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TOWARDS A PROOF OF THE TWIN PRIME CONJECTURE

Badih Ghusayni
Department of Mathematics
Faculty of Science-1
Lebanese University
Hadath, LEBANON
e-mail: badih@futureintech.net

Abstract: Prime numbers differing by 2 are called twin primes. The twin prime conjecture states that the number of twin primes is infinite. Many attempts to prove or disprove this 2300-year old conjecture have failed. The objective of this paper is two-fold. We first tie the twin prime conjecture to complex variable theory. We then look at some of the most recent progress on it.

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1. Introduction

Whether twin primes are finite or infinite in number is one of the most famous problems in number theory. Brun [4] showed that $\sum_n \frac{1}{p_n}$, where p_n and p_n+2 are prime numbers, is finite. In fact, the sum $\sum_n (\frac{1}{p_n} + \frac{1}{p_n+2})$, known as Brun's constant B, has been calculated by Shanks and Wrench [22] and by Brent [3] to be approximately 1.90216054.

Definition. Let $f(z)$ be an entire function. The *maximum modulus function*, denoted by $M(r)$, is defined by $M(r) = \max\{|f(z)| : |z| = r\}$.

Definition. Let $f(z)$ be a non-constant entire function. The *order* ρ of $f(z)$ is defined by

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$$\rho = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r}\,.$$

The order of any constant function is 0, by convention.

Definition. Let $f(z)$ be an entire function. If $|f(z)| \le Ce^{A|z|}$ for some positive constants A and C and all values of z, we say that $f(z)$ is of *exponential type*.

Remark 1.1. If an entire function $f(z)$ is of exponential type, then $f(z)$ is of order ≤ 1.

Definition. *The exponential type* σ of an entire function $f(z)$ of exponential type is defined by

$$\sigma = \limsup_{r\to\infty} \frac{\log M(r)}{r}\,.$$

The zero function has exponential type 0, by convention.

Definition. An entire function $f(z)$ of positive order ρ is said to be of *type* τ if

$$\tau = \limsup_{r\to\infty} \frac{\log M(r)}{r^{\rho}}\,.$$

Terminology. If $0 \le \tau < \infty$, then $f(z)$ is said to be of *finite type*.

If $\tau = 0$, then $f(z)$ is said to be of *minimal type*.

If $0 < \tau < \infty$, then $f(z)$ is said to be of *normal type*.

If $\tau = \infty$, then $f(z)$ is said to be of *infinite or maximal type*.

Definition. Let $z_1, z_2, \cdots$ be a sequence of non-zero complex numbers. The greatest lower bound of positive numbers α for which $\sum_0^\infty \frac{1}{|z_n|^\alpha}$ is convergent is called the *exponent of convergence* of the sequence $\{z_n\}$ and is denoted by ρ_1. The smallest positive integer α for which the series is convergent is denoted by $p+1$ and p is called the *genus* of $\{z_n\}$.

The infinite product $\prod G(\frac{z}{z_n}, p)$, where

$$G(u,p) = (1-u)e^{u+\frac{u^2}{2}+\cdots+\frac{u^p}{p}}$$

and

$$G(u,0) = 1-u$$

is called a *canonical product* of genus p.

The original idea appeared in [7] where the author related the twin prime conjecture to complex variable theory as follows:

Since B $< \infty$ the canonical product $\prod_n (1 - \frac{z}{p_n})$ is an entire function (see for Example [25], p. 54). Moreover, if the number of twin primes was finite, then $\prod_n (1 - \frac{z}{p_n})$ would be a real polynomial and thus of order (see for Example [25], p. 64). 0. Consequently, if $\prod_n (1 - \frac{z}{p_n})$ has a nonzero order, then the number of twin primes is infinite.

2. Auxiliary Results

We will use the following results.

Theorem 2.1. (Borel's Theorem, see [8], p. 27, for a proof) *A canonical product of genus* 0 *is an entire function of exponential type* 0*.*

Theorem 2.2. (Hadamard Factorization Theorem, See [8], pp. 42-45, for a proof) *If $f(z)$ is an entire function of order ρ with a zero at* 0 *of multiplicity m, then*

$$f(z) = z^m e^{Q(z)} \prod_1^\infty G(\frac{z}{z_n}, p),$$

where $Q(z)$ is a polynomial of degree $q < \rho$.

Definition. The *genus of the function* $f(z)$ in the Hadamard Factorization Theorem is defined as $\max(p, q)$.

Theorem 2.3. (see [18], Theorem 12, p. 22) *a) The order of the product of two entire functions of different orders is equal to the larger of the orders of the factors, and the type is equal to the type of the function that has the larger order.*

b) The product of two entire functions of the same order, one having normal type σ and the other having minimal type, is an entire function of the same order and type σ.

c) The product of two entire functions of the same order ρ, one having maximal type and the other having at most normal type, is an entire function of order ρ and maximal type.

Suppose $\{p_n\}$ is a sequence of positive numbers (the sequence may be finite). For infinite sequences we suppose further that $\sum_{n=1}^\infty \frac{1}{p_n}$ converges (if $\{p_n\}_{n=1}^k$ is a finite sequence, the corresponding convergence condition that $\sum_{n=1}^k \frac{1}{p_n}$

converges is clearly there). Finally, let A $= \sum_n \frac{1}{p_n}$. The canonical product

$$\prod_n (1 - \frac{z}{p_n}) e^{\frac{z}{p_n}}$$

is an entire function [25], p. 55. The following lemma was proved by the author in [7]. Since we need something from its proof for the current paper, we repeat the proof here:

Lemma 2.4. *The order of* $\prod_n (1 - \frac{z}{p_n}) e^{\frac{z}{p_n}}$ *is 1.*

Proof. The canonical product $\prod_n (1 - \frac{z}{p_n}) e^{\frac{z}{p_n}}$ is an entire function [25], p. 55. Let $a > 1$. Since $p_n^a > p_n$, $\sum_n \frac{1}{p_n^a} < \infty$. Thus $\sum_n \frac{1}{p_n^a} < \infty$ when a ≥ 1. Consequently, the genus of the zeros of $\prod_n (1 - \frac{z}{p_n})$ is 0. By the Hadamard Factorization Theorem and the definition of the genus of a function it follows that the genus of $\prod_n (1 - \frac{z}{p_n})$ is also 0. By Borel's Theorem, $\prod_n (1 - \frac{z}{p_n})$ is of exponential type 0. In particular, since $\prod_n (1 - \frac{z}{p_n})$ is of exponential type, it is of order k ≤ 1 (this is what we need for the current paper to conjecture that the order of $\prod_n (1 - \frac{z}{p_n})$ is 1). We now consider two cases:

Case 1. $k < 1$: Since the order of e^{Az} is 1, it follows from Theorem 2.2(a) that

$$\prod_n (1 - \frac{z}{p_n}) e^{\frac{z}{p_n}} = e^{Az} \prod_n (1 - \frac{z}{p_n})$$

is 1.

Case 2. $k = 1$: Since the definition of type and exponential type agree for functions of order 1, the type of $\prod_n (1 - \frac{z}{p_n})$ is 0. Clearly, the type of e^{Az} is $A > 0$. Thus, from Theorem 2.3(b), the order of

$$\prod_n (1 - \frac{z}{p_n}) e^{\frac{z}{p_n}} = e^{Az} \prod_n (1 - \frac{z}{p_n})$$

is 1. The proof of the lemma is complete.

Lemma 2.5. *The type of* $\prod_n (1 - \frac{z}{p_n}) e^{\frac{z}{p_n}}$ *is Brun's Constant B, which is approximately 1.9.*

Proof.

$$\prod_n (1 - \frac{z}{p_n}) e^{\frac{z}{p_n}} = e^{Bz} \prod_n (1 - \frac{z}{p_n}) .$$

Using the generalized triangle inequality if the number of twin primes is finite and both a limiting process and the generalized triangle inequality if the number of twin primes is infinite, we get

$$|f(z)| = \prod_n |1 - \frac{z}{p_n}| e^{\frac{Re z}{p_n}} \leq \prod_n (1 + \frac{|z|}{p_n}) e^{\frac{|z|}{p_n}} .$$

As a result

$$M(r) = \prod_n (1 + \frac{r}{p_n}) e^{\frac{r}{p_n}} .$$

Thus the type

$$\tau = \limsup_{r\to\infty} \frac{\log M(r)}{r} = \sum_n \frac{1}{p_n} + \limsup_{r\to\infty} \sum_n \frac{\log(1+\frac{r}{p_n})}{r}$$
$$= B + \limsup_{r\to\infty} \sum_n \frac{\log(1+\frac{r}{p_n})}{r}.$$

Since $\log(1+x) \leq x$ for $x > 0$, we have $\log |\frac{1+\frac{r}{p_n}}{r}| \leq \frac{1}{p_n}$ for all $r > 0$. Since $\sum_n \frac{1}{p_n} < \infty$ (indeed is B) it follows from Weierstrass criterion for uniform convergence that $\sum_n \frac{\log(1+\frac{r}{p_n})}{r}$ is uniformly convergent for all $r > 0$.

Now

$$\tau = \lim_{r\to\infty} \frac{\log M(r)}{r} = B + \sum_n \lim_{r\to\infty} \frac{\log(1+\frac{r}{p_n})}{r} = B . \qquad \square$$

3. The Other Half

Let $\pi(x)$ denote the number of primes $p \leq x$. The prime number theorem (see for instance [8]) states that

$$\lim_{x\to\infty} \frac{\pi(x)\log x}{x} = 1 .$$

Theorem 3.1. $\lim_{x\to\infty} \frac{\pi(x)\log x}{x} = 1 \Leftrightarrow \lim_{n\to\infty} \frac{p_n}{n \log n} = 1$.

Proof. ($\Rightarrow$) Let $x = p_n$. Then $\lim_{n\to\infty} \frac{\pi(p_n)}{p_n/\log p_n} = 1$. Since $\pi(p_n) = n, \lim_{n\to\infty} \frac{n}{p_n/\log p_n} = 1$. Then there is $n_0 \in N$ such that

$$n \geq n_0 \Rightarrow n \geq \frac{1}{2} \frac{p_n}{\log p_n} .$$

Then

$$n \geq n_0 \Rightarrow p_n \leq 2n \log p_n .$$

Therefore

$$n \geq n_0 \Rightarrow \log p_n \leq \log 2 + \log n + \log\log p_n \,.$$

Hence

$$n \geq n_0 \Rightarrow \log p_n (1 - \frac{\log\log p_n}{\log p_n}) \leq \log 2 + \log n \,,$$

or

$$n \geq n_0 \Rightarrow \frac{\log p_n}{\log n}(1 - \frac{\log\log p_n}{\log p_n}) \leq \frac{\log 2}{\log n} + 1 \,.$$

Now, since $\lim_{n\to\infty} \frac{\log\log p_n}{\log p_n} = 0$ and $\lim_{n\to\infty} \frac{\log 2}{\log n}$, given $\epsilon > 0$, there exists $n_1 \in N$ such that

$$n \geq n_1 \Rightarrow 1 - \epsilon \leq \frac{\log p_n}{\log n} \leq 1 + \epsilon \,.$$

Since ϵ was arbitrary it follows that

$$\lim_{n\to\infty} \frac{\log p_n}{\log n} = 1 \,.$$

Using this we can now write

$$\lim_{n\to\infty} \frac{n}{p_n / \log n} = 1 \,.$$

That is,

$$\lim_{n\to\infty} \frac{p_n}{n \log n} = 1 \,.$$

($\Leftarrow$) For any x, there exist prime numbers p_n and p_{n+1} such that $p_n \leq x < p_{n+1}$. Hence $\pi(x) = n$. Since the function $f(x) = \frac{x}{\log x}$ is increasing,

$$\frac{p_n}{\log p_n} \leq \frac{x}{\log x} \leq \frac{p_{n+1}}{\log p_{n+1}} \,.$$

Multiplying by $\frac{1}{n}$ we get

$$\frac{p_n}{n \log p_n} \leq \frac{x}{\pi(x) \log x} \leq \frac{p_{n+1}}{n \log p_{n+1}} \,.$$

Taking limits as $n \to \infty$ (and hence $x \to \infty$) and using the hypothesis, the result follows and the proof is complete. $\square$

By the above theorem it follows that the prime number theorem implies that $p_n \sim n \log n, n \to \infty$ from which we can deduce that

$$L := \liminf_{n\to\infty} \frac{p_{n+1} - p_n}{\log p_n} \leq 1 \,.$$

Paul Erdös [6] had shown in 1940 that $L < 1$, improved in 1954 by Ricci [21] to $L \leq \frac{15}{16}$, improved in 1966 by Bombieri and Davenport [2] to $L \leq 0.4665\ldots$, improved in 1972 by Pilt'ai [20] to $L \leq 0.4571$, improved in 1975 by Uchiyama

[23] to $L \leq 0.4542$, improved in 1977 by Huxley in several steps [22,23] to $L \leq 0.44254\ldots$, improved in 1984 by Huxley [17] to $L \leq 0.4393$, improved in 1988 by Maier [19] to $L \leq 0.2486\ldots$ Recently, Goldston, Pintz and Yildirim [14], [12], [13], [10] have obtained the following major and deep result: major because it is an approximation to the twin prime conjecture which can be expressed as $\liminf_{n\to\infty}(p_{n+1} - p_n) = 2$ and deep because its proof required 32 pages.

Theorem 3.2.

$$L := \liminf_{n\to\infty} \frac{p_{n+1} - p_n}{\log p_n} = 0.$$

Remark 3.3. A simpler proof was later obtained (still 8 pages long) with the help of Motohashi [11].

Remark 3.4. It is interesting to note that in 1931 Westzynthius [24] proved that

$$\limsup_{n\to\infty} \frac{p_{n+1} - p_n}{\log p_n} = \infty\,.$$

Remark 3.5. In the abstract to his talk “Revenge of the twin prime conjecture” at MathFest 2007, Daniel Goldston states that “two years ago Pintz, Yildirim, and I proved that there always exist primes that are very close together - very close meaning much closer than the average distance between neighboring primes. Our method also proves that if the primes are well distributed in arithmetic progressions then one can obtain results not too far from the twin prime conjecture. For example, if the Elliott-Halberstam Conjecture is true then there are infinitely many pairs of primes with difference 16 or less. With these successes I was hopeful that before too long our method could be pushed to unconditionally show that there are infinitely often pairs of primes closer than some fixed bounded distance, i.e. bounded gaps, a giant step towards the twin prime conjecture. In this talk I will discuss the method and why perhaps further progress towards bounded gaps and the twin prime conjecture is going to be difficult, although I will be delighted to be proved wrong.”

This remark motivates the next section.

4. Future Work

For an arbitrary sequence $\{a_n\}$ in $[-\infty, \infty]$, we have

$$\limsup_{n\to\infty}(-a_n) = \inf_n \sup_{k\geq n}\{-a_k\} = \inf_n\{-\inf_{k\geq n} a_k)$$
$$= -\sup_n(\inf_{k\geq n} a_k) = -\liminf_{n\to\infty} a_n.$$

It follows from Goldston, Pintz and Yildirim Theorem that

$$\limsup_{n\to\infty} \frac{p_n - p_{n+1}}{\log p_n} = 0.$$

Can this be related to

$$\limsup_{r\to\infty} \frac{\log\log M(r)}{\log r}$$

with the latter being, by definition,

$$\lim_{r\to\infty}\{\sup_{t\geq r} \frac{\log\log M(t)}{\log t}\}$$

to show that the order of $\prod_n(1-\frac{z}{p_n})$ is ≥ 1 thus proving the twin prime conjecture?

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