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CIRCULANT DETERMINANTS FOR THE STATIONARY SEQUENCES

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Abstract: In this paper we prove two results concerning the stationary sequences of random variables generated by circulant matrix $C_n = \text{circ}\,\{R_0, R_1, ..., R_{n-1}\}$, where R_m; $m = 0, 1, ..., n-1$ are correlation coefficients.

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1. Introduction

Let be given a sequence

$$(x_m)_{m=-\infty}^{m=+\infty}\,, \tag{$*$}$$

of random variables satisfying the following two conditions:

(1) all terms of the sequence $(*)$ have the same expected values and the same dispersions,

(2) the correlation coefficients r_{ik} of the random variables x_i and x_k are dependent only of the number $|i-k|$.

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Definition 1. The sequence $(*)$ is a stationary sequence if the conditions (1) and (2) are satisfied.

Definition 2. Let $E\left(X\right)$ denote the expected value of the random variable X. The correlation coefficients r_{ik} of the stationary sequence $(*)$ satisfy the following conditions:

(3) $E\left(x_i\right)=0,\ \ E\left(x_i^2\right)=1, \quad E\left(x_i x_k\right)=r_{ik}=R_{|i-k|}$.

Denote by $C_n=\text{circ}\left\{R_0,R_1,\ldots,R_{n-1}\right\}$ circulant matrix of the correlation coefficients $r_{ik}=R_{|i-k|}=R_m, m=0,1,2,\ldots,n-1$.

We prove the following two theorems.

Theorem 1. *Let* $W_n=\det C_n$. *Then* $W_n=0$ *if and only if for some* $j=0,1,\ldots,n-1$*we have*

$$R_1\cos\frac{2\pi j}{n}+R_2\cos\frac{2\pi 2j}{n}+\cdots+R_{n-1}\cos\frac{2\pi\left(n-1\right)j}{n}=-1\,, \tag{1.1}$$

$$R_1\sin\frac{2\pi j}{n}+R_2\sin\frac{2\pi 2j}{n}+\cdots+R_{n-1}\sin\frac{2\pi\left(n-1\right)j}{n}=0. \tag{1.2}$$

Theorem 2. *Let* $C_n=\text{circ}\left\{R_0,R_1,\ldots,R_{n-1}\right\}$ *be circulant non-singular matrix such that*

$$R_1+R_2+\cdots+R_{n-1}>\sqrt{2}-1. \tag{1.3}$$

If the equation

$$C_n^r+C_n^s+C_n^t, \tag{1.4}$$

has a solutions in positive integers r,s,t *then*

$$t=r+1 \ \text{ or } \ t=s+1. \tag{1.5}$$

We note that some interesting properties connected with the correlation coefficients for stationary sequences has been proved by Gelfond in the paper [2]. Namely, Gelfond proved that if G_n is the Gram determinant with the correlation coefficients $R_m, m=0,1,2,\ldots,n-1$ such that $G_n=0$ and $G_k>0$ for each $k<n$ then

$$R_m=\sum_{k=1}^{s}C_k\cos m\psi_k,\ \ s<n;\quad C_k,\psi_k\in R, \tag{$*$}$$

but the real constants C_k,ψ_k are not determined in explicit form. Moreover, Gelfond proved in the same paper that if Gram's determinant G_n, satisfies the inequality $G_n>0$ for every natural number n then we have:

$$\frac{G_n^2}{G_{n-1}G_{n+1}}\geq 1. \tag{$**$}$$

2. Proof of Theorem 1

It is known (see [5], p.127) that if $W_n = \det C_n = \det \operatorname{circ}\{R_0, R_1, \ldots, R_{n-1}\}$ then

$$W_n = \varphi(\varepsilon_0)\,\varphi(\varepsilon_1)\ldots\varphi(\varepsilon_{n-1}), \tag{2.1}$$

where

$$\varphi(x) = R_0 + R_1 x + \cdots + R_{n-1}x^{n-1}, \tag{2.2}$$

and

$$\varepsilon_j = \cos\frac{2\pi j}{n} + i\sin\frac{2\pi j}{n}, \quad j = 0, 1, \ldots, n-1. \tag{2.3}$$

Suppose that $W_n = 0$. Then by (2.1) and (2.2) it follows that for some $j = 0, 1, \ldots, n-1$ we have

$$\varphi(\varepsilon_j) = R_0 + R_1\varepsilon_j + \cdots + R_{n-1}\varepsilon_j^{n-1} = 0. \tag{2.4}$$

From (2.4) and (2.3) we obtain

$$R_0 + R_1\cos\frac{2\pi j}{n} + \cdots + R_{n-1}\cos\frac{2\pi(n-1)j}{n} = 0 \tag{2.5}$$

and

$$R_1\sin\frac{2\pi j}{n} + \cdots + R_{n-1}\sin\frac{2\pi(n-1)j}{n} = 0. \tag{2.6}$$

By the assumption (3) it follows that $R_0 = 1$ and consequently equality (2.5) and (2.6) implies (1.1) and (1.2).

The proof of Theorem 1 is complete. □

3. Proof of Theorem 2

Denote by F^* matrix of the form

$$F^* = \frac{1}{\sqrt{n}}\begin{pmatrix} 1 & 1 & 1 & . & . & . & 1 \\ 1 & \varepsilon_1 & \varepsilon_1^2 & . & . & . & \varepsilon_1^{n-1} \\ 1 & \varepsilon_1^2 & \varepsilon_1^3 & . & . & . & \varepsilon_1^{n-2} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 1 & \varepsilon_1^{n-1} & \varepsilon_1^{n-2} & . & . & . & \varepsilon_1 \end{pmatrix}, \quad \varepsilon_1 = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}. \tag{3.1}$$

Then the matrix F satisfying the condition

$$F = \overline{\overline{F^*}} \tag{3.2}$$

is the Fourier matrix.

Now, let

$$SpecC_n = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \quad \text{and} \quad \Lambda = diag\{\lambda_1, \lambda_2, \ldots, \lambda_n\}. \tag{3.3}$$

Then from Theorems 3.2.2, 3.2.3 and 3.2.4 and formula (3.4.2) given in [1] on pp. 72-74 and p. 92 it folows that

$$C_n^k = F^* \Lambda^k F, \tag{3.4}$$

for any integer k.

Suppose that there are positive integers r, s, t such that the equation (1.4) is satisfied. Then by (3.3) and (3.4) it follows that

$$\Lambda^r + \Lambda^s = \Lambda^t, \quad \lambda_j^r + \lambda_j^s = \lambda_j^t, \tag{3.5}$$

for every $j = 1, 2, \ldots, n$.

Now we observe that

$$\lambda_1 = \varphi(\varepsilon_0) = R_0 + R_1\varepsilon_0 + \cdots + R_{n-1}\varepsilon_0^{n-1} = R_0 + R_1 + \cdots + R_{n-1}, \tag{3.6}$$

because $\varepsilon_0 = 1$.

By the assumption of the Theorem 2 and (3.6) it follows that

$$\lambda_1 > \sqrt{2}. \tag{3.7}$$

Since the matrix C_n is non-singular then all eigenvalues $\lambda_j \neq 0$ and consequently for $j = 1$ we obtain from (3.5) that

$$\lambda_1^{r-t} + \lambda_1^{s-t} = 1. \tag{3.8}$$

From (3.7) immediately follows that if $r - t \geq 0$ or $s - t \geq 0$ then $\lambda_1^{r-t} \geq 1$ or $\lambda_1^{s-t} \geq 1$ and (3.8) is impossible. Hence, we get

$$r - t < 0, \text{ and } s - t < 0. \tag{3.9}$$

On the other hand by (3.7) it follows that

$$\lambda_1^{-2} < \frac{1}{2}. \tag{3.10}$$

If $r - t \leq -2$ and $s - t \leq -2$ then by (3.10) it follows that the equation (3.8) is impossible. Hence, in virtue of (3.9) we get that if the equation (1.4) with the condition (1.3) has a solution in positive integers r, s and t then $r - t = -1$ or $s - t = -1$.

The proof of Theorem 2 is complete. □

4. Corollary and Remark

Applying the equality (2.1) we obtain the following corollary.

Corollary. *Let* $W_k = \det C_k = \text{circ}\,\{R_0, R_1, \ldots, R_{k-1}\} \neq 0$ *for all natural number* k*. Then we have*

$$\frac{W_n^2}{W_{n-1}W_{n+1}} = \frac{R_0 + R_1\varepsilon_{n-1} + \cdots + R_{n-1}\varepsilon_{n-1}^{n-1}}{R_0 + R_1 + \cdots + R_{n-1}}. \tag{4.1}$$

Remark. The matrix equation of the form $A^r + A^s = A^t$, where $A = (a_{ij})_{n\times n}$ is the matrix with some restrictions on the elements a_{ij} has been investigated by A. Grytczuk in the papers [3], [4].

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