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SUBORDINATION RESULTS AND INTEGRAL MEANS FOR CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS

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Abstract: In this paper, we introduce a generalized class of starlike functions and obtain the subordination results for various subclasses of starlike functions. Further, we obtain the integral means inequalities for various subclasses of starlike functions. Some interesting consequences of our results are also pointed out.

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1. Introduction and Preliminaries

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \,, \tag{1.1}$$

which are analytic and univalent in the open disc $U = \{z : |z| < 1\}$. For functions $\Phi \in A$ given by $\Phi(z) = z + \sum\limits_{n=2}^{\infty} \phi_n z^n$ and $\Psi \in A$ given by $\Psi(z) =$

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$z+\sum_{n=2}^{\infty}\psi_n z^n$, we define the Hadamard product (or convolution) of Φ and Ψ by

$$(\Phi * \Psi)(z) = z + \sum_{n=2}^{\infty} \phi_n \psi_n z^n, \quad z \in U. \tag{1.2}$$

For complex parameters $\alpha_1, \ldots, \alpha_l$ and $\beta_1, \ldots, \beta_m$ ($\beta_j \neq 0, -1, \ldots; j = 1, 2, \ldots, m$) the *generalized hypergeometric function* ${}_lF_m(z)$ is defined by

$$\begin{aligned} {}_lF_m(z) \equiv {}_lF_m(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m; z) \quad &:= \quad \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_m)_n} \, \frac{z^n}{n!} \quad (1.3) \\ (l \leq m+1;\ l, m \in N_0 \quad &:= \quad N \cup \{0\}; z \in U), \end{aligned}$$

where N denotes the set of all positive integers and $(\alpha)_n$ is the Pochhammer symbol defined by

$$(\alpha)_n = \begin{cases} 1, & n = 0, \\ \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n-1), & n \in N. \end{cases} \tag{1.4}$$

The notation ${}_lF_m$ is quite useful for representing many well-known functions such as the exponential, the Binomial, the Bessel, the Laguerre polynomial and others; for example see [8]. For positive real values of $\alpha_1, \ldots, \alpha_l$ and $\beta_1, \ldots, \beta_m$ ($\beta_j \neq 0, -1, \ldots; j = 1, 2, \ldots, m$), let $H(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m) : A \rightarrow A$ be a linear operator defined by

$$\begin{aligned} [(H(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m))(f)](z) := z \ {}_lF_m(\alpha_1, \alpha_2, \ldots \alpha_l; \beta_1, \beta_2 \ldots, \beta_m; z) \\ * f(z) = z + \sum_{n=2}^{\infty} \Gamma_n \ a_n z^n, \quad (1.5) \end{aligned}$$

where

$$\Gamma_n = \frac{(\alpha_1)_{n-1} \ldots (\alpha_l)_{n-1}}{(\beta_1)_{n-1} \ldots (\beta_m)_{n-1}} \, \frac{1}{(n-1)!}. \tag{1.6}$$

For notational simplicity, we can use a shorter notation $H_m^l[\alpha_1, \beta_1]$ for $H(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)$ in the sequel. The linear operator $H_m^l[\alpha_1, \beta_1]$ is called Dziok-Srivastava operator [3], includes various other linear operators introduced and studied by Bernardi, see [1], Carlson and Shaffer [2], Libera [5], Livingston [7], Ruscheweyh [9] and Srivastava-Owa [14].

For $0 \leq \lambda < 1$, $0 \leq \gamma < 1$ and $k \geq 0$, we let $\mathcal{H}_m^l(\lambda, \gamma, k)$ be the subclass of A consisting of functions of the form (1.1) and satisfying the analytic criterion

$$\mathrm{Re} \left\{ \frac{z(H_m^l[\alpha_1, \beta_1]f(z))'}{(1-\lambda)H_m^l[\alpha_1, \beta_1]f(z) + \lambda z(H_m^l[\alpha_1, \beta_1]f(z))'} - \gamma \right\}$$

$$> k\left|\frac{z(H_m^l[\alpha_1,\beta_1]f(z))'}{(1-\lambda)H_m^l[\alpha_1,\beta_1]f(z)+\lambda z(H_m^l[\alpha_1,\beta_1]f(z))'}-1\right|, \quad z\in U, \quad (1.7)$$

where $H_m^l[\alpha_1,\beta_1]f(z)$ is given by (1.5). We further let $T\mathcal{H}_m^l(\lambda,\gamma,k)=\mathcal{H}_m^l(\lambda,\gamma,k)\cap T$, where

$$T \;:=\; \left\{f\in A : f(z)=z-\sum_{n=2}^{\infty}|a_n|z^n, \quad z\in U\right\}, \tag{1.8}$$

a subclass of A introduced and studied by Silverman [10].

By suitably specializing the values of l, m, $\alpha_1,\alpha_2,\ldots,\alpha_l$, $\beta_1,\beta_2,\ldots,\beta_m$, λ, γ and k the class $\mathcal{H}_m^l(\lambda,\gamma,k)$, leads to various new subclasses. We present some examples below:

Example 1. If $l=2$ and $m=1$ with $\alpha_1=1$, $\alpha_2=1$, $\beta_1=1$ then

$$\mathcal{H}_1^2(\lambda,\gamma,k)\equiv\mathbb{S}(\lambda,\gamma,k):=\left\{f\in A:\mathrm{Re}\left\{\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)}-\gamma\right\}\right.$$
$$\left. > k\left|\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)}-1\right|, \ z\in U\right\}.$$

Further $T\mathbb{S}(\lambda,\gamma,k)=\mathbb{S}(\lambda,\gamma,k)\cap T$, where T is given by (1.8).

Example 2. If $l=2$ and $m=1$ with $\alpha_1=\delta+1\,(\delta>-1)$, $\alpha_2=1$, $\beta_1=1$, then

$$\mathcal{H}_1^2(\lambda,\gamma,k)\equiv R_\delta(\lambda,\gamma,k):=\left\{f\in A:\mathrm{Re}\left\{\frac{z(D^\delta f(z))'}{(1-\lambda)D^\delta f(z)+\lambda z(D^\delta f(z))'}-\gamma\right\}\right.$$
$$\left. > k\left|\frac{z(D^\delta f(z))'}{(1-\lambda)D^\delta f(z)+\lambda z(D^\delta f(z))'}-1\right|, \ z\in U\right\},$$

where D^δ is called Ruscheweyh derivative of order δ $(\delta>-1)$ defined by

$$D^\delta f(z):=\frac{z}{(1-z)^{\delta+1}}*f(z)\equiv H_1^2(\delta+1,1;1)f(z).$$

Also $TR_\delta(\lambda,\gamma,k)=R_\delta(\lambda,\gamma,k)\cap T$, where T is given by (1.8).

Example 3. If $l=2$ and $m=1$ with $\alpha_1=\mu+1(\mu>-1)$, $\alpha_2=1$, $\beta_1=\mu+2$, then

$$\mathcal{H}_1^2(\lambda,\gamma,k)\equiv B_\mu(\lambda,\gamma,k):=\left\{f\in A:\mathrm{Re}\left(\frac{z(J_\mu f(z))'}{(1-\lambda)J_\mu f(z)+\lambda z(J_\mu f(z))'}-\gamma\right)\right.$$
$$\left. > k\left|\frac{z(J_\mu f(z))'}{(1-\lambda)J_\mu f(z)+\lambda z(J_\mu f(z))'}-1\right|, \ z\in U\right\},$$

where J_μ is a Bernardi operator (see [1]), defined by

$$J_\mu f(z) := \frac{\mu+1}{z^\mu}\int_0^z t^{\mu-1} f(t)dt \equiv H_1^2(\mu+1, 1; \mu+2)f(z).$$

Note that the operator J_1 was studied earlier by Libera [5] and Livingston [7]. Further, $TB_\mu(\lambda,\gamma,k) = B_\mu(\lambda,\gamma,k)\cap T$, where T is given by (1.8).

Example 4. If $l=2$ and $m=1$ with $\alpha_1 = a\,(a>0)$, $\alpha_2 = 1$, $\beta_1 = c\,(c>0)$, then

$$\begin{aligned}\mathcal{H}_1^2(\lambda,\gamma,k) &\equiv L_c^a(\lambda,\gamma,k)\\ &:= \left\{f\in A : \operatorname{Re}\left(\frac{z(L(a,c)f(z))'}{(1-\lambda)L(a,c)f(z)+\lambda z(L(a,c)f(z))'}-\gamma\right)\right.\\ &\qquad\left. > k\left|\frac{z(L(a,c)f(z))'}{(1-\lambda)L(a,c)f(z)+\lambda z(L(a,c)f(z))'}-1\right|,\quad z\in U\right\},\end{aligned}$$

where $L(a,c)$ is a well-known Carlson-Shaffer linear operator (see [2]), defined by

$$L(a,c)f(z) := \left(\sum_{k=0}^{\infty}\frac{(a)_k}{(c)_k}z^{k+1}\right) * f(z) \equiv H_1^2(a,1;c)f(z).$$

Further, $TL_c^a(\lambda,\gamma,k) = L_c^a(\lambda,\gamma,k)\cap T$, where T is given by (1.8).

Now we recall the following results which are very much needed for our study.

Definition 1.1. (Subordination Principle) For analytic functions g and h with $g(0)=h(0)$, g is said to be subordinate to h, denoted by $g\prec h$, if there exists an analytic function w such that $w(0)=0$, $|w(z)|<1$ and $g(z)=h(w(z))$, for all $z\in U$.

Definition 1.2. (Subordinating Factor Sequence) A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is said to be a subordinating sequence if, whenever $f(z) = \sum\limits_{n=1}^{\infty} a_n z^n$, $a_1 = 1$ is regular, univalent and convex in U, we have

$$\sum_{n=1}^{\infty} b_n a_n z^n \prec f(z),\quad z\in U. \tag{1.9}$$

Lemma 1.1. (see [15]) *The sequence $\{b_n\}_{n=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$\operatorname{Re}\left\{1+2\sum_{n=1}^{\infty} b_n z^n\right\} > 0,\quad z\in U. \tag{1.10}$$

Lemma 1.2. (see [6]) *If the functions f and g are analytic in U with $g \prec f$, then for $\eta > 0$, and $0 < r < 1$,*

$$\int_0^{2\pi} \left|g(re^{i\theta})\right|^\eta d\theta \leq \int_0^{2\pi} \left|f(re^{i\theta})\right|^\eta d\theta. \tag{1.11}$$

In [10], Silverman found that the function $f_2(z) = z - \frac{z^2}{2}$ is often extremal over the family T. He applied this function to resolve his integral means inequality, conjectured in [11] and settled in [12], that

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^\eta d\theta \leq \int_0^{2\pi} \left|f_2(re^{i\theta})\right|^\eta d\theta,$$

for all $f \in T$, $\eta > 0$ and $0 < r < 1$. In [12], he also proved his conjecture for the subclasses $T^*(\gamma)$ and $C(\gamma)$ of T.

Motivated by earlier works of [11], [15] in this paper, we investigate certain characteristic properties and obtain the subordination results for the class of functions $f \in \mathcal{H}_m^l(\lambda, \gamma, k)$ and integral means results for the class of functions $f \in T\mathcal{H}_m^l(\lambda, \gamma, k)$. We state some interesting results for functions in those classes defined in Examples 1 to 4.

2. Basic Properties

In this section we obtain the characterization properties for the classes $\mathcal{H}_m^l(\lambda, \gamma, k)$ and $T\mathcal{H}_m^l(\lambda, \gamma, k)$.

Theorem 2.1. *A function $f(z)$ of the form (1.1) is in $\mathcal{H}_m^l(\lambda, \gamma, k)$ if*

$$\sum_{n=2}^{\infty} [n(1+k) - (\gamma + k)(1 + n\lambda - \lambda)]\Gamma_n \, |a_n| \leq 1 - \gamma, \tag{2.1}$$

where $0 \leq \lambda < 1$, $0 \leq \gamma < 1$ and $k \geq 0$, Γ_n is given by (1.6).

Proof. It suffices to show that

$$k\left|\frac{z(H_m^l[\alpha_1, \beta_1]f(z))'}{(1-\lambda)H_m^l[\alpha_1, \beta_1]f(z) + \lambda z(H_m^l[\alpha_1, \beta_1]f(z))'} - 1\right|$$
$$- \operatorname{Re}\left\{\frac{z(H_m^l[\alpha_1, \beta_1]f(z))'}{(1-\lambda)H_m^l[\alpha_1, \beta_1]f(z) + \lambda z(H_m^l[\alpha_1, \beta_1]f(z))'} - \gamma\right\} \leq 1 - \gamma.$$

We have

$$\begin{aligned}
&k\left|\frac{z(H_m^l[\alpha_1,\beta_1]f(z))'}{(1-\lambda)H_m^l[\alpha_1,\beta_1]f(z)+\lambda z(H_m^l[\alpha_1,\beta_1]f(z))'}-1\right| \\
&\qquad -\operatorname{Re}\left\{\frac{z(H_m^l[\alpha_1,\beta_1]f(z))'}{(1-\lambda)H_m^l[\alpha_1,\beta_1]f(z)+\lambda z(H_m^l[\alpha_1,\beta_1]f(z))'}-\gamma\right\} \\
&\le (1+k)\left|\frac{z(H_m^l[\alpha_1,\beta_1]f(z))'}{(1-\lambda)H_m^l[\alpha_1,\beta_1]f(z)+\lambda z(H_m^l[\alpha_1,\beta_1]f(z))'}-1\right| \\
&\le \frac{(1+k)\sum_{n=2}^{\infty}(n-1-n\lambda+\lambda)\Gamma_n|a_n||z|^{n-1}}{1-\sum_{n=2}^{\infty}(1+n\lambda-\lambda)\Gamma_n|a_n||z|^{n-1}}.
\end{aligned}$$

The last expression is bounded above by $(1-\gamma)$ if

$$\sum_{n=2}^{\infty}[n(1+k)-(\gamma+k)(1+n\lambda-\lambda)]\Gamma_n|a_n| \le 1-\gamma$$

and the proof is complete. □

In the view of Examples 1 to 4, we state the following corollaries.

Corollary 2.1. *A function $f(z)$ of the form (1.1) is in $\mathbb{S}(\lambda,\gamma,k)$ if*

$$\sum_{n=2}^{\infty}[n(1+k)-(\gamma+k)(1+n\lambda-\lambda)]\ |a_n| \le 1-\gamma, \tag{2.2}$$

where $0\le\lambda<1$, $0\le\gamma<1$ and $k\ge 0$.

Corollary 2.2. *A function $f(z)$ of the form (1.1) is in $R_\delta(\lambda,\gamma,k)$ if*

$$\sum_{n=2}^{\infty}[n(1+k)-(\gamma+k)(1+n\lambda-\lambda)]\frac{(\delta+1)\ldots(\delta+n-1)}{(n-1)!}|a_n| \le 1-\gamma, \tag{2.3}$$

where $0\le\lambda<1$, $0\le\gamma<1$, $k\ge 0$ and $\delta>-1$.

Corollary 2.3. *A function $f(z)$ of the form (1.1) is in $B_\mu(\lambda,\gamma,k)$ if*

$$\sum_{n=2}^{\infty}[n(1+k)-(\gamma+k)(1+n\lambda-\lambda)]\left(\frac{\mu+1}{\mu+n}\right)|a_n| \le 1-\gamma, \tag{2.4}$$

where $0\le\lambda<1$, $0\le\gamma<1$, $k\ge 0$ and $\mu>-1$.

Corollary 2.4. *A function $f(z)$ of the form (1.1) is in $L_c^a(\lambda,\gamma,k)$ if*

$$\sum_{n=2}^{\infty}[n(1+k)-(\gamma+k)(1+n\lambda-\lambda)]\frac{(a)_{n-1}}{(c)_{n-1}}|a_n| \le 1-\gamma, \tag{2.5}$$

where $0\le\lambda<1$, $0\le\gamma<1$, $k\ge 0$ and $a>0$, $c>0$.

Theorem 2.2. *Let $0 \leq \lambda < 1$, $0 \leq \gamma < 1$, $k \geq 0$, then a function f of the form (1.8) is in the class $T\mathcal{H}_m^l(\lambda, \gamma, k)$ if and only if*

$$\sum_{n=2}^{\infty}[n(1+k) - (\gamma+k)(1+n\lambda-\lambda)]\Gamma_n \; |a_n| \leq 1-\gamma, \tag{2.6}$$

where Γ_n is given by (1.6).

Proof. In view of Theorem 2.1, we need only to prove the necessity. If $f \in T\mathcal{H}_m^l(\lambda, \gamma, k)$ and z is real then

$$\frac{1 - \sum\limits_{n=2}^{\infty} n\Gamma_n \; a_n z^{n-1}}{1 - \sum\limits_{n=2}^{\infty}[1+n\lambda-\lambda]\Gamma_n \; a_n z^{n-1}} - \gamma > k \left| \frac{\sum\limits_{n=2}^{\infty}(n-1-n\lambda+\lambda)\Gamma_n \; a_n z^{n-1}}{1 - \sum\limits_{n=2}^{\infty}[1+n\lambda-\lambda]\Gamma_n \; a_n z^{n-1}} \right|.$$

Letting $z \to 1$ along the real axis, we obtain the desired inequality

$$\sum_{n=2}^{\infty}[n(1+k) - (\gamma+k)(1+n\lambda-\lambda)]\Gamma_n \; |a_n| \leq 1-\gamma,$$

where $0 \leq \lambda < 1$, $0 \leq \gamma < 1$, $k \geq 0$ and Γ_n is given by (1.6). □

Corollary 2.5. *If $f \in T\mathcal{H}_m^l(\lambda, \gamma, k)$, then*

$$|a_n| \leq \frac{1-\gamma}{[n(1+k) - (\gamma+k)(1+n\lambda-\lambda)]\Gamma_n}, \;\; 0 \leq \lambda < 1, \;\; 0 \leq \gamma < 1, k \geq 0, \tag{2.7}$$

where Γ_n is given by (1.6). Equality holds for the function

$$f(z) = z - \frac{1-\gamma}{[n(1+k) - (\gamma+k)(1+n\lambda-\lambda)]\Gamma_n} z^n.$$

Remark 2.1. For specific choices of the parameters l, m, $\alpha_1, \alpha_2, \ldots, \alpha_l$, $\beta_1, \beta_2, \ldots, \beta_m$, λ, γ and k, one can state the coefficient inequalities for the subclasses of functions $T\mathbb{S}(\lambda, \gamma, k)$, $TR_\delta(\lambda, \gamma, k)$, $TB_\mu(\lambda, \gamma, k)$ and $TL_c^a(\lambda, \gamma, k)$.

Theorem 2.3. (Extreme Points) *Let*

$$f_1(z) = z \quad \text{and}$$

$$f_n(z) = z - \frac{1-\gamma}{[n(1+k) - (\gamma+k)(1+n\lambda-\lambda)]\Gamma_n} z^n, \;\; n \geq 2, \tag{2.8}$$

for $0 \leq \gamma < 1$, $0 \leq \lambda < 1$, $k \geq 0$ and Γ_n be given by (1.6). Then $f(z)$ is in the class $T\mathcal{H}_m^l(\lambda, \gamma, k)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z), \tag{2.9}$$

where $\mu_n \geq 0$ *and* $\sum\limits_{n=1}^{\infty} \mu_n = 1$.

Proof. Suppose $f(z)$ can be written as in (2.9). Then

$$f(z) \quad = \quad z - \sum_{n=2}^{\infty} \mu_n \frac{1-\gamma}{[n(1+k)-(\gamma+k)(1+n\lambda-\lambda)]\Gamma_n} z^n.$$

Now,

$$\sum_{n=2}^{\infty} \frac{[n(1+k)-(\gamma+k)(1+n\lambda-\lambda)]\Gamma_n}{1-\gamma} \mu_n \frac{1-\gamma}{[n(1+k)-(\gamma+k)(1+n\lambda-\lambda)]\Gamma_n}$$

$$= \sum_{n=2}^{\infty} \mu_n = 1-\mu_1 \leq 1.$$

Thus $f \in T\mathcal{H}_m^l(\lambda,\gamma,k)$. Conversely, let us have $f \in T\mathcal{H}_m^l(\lambda,\gamma,k)$. Then by using (2.7), we set

$$\mu_n = \frac{[n(1+k)-(\gamma+k)(1+n\lambda-\lambda)]\Gamma_n}{1-\gamma} a_n, \quad n \geq 2$$

and $\mu_1 = 1 - \sum\limits_{n=2}^{\infty} \mu_n$. Then we have $f(z) = \sum\limits_{n=1}^{\infty} \mu_n f_n(z)$ and hence this completes the proof of Theorem 2.3. □

Remark 2.2. For specific choices of the parameters l, m, $\alpha_1, \alpha_2, \ldots, \alpha_l$, $\beta_1, \beta_2, \ldots, \beta_m$, λ, γ and k, one can state extreme points results for the subclasses of functions $T\mathbb{S}(\lambda,\gamma,k)$, $TR_\delta(\lambda,\gamma,k)$, $TB_\mu(\lambda,\gamma,k)$ and $TL_c^a(\lambda,\gamma,k)$.

3. Subordination Results

Let $\mathcal{H}_m^{*l}(\lambda,\gamma,k)$ denote the subclass of functions f in A whose coefficients a_n satisfy the condition (2.1). We note that $\mathcal{H}_m^{*l}(\lambda,\gamma,k) \subseteq \mathcal{H}_m^l(\lambda,\gamma,k)$.

Theorem 3.1. *Let* $f \in \mathcal{H}_m^{*l}(\lambda,\gamma,k)$ *and* $g(z)$ *be any function in the usual class of convex functions* C*, then*

$$\frac{(2+k-\gamma-\lambda(k+\gamma))\Gamma_2}{2[1-\gamma+(2+k-\gamma-\lambda(k+\gamma))\Gamma_2]}(f*g)(z) \prec g(z)\,, \tag{3.1}$$

where $0 \leq \gamma < 1$; $k \geq 0$ *and* $0 \leq \lambda < 1$, *with*

$$\Gamma_2 = \frac{\alpha_1 \ldots \alpha_l}{\beta_1 \ldots \beta_m} \tag{3.2}$$

and

$$\text{Re}\ \{f(z)\} > -\frac{[1-\gamma+(2+k-\gamma-\lambda(k+\gamma))\Gamma_2]}{(2+k-\gamma-\lambda(k+\gamma))\Gamma_2}, \quad z \in U. \tag{3.3}$$

The constant factor $\frac{(2+k-\gamma-\lambda(k+\gamma))\Gamma_2}{2[1-\gamma+(2+k-\gamma-\lambda(k+\gamma))\Gamma_2]}$ *in (3.1) cannot be replaced by a larger number.*

Proof. Let $f \in \mathcal{H}_m^{*l}(\lambda,\gamma,k)$ and suppose that $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in C$. Then

$$\frac{(2+k-\gamma-\lambda(k+\gamma))\Gamma_2}{2[1-\gamma+(2+k-\gamma-\lambda(k+\gamma))\Gamma_2]}(f*g)(z)$$

$$= \frac{(2+k-\gamma-\lambda(k+\gamma))\Gamma_2}{2[1-\gamma+(2+k-\gamma-\lambda(k+\gamma))\Gamma_2]}\left(z+\sum_{n=2}^{\infty} b_n a_n z^n\right). \quad (3.4)$$

Thus, by Definition 1.2, the subordination result holds true if

$$\left\{\frac{(2+k-\gamma-\lambda(k+\gamma))\Gamma_2}{2[1-\gamma+(2+k-\gamma-\lambda(k+\gamma))\Gamma_2]}\right\}_{n=1}^{\infty}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 1.1, this is equivalent to the following inequality

$$\text{Re}\left\{1+\sum_{n=1}^{\infty}\frac{(2+k-\gamma-\lambda(k+\gamma))\Gamma_2}{[1-\gamma+(2+k-\gamma-\lambda(k+\gamma))\Gamma_2]}a_n z^n\right\} > 0, \quad z\in U. \quad (3.5)$$

By noting the fact that $\frac{(n(1+k)-(\gamma+k)(1+n\lambda-\lambda))\Gamma_n}{(1-\gamma)}$ is increasing function for $n \geq 2$ and in particular

$$\frac{(2+k-\gamma-\lambda(k+\gamma))\Gamma_2}{(1-\gamma)} \leq \frac{(n(1+k)-(\gamma+k)(1+n\lambda-\lambda))\Gamma_n}{(1-\gamma)}, \quad n\geq 2,$$

therefore, for $|z| = r < 1$, we have

$$\text{Re}\left\{1+\frac{(2+k-\gamma-\lambda(k+\gamma))\Gamma_2}{[1-\gamma+(2+k-\gamma-\lambda(k+\gamma))\Gamma_2]}\sum_{n=1}^{\infty}a_n z^n\right\}$$

$$= \text{Re}\left\{1+\frac{(2+k-\gamma-\lambda(k+\gamma))\Gamma_2}{[1-\gamma+(2+k-\gamma-\lambda(k+\gamma))\Gamma_2]}z + \frac{\sum_{n=2}^{\infty}(2+k-\gamma-\lambda(k+\gamma))\Gamma_2 a_n z^n}{[1-\gamma+(2+k-\gamma-\lambda(k+\gamma))\Gamma_2]}\right\}$$

$$\geq 1-\frac{(2+k-\gamma-\lambda(k+\gamma))\Gamma_2}{[1-\gamma+(2+k-\gamma-\lambda(k+\gamma))\Gamma_2]}r -$$

$$\frac{1}{[1-\gamma+(2+k-\gamma-\lambda(k+\gamma))\Gamma_2]}\sum_{n=2}^{\infty}|[n(1+k)-(\gamma+k)(1+n\lambda-\lambda)]\Gamma_n a_n|\, r^n$$

$$\geq 1-\frac{(2+k-\gamma-\lambda(k+\gamma))\Gamma_2}{[1-\gamma+(2+k-\gamma-\lambda(k+\gamma))\Gamma_2]}r-\frac{1-\gamma}{[1-\gamma+(2+k-\gamma-\lambda(k+\gamma))\Gamma_2]}r$$
$$> 0, \quad |z| = r < 1,$$

where we have also made use of the assertion (2.1) of Theorem 2.1. This evidently proves the inequality (3.5) and hence also the subordination result (3.1) asserted by Theorem 3.1. The inequality (3.3) follows from (3.1) by taking

$$g(z) = \frac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n \in C.$$

Next we consider the function

$$F(z) := z - \frac{1-\gamma}{(2+k-\gamma-\lambda(k+\gamma))\Gamma_2}z^2,$$

where $0 \leq \gamma < 1,\ k \geq 0, 0 \leq \lambda < 1$ and Γ_2 is given by (3.2). Clearly $F \in \mathcal{H}_m^{*l}(\lambda,\gamma,k)$. For this function (3.1) becomes

$$\frac{(2+k-\gamma-\lambda(k+\gamma))\Gamma_2}{2[1-\gamma+(2+k-\gamma-\lambda(k+\gamma))\Gamma_2]}F(z) \prec \frac{z}{1-z}.$$

It is easily verified that

$$\min\left\{\mathrm{Re}\ \left(\frac{(2+k-\gamma-\lambda(k+\gamma))\Gamma_2}{2[1-\gamma+(2+k-\gamma-\lambda(k+\gamma))\Gamma_2]}F(z)\right)\right\} = -\frac{1}{2}, \quad z \in U.$$

This shows that the constant $\frac{(2+k-\gamma-\lambda(k+\gamma))\Gamma_2}{2[1-\gamma+(2+k-\gamma-\lambda(k+\gamma))\Gamma_2]}$ cannot be replaced by any larger one. □

By taking different choices of l, m, $\alpha_1,\alpha_2,\ldots,\alpha_l$, $\beta_1,\beta_2,\ldots,\beta_m$, λ, γ and k in the above theorem and in view of the Examples 1 to 4 in Section 1, we state the following corollaries for the subclasses defined in those examples.

Let $\mathbb{S}^*(\lambda,\gamma,k)$ denote the subclass of functions f in A whose coefficients a_n satisfy the condition (2.2). We note that $\mathbb{S}^*(\lambda,\gamma,k) \subseteq \mathbb{S}(\lambda,\gamma,k)$.

Corollary 3.1. *If $f \in \mathbb{S}^*(\lambda,\gamma,k)$, then*

$$\frac{[2+k-\gamma-\lambda(\gamma+k)]}{2[3+k-2\gamma-\lambda(k+\gamma)]}(f*g)(z) \prec g(z), \tag{3.6}$$

where $0 \leq \gamma < 1,\ 0 \leq \lambda < 1,\ k \geq 0,\ g \in C$ and

$$Re\{f(z)\} > -\frac{[3+k-2\gamma-\lambda(k+\gamma)]}{[2+k-\gamma-\lambda(k+\gamma)]}, \ \ z \in U.$$

The constant factor

$$\frac{[2+k-\gamma-\lambda(\gamma+k)]}{2[3+k-2\gamma-\lambda(k+\gamma)]}$$

in (3.6) cannot be replaced by a larger one.

Let $R_\delta^*(\lambda,\gamma,k)$ denote the subclass of functions f in A whose coefficients a_n satisfy the condition (2.3). We note that $R_\delta^*(\lambda,\gamma,k) \subseteq R_\delta(\lambda,\gamma,k)$.

Corollary 3.2. *If $f \in R_\delta^*(\lambda,\gamma,k)$, then*

$$\frac{(\delta+1)[2+k-\gamma-\lambda(k+\gamma)]}{2[(1-\gamma)+(\delta+1)(2+k-\gamma-\lambda(k+\gamma))]}(f*g)(z) \prec g(z), \tag{3.7}$$

where $0 \le \gamma < 1,\ 0 \le \lambda < 1,\ k \ge 0,\ \delta > -1,\ g \in C$ and

$$Re\{f(z)\} > -\frac{[(1-\gamma)+(\delta+1)(2+k-\gamma-\lambda(k+\gamma))]}{(\delta+1)[2+k-\gamma-\lambda(k+\gamma)]},\ \ z \in U.$$

The constant factor

$$\frac{(\delta+1)[2+k-\gamma-\lambda(k+\gamma)]}{2[(1-\gamma)+(\delta+1)(2+k-\gamma-\lambda(k+\gamma))]}$$

in (3.7) cannot be replaced by a larger one.

Let $B_\mu^*(\lambda,\gamma,k)$ denote the subclass of functions f in A whose coefficients a_n satisfy the condition (2.4). We note that $B_\mu^*(\lambda,\gamma,k) \subseteq B_\mu(\lambda,\gamma,k)$.

Corollary 3.3. *If $f \in B_\mu^*(\lambda,\gamma,k)$, then*

$$\frac{(\mu+1)[2+k-\gamma-\lambda(k+\gamma)]}{2[(\mu+2)(1-\gamma)+(\mu+1)[2+k-\gamma-\lambda(k+\gamma)]]}(f*g)(z) \prec g(z), (2+k-\gamma-\lambda(k+\gamma)) \tag{3.8}$$

where $0 \le \gamma < 1,\ 0 \le \lambda < 1,\ k \ge 0,\ \mu > -1,\ g \in C$ and

$$Re\{f(z)\} > -\frac{[(\mu+2)(1-\gamma)+(\mu+1)(2+k-\gamma-\lambda(k+\gamma))]}{(\mu+1)(2+k-\gamma-\lambda(k+\gamma))},\ \ z \in U.$$

The constant factor

$$\frac{(\mu+1)[2+k-\gamma-\lambda(k+\gamma)]}{2[(\mu+2)(1-\gamma)+(\mu+1)[2+k-\gamma-\lambda(k+\gamma)]]}$$

in (3.8) cannot be replaced by a larger one.

Let $L_c^{*a}(\lambda,\gamma,k)$ denote the subclass of functions f in A whose coefficients a_n satisfy the condition (2.5). We note that $L_c^{*a}(\lambda,\gamma,k) \subseteq L_c^a(\lambda,\gamma,k)$.

Corollary 3.4. *If $f \in L_c^{*a}(\lambda,\gamma,k)$, then*

$$\frac{a[2+k-\gamma-\lambda(k+\gamma)]}{2[c(1-\gamma)+a[2+k-\gamma-\lambda(k+\gamma)]]}(f*g)(z) \prec g(z), \tag{3.9}$$

where $0 \le \gamma < 1,\ 0 \le \lambda < 1,\ k \ge 0,\ a > 0, c > 0,\ g \in C$ and

$$Re\{f(z)\} > -\frac{[c(1-\gamma)+a[2+k-\gamma-\lambda(k+\gamma)]]}{a[2+k-\gamma-\lambda(k+\gamma)]},\ \ z \in U.$$

The constant factor

$$\frac{a[2+k-\gamma-\lambda(k+\gamma)]}{2[c(1-\gamma)+a[2+k-\gamma-\lambda(k+\gamma)]]}$$

in (3.9) cannot be replaced by a larger one.

Remark 3.1. We observe that Corollary 3.1, yields the results obtained by Frasin [4] and Singh [13] for the special values of λ, γ and k.

4. Integral Means Inequalities

In this section, we obtain integral means inequalities for the functions in the family $T\mathcal{H}_m^l(\lambda,\gamma,k)$. By taking appropriate choices of the parameters l, m, $\alpha_1,\ldots,\alpha_l,\beta_1,\ldots,\beta_m$, λ, γ, k, we obtain the integral means inequalities for several known as well as new subclasses.

Applying Lemma 1.2, Theorem 2.2 and Theorem 2.3, we prove the following result.

Theorem 4.1. *Suppose* $f \in T\mathcal{H}_m^l(\lambda,\gamma,k)$, $\eta > 0$, $0 \le \lambda < 1$, $0 \le \gamma < 1$, $k \ge 0$ *and* $f_2(z)$ *is defined by*

$$f_2(z) = z - \frac{1-\gamma}{\Phi(\lambda,\gamma,k,2)} z^2,$$

where

$$\Phi(\lambda,\gamma,k,2) = [2+k-\gamma-\lambda(k+\gamma)]\Gamma_2 \tag{4.1}$$

and Γ_2 *is given by (3.2). Then for* $z = re^{i\theta}$, $0 < r < 1$, *we have*

$$\int_0^{2\pi} |f(z)|^\eta \, d\theta \le \int_0^{2\pi} |f_2(z)|^\eta \, d\theta. \tag{4.2}$$

Proof. For $f(z) = z - \sum\limits_{n=2}^{\infty} |a_n| z^n$, (4.2) is equivalent to proving that

$$\int_0^{2\pi} \left|1 - \sum_{n=2}^{\infty} |a_n| z^{n-1}\right|^\eta d\theta \le \int_0^{2\pi} \left|1 - \frac{(1-\gamma)}{\Phi(\lambda,\gamma,k,2)} z\right|^\eta d\theta.$$

By Lemma 1.2, it suffices to show that

$$1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} \prec 1 - \frac{1-\gamma}{\Phi(\lambda,\gamma,k,2)} z.$$

Setting

$$1-\sum_{n=2}^{\infty}|a_n|z^{n-1}=1-\frac{1-\gamma}{\Phi(\lambda,\gamma,k,2)}w(z), \tag{4.3}$$

and using (2.6), we obtain

$$|w(z)|=\left|\sum_{n=2}^{\infty}\frac{\Phi(\lambda,\gamma,k,n)}{1-\gamma}|a_n|z^{n-1}\right|\leq |z|\sum_{n=2}^{\infty}\frac{\Phi(\lambda,\gamma,k,n)}{1-\gamma}|a_n|\leq |z|,$$

where $\Phi(\lambda,\gamma,k,n)=[n(1+k)-(\gamma+k)(1+n\lambda-\lambda)]\Gamma_n$ and Γ_n is given by (1.6). This completes the proof by Theorem 2.2. □

In view of the Examples 1 to 4 in Section 1 and Theorem 4.1, we have following corollaries for the classes defined in those examples.

Corollary 4.1. *If $f\in T\mathbb{S}(\lambda,\gamma,k)$, $0\leq\lambda<1$, $0\leq\gamma<1$, $k\geq 0$ and $\eta>0$, then the assertion (4.2) holds true, where*

$$f_2(z)=z-\frac{1-\gamma}{[2+k-\gamma-\lambda(k+\gamma)]}z^2.$$

Corollary 4.2. *If $f\in TR_\delta(\lambda,\gamma,k)$, $0\leq\lambda<1$, $\delta>-1$, $0\leq\gamma<1$, $k\geq 0$ and $\eta>0$, then the assertion (4.2) holds true, where*

$$f_2(z)=z-\frac{(1-\gamma)}{(\delta+1)[2+k-\gamma-\lambda(k+\gamma)]}z^2\ .$$

Corollary 4.3. *If $f\in TB_\mu(\lambda,\gamma,k)$, $\mu>-1$, $0\leq\lambda<1$, $0\leq\gamma<1$, $k\geq 0$ and $\eta>0$, then the assertion (4.2) holds true, where*

$$f_2(z)=z-\frac{(1-\gamma)(\mu+2)}{(\mu+1)[2+k-\gamma-\lambda(k+\gamma)]}z^2\ .$$

Corollary 4.4. *If $f\in TL_c^a(\lambda,\gamma,k)$, $a>0$, $c>0$, $0\leq\lambda<1$, $0\leq\gamma<1$, $k\geq 0$ and $\eta>0$, then the assertion (4.2) holds true, where*

$$f_2(z)=z-\frac{c(1-\gamma)}{a[2+k-\gamma-\lambda(k+\gamma)]}z^2.$$

Remark 4.1. Fixing $\lambda=0$ and $k=0$, Corollary 4.1 gives the integral means inequality for the class $T^*(\gamma)$ obtained in [12].

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