
	$IG(0.5)$	96	97	8	90	86	98	98	94	**99***	88
	$IG(1.5)$	54	62	2	63	19	66	69	37	69	36
	$LN(0.8)$	65	72	2	70	31	76	77	49	78	44
1.7	$LN(1.5)$	99	99	8	95	97	**100***	**100***	**100***	**100***	99
	$GO(0.5)$	49	53	10	36	32	77	67	70	78	61
	$GO(1.5)$	16	16	8	10	13	46	28	46	43	40
	$PW(1.0)$	14	6	20	9	13	41	15	46	26	47
	$PW(2.0)$	78	61	50	21	76	98	90	98	97	96
	$LF(2.0)$	32	37	7	27	19	60	50	53	61	45
	$LF(4.0)$	21	26	5	20	12	47	36	41	47	35
	$EP(1.0)$	27	29	9	18	19	59	44	56	58	49
	$EP(2.0)$	34	34	13	21	29	1	18	1	8	1
	Av.	50.7	51.9	12.3	41.3	37.8	61.5	57.0	54.0	62.3	49.7
	$W(1.0)$	81	82	13	64	63	95	94	88	96	83
	$W(3.0)$	**55**	43	47	12	58	0	14	0	2	0
	$G(1.5)$	54	60	6	49	29	74	73	57	**78***	49
	$G(2.0)$	30	39	3	36	11	47	48	30	51	27
	$IG(0.5)$	96	96	11	85	87	98	**99***	93	**99***	89
	$IG(1.5)$	55	64	3	61	25	67	69	37	68	40
	$LN(0.8)$	65	72	4	67	36	76	78	49	78	47
2.0	$LN(1.5)$	99	99	11	91	97	**100***	**100***	**100***	**100***	99
	$GO(0.5)$	45	47	11	31	30	78	72	70	80	62
	$GO(1.5)$	13	14	7	8	10	47	33	46	46	41
	$PW(1.0)$	12	6	15	10	10	41	17	46	29	46
	$PW(2.0)$	70	47	50	15	69	98	93	97	97	96
	$LF(2.0)$	29	34	7	24	17	61	54	53	63	47
	$LF(4.0)$	19	24	5	18	10	48	40	42	49	36
	$EP(1.0)$	24	25	9	16	17	60	49	56	61	50
	$EP(2.0)$	35	32	21	14	34	1	18	1	2	2
	Av.	48.9	49.1	14.2	37.6	37.6	61.9	59.4	54.1	62.5	50.9

Table 17: Powers of 5% tests when $n = 20$ ($k = 3$), based on 50,000 simulations using critical values obtained from 100,000 samples

r	Alt.	$\hat{T}_n^{(r,4)}$	$\hat{T}_{n;c_1}^{(r,4)}$	$\hat{T}_{n;c_2}^{(r,4)}$	$\hat{T}_{n;c_3}^{(r,4)}$	$\hat{T}_{n;c_4}^{(r,4)}$	$\hat{T}_n^{*(r,4)}$	$\hat{T}_{n;c_1}^{*(r,4)}$	$\hat{T}_{n;c_2}^{*(r,4)}$	$\hat{T}_{n;c_3}^{*(r,4)}$	$\hat{T}_{n;c_4}^{*(r,4)}$
	$W(1.0)$	95	89	66	95	76	62	78	38	47	83
	$W(3.0)$	47	0	**55**	53	3	22	14	21	17	39
	$G(1.5)$	65	52	54	66	33	43	58	27	32	60
	$G(2.0)$	28	20	29	31	11	28	40	19	22	40
	$IG(0.5)$	95	79	96	96	43	84	94	67	77	95
	$IG(1.5)$	28	3	38	32	1	44	63	29	35	61
	$LN(0.8)$	48	17	57	52	4	54	71	37	45	70
-0.45	$LN(1.5)$	**100***	99	64	**100***	96	94	98	83	91	98
	$GO(0.5)$	78	74	49	76	60	41	43	31	31	49
	$GO(1.5)$	52	56	26	44	47	31	12	31	29	14
	$PW(1.0)$	43	**52**	18	29	46	29	1	30	31	3
	$PW(2.0)$	**99***	**99***	75	98	97	57	34	52	50	42
	$LF(2.0)$	63	62	37	58	49	34	33	27	27	36
	$LF(4.0)$	50	52	28	43	42	27	24	24	23	26
	$EP(1.0)$	63	64	35	58	53	35	22	32	31	26
	$EP(2.0)$	21	2	26	24	5	13	9	12	10	26
	Av.	61.0	51.3	47.0	59.7	41.6	43.6	43.4	34.9	37.4	47.9
	$W(1.0)$	94	90	51	94	77	72	79	32	49	83
	$W(3.0)$	44	0	**54**	**54**	2	23	13	20	13	37
	$G(1.5)$	63	54	43	66	34	51	58	22	33	61
	$G(2.0)$	27	21	25	31	11	33	41	16	22	40
	$IG(0.5)$	95	81	86	96	47	89	94	62	80	94
	$IG(1.5)$	27	3	43	32	0	50	64	22	38	61
	$LN(0.8)$	46	19	58	53	4	60	72	29	47	71
-0.4	$LN(1.5)$	**100***	99	59	**100***	96	96	98	81	92	97
	$GO(0.5)$	78	76	38	75	62	48	43	28	29	49
	$GO(1.5)$	52	57	23	42	48	30	12	30	27	15
	$PW(1.0)$	45	**52**	18	27	47	28	1	32	33	2
	$PW(2.0)$	**99***	**99***	80	97	97	59	36	49	45	41
	$LF(2.0)$	62	63	30	57	50	38	33	25	25	37
	$LF(4.0)$	50	53	23	42	43	30	24	23	22	26
	$EP(1.0)$	63	65	29	56	53	37	22	31	28	27
	$EP(2.0)$	20	2	27	25	5	13	9	11	7	25
	Av.	60.3	52.0	42.9	59.2	42.4	47.5	43.7	32.0	36.8	48.0
	$W(1.0)$	93	95	54	91	86	90	86	69	73	78
	$W(3.0)$	43	38	33	**58**	2	38	42	1	0	43
	$G(1.5)$	59	67	14	61	44	69	66	43	48	62
	$G(2.0)$	23	30	7	30	15	44	41	28	31	39
	$IG(0.5)$	93	94	29	97	74	98	77	92	93	62
	$IG(1.5)$	22	18	23	42	3	67	56	52	56	47
	$LN(0.8)$	40	41	13	59	12	76	63	61	65	53
0.1	$LN(1.5)$	**100***	**100***	91	**100***	99	**100***	70	98	98	65
	$GO(0.5)$	77	82	38	64	70	62	63	31	34	64
	$GO(1.5)$	53	58	29	27	54	23	24	16	14	27
	$PW(1.0)$	**51**	48	37	12	**55**	7	8	36	34	8
	$PW(2.0)$	**99***	**99***	93	90	98	75	75	45	47	77
	$LF(2.0)$	61	68	28	44	57	45	46	22	24	47
	$LF(4.0)$	49	55	23	30	48	32	33	17	18	34
	$EP(1.0)$	63	69	32	41	60	38	39	18	18	42
	$EP(2.0)$	24	14	24	31	7	28	31	2	3	30
	Av.	59.3	61.1	35.6	54.9	49.1	55.7	51.2	39.5	41.0	48.6
	$W(1.0)$	94	95	50	90	87	90	50	82	78	63
	$W(3.0)$	47	47	25	**59***	2	38	39	2	0	41
	$G(1.5)$	61	67	11	60	45	66	40	58	53	47

	$G(2.0)$	24	31	5	30	15	42	26	37	34	29
	$IG(0.5)$	94	95	35	97	80	98	57	96	95	56
	$IG(1.5)$	25	25	15	45	5	65	28	64	60	33
	$LN(0.8)$	44	47	8	61	16	74	36	72	68	39
0.3	$LN(1.5)$	**100***	**100***	92	**100***	99	**100***	74	99	99	59
	$GO(0.5)$	76	80	32	61	69	62	52	46	40	57
	$GO(1.5)$	52	52	25	23	51	25	33	14	13	29
	$PW(1.0)$	49	40	37	9	**54**	11	16	20	29	13
	$PW(2.0)$	**99***	**99***	92	86	98	83	80	63	57	85
	$LF(2.0)$	60	64	23	41	55	45	40	32	28	42
	$LF(4.0)$	47	50	18	27	44	32	31	22	19	31
	$EP(1.0)$	62	65	27	37	58	39	43	24	21	42
	$EP(2.0)$	27	19	22	33	6	29	26	9	4	29
	Av.	60.0	61.0	32.4	53.6	49.1	56.2	41.9	46.2	43.6	43.4
	$W(1.0)$	93	95	43	88	85	89	45	87	82	61
	$W(3.0)$	50	52	18	**59***	2	35	22	32	1	36
	$G(1.5)$	60	67	8	59	42	66	32	62	57	43
	$G(2.0)$	24	31	4	30	13	41	21	39	37	26
	$IG(0.5)$	95	96	38	97	82	98	73	97	96	63
	$IG(1.5)$	26	30	9	48	6	64	33	63	63	31
	$LN(0.8)$	45	52	5	63	18	73	42	72	71	36
0.5	$LN(1.5)$	**100***	**100***	91	**100***	99	**100***	88	**100***	99	83
	$GO(0.5)$	73	77	25	57	65	63	34	55	45	54
	$GO(1.5)$	46	45	20	20	45	28	31	19	14	31
	$PW(1.0)$	44	31	35	8	50	15	28	11	24	18
	$PW(2.0)$	98	98	88	80	98	88	56	77	65	89
	$LF(2.0)$	56	59	17	38	49	45	29	38	31	40
	$LF(4.0)$	42	45	13	25	37	32	25	26	21	30
	$EP(1.0)$	57	59	21	33	53	41	34	32	24	42
	$EP(2.0)$	29	24	19	35	5	26	13	31	6	25
	Av.	58.5	60.1	28.3	52.4	46.8	56.6	37.8	52.7	46.0	44.2

Table 18: Powers of 5% tests when $n = 20$ ($k = 4$), based on 100,000 simulations using critical values obtained from 100,000 samples

Here test with the greatest Av. power (65.0) is $\hat{T}_n^{(0.1,3)}$. But a recommended test is also $\hat{T}_{n;c_1}^{(-0.1,3)}$ with maximum powers for 6 alternatives. We should mention here that for $k = 2$ we have tests with Av. power ≥ 66.0: $\hat{T}_n^{(-0.1,2)}$ (66.5), $\hat{T}_n^{(0.1,2)}$ (67.0), $\hat{T}_{n;c_3}^{*(1,2)}$ (66.1). Other favorable tests are: $\hat{T}_{n;c_1}^{(-0.1,2)}$ with Av. power $= 64.0$ and having maximum powers for 7 alternatives. $\hat{T}_{n;c_3}^{*(1,2)}$ with Av. power $= 66.1$ and having maximum powers for 4 alternatives, and $\hat{T}_{n;c_3}^{*(1.5,2)}$ with Av. power $= 65.3$ and having maximum powers for 7 alternatives. So the simplest recommended tests are:

$$\hat{T}_{n;c_3}^{*(1,2)} = \frac{27n(n-1)}{\left(116 - 3\left(12 - \pi^2\right)^2\right)(n-1) + 154}$$

$$\cdot \left[\frac{4}{n(n-1)} \sum_{i=1}^{n} (n-i) \log^2 \frac{1}{1-\exp\left(X^2_{n-i+1}/\overline{X^2_n}\right)} - 1 \right]^2$$

and $\hat{T}^{*(1.5,2)}_{n;c_3}$ with Av. powers: 66.1 and 65.3, respectively (cf. Morris et al [12]).

r	-0.1			0.1			0.5		
Alt.↓ Tests→	$\hat{T}^{(r,2)}_n$	$\hat{T}^{(r,3)}_n$	$\hat{T}^{(r,4)}_{n;c_1}$	$\hat{T}^{(r,2)}_n$	$\hat{T}^{(r,3)}_n$	$\hat{T}^{(r,4)}_{n;c_1}$	$\hat{T}^{*(r,2)}_{n;c_3}$	$\hat{T}^{(r,3)}_n$	$\hat{T}^{(r,4)}_{n;c_1}$
$W(1.0)$	96	95	94	96	95	95	96	94	95
$W(3.0)$	31	38	21	33	41	38	33	47	52
$\Gamma(1.5)$	72	66	64	74	68	67	**78**	67	67
$\Gamma(2.0)$	41	31	28	42	34	30	49	33	31
$IG(0.5)$	97	96	91	98	97	94	**99***	97	96
$IG(1.5)$	59	43	12	60	47	18	66	49	30
$LN(0.8)$	69	57	34	70	60	41	77	62	52
$LN(1.5)$	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***	**100***
$GO(0.5)$	81	79	82	82	80	82	80	75	77
$GO(1.5)$	56	56	**61***	55	56	58	45	45	45
$PW(1.0)$	**54**	**52**	**54**	**54**	**54**	48	26	45	31
$PW(2.0)$	**99***	**99***	**99***	**99***	**99***	**99***	97	98	98
$LF(2.0)$	67	65	68	67	65	68	63	57	59
$LF(4.0)$	54	53	57	54	52	55	49	43	45
$EP(1.0)$	67	66	70	67	66	69	61	57	59
$EP(2.0)$	21	22	8	21	25	14	22	27	24
Av.	66.5	63.6	58.8	67.0	65.0	61.1	65.1	62.3	60.1

r	1.0			1.5		
Alt.↓ Tests→	$\hat{T}^{*(r,2)}_{n;c_3}$	$\hat{T}^{*(r,3)}_n$	$\hat{T}^{*(r,4)}_n$	$\hat{T}^{*(r,2)}_{n;c_3}$	$\hat{T}^{*(r,3)}_{n;c_3}$	$\hat{T}^{*(r,4)}_n$
$W(1.0)$	**97***	94	91	**97***	95	92
$W(3.0)$	24	2	16	4	13	2
$\Gamma(1.5)$	**79***	73	67	**79***	76	69
$\Gamma(2.0)$	50	47	43	49	50	44
$IG(0.5)$	**99***	98	98	**99***	**99***	98
$IG(1.5)$	63	67	65	59	69	66
$LN(0.8)$	75	76	74	73	78	75
$LN(1.5)$	**100***	**100***	**100***	**100***	**100***	**100***
$GO(0.5)$	83	74	66	**85***	77	69
$GO(1.5)$	52	42	33	57	40	36
$PW(1.0)$	35	37	25	42	23	32
$PW(2.0)$	98	97	93	**99***	96	95
$LF(2.0)$	68	57	49	**71***	60	51
$LF(4.0)$	54	44	36	57	45	38
$EP(1.0)$	67	56	47	71	56	49
$EP(2.0)$	13	1	10	2	12	1
Av.	66.1	60.2	57.1	65.3	61.9	57.2

Table 19: Powers of 5% selected tests using empirical values; $n = 20$

The role of k (=2,3,4) in powers of these tests is illustrated in Tables 19.

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