International Journal of Pure and Applied Mathematics
Volume 47 No. 3 2008, 411-428

EXACT SOLUTIONS FOR THE MAGNETOHYDRODYNAMIC STATIONARY FLOW OF A NEWTONIAN FLUID PAST A ROTATING PLANE

Alessandra Borrelli[1 §], Maria Cristina Patria[2]
[1,2]Department of Mathematics
University of Ferrara
35, via Machiavelli, Ferrara, 44100, ITALY
[1]e-mail: brs@unife.it
[2]e-mail: pat@unife.it

Abstract: Exact solutions are given for the steady flow of a Newtonian fluid occupying the halfspace $\mathcal{S}$ past the plane $z = 0$ uniformly rotating about a fixed normal axis ($\equiv z-$axis) when a uniform magnetic field $\mathbf{H}_0$ orthogonal to the plane is impressed. The plane is supposed to be electrically non conducting. The solutions are obtained in a velocity field of the form considered by Berker in [1] and supposing the induced magnetic field depending only on z. The results are compared with those corresponding to the Newtonian non-electrically conducting case and can be deduced as a limiting case, as $l \to +\infty$, of the solution to the problem relative to the strip $0 \leq z \leq l$.

AMS Subject Classification: 76W05, 76U05
Key Words: magnetohydrodynamics, exact solutions, rotating-disk flow

1. Introduction

In the past few years there has been a considerable interest in rotating magnetohydrodynamic viscous fluid flows. Actually magnetohydrodynamics finds practical use in many areas of engineering and pure sciences as, for example, pumping and levitation of liquid metals.

Received: June 23, 2008

§Correspondence author

The flow induced by an infinite disk (the plane $z = 0$) rotating in its own plane in a fluid occupies a central position in fluid dynamics beginning from the work of T. von Karman [8] (swirling flow) because it has immediate technical applications (i.e. rotating machines) and, from a mathematical point of view, the geometry of the flow is so simple to make possible the determination of an exact solution. The solutions relative to this problem, apart from the rigid body motion, can be divided into two classes: solutions with the velocity field symmetric and solutions with the velocity field asymmetric about the $z-$ axis (i.e. the rotation axis): von Karman flow belongs to the first family while the flow we are going to study belongs to the second one. So we are interested to velocity fields which are not symmetric about the rotation axis and we suppose the induced magnetic field depending only on z.

In order to clarify the characteristics of the motion taken into consideration, we recall that asymmetric solutions were introduced in [1] in the study of the steady motion for a Newtonian incompressible fluid confined between two parallel planes rotating about a fixed normal axis with the same angular velocity. These flow problems have relevance to the determination of the material moduli in viscometric experiments (rheometers).

We follow the procedure outlined in [7] (Newtonian non-electrically conducting fluids) according to which the problem is approached in a more significative physical manner than in [1]. More precisely a pressure field is assigned in order to generalize the form of the rigid motion pressure. Therefore we assume the pressure independent of z so that the streamlines are concentric circles in planes parallel to rigid boundary and the locus of the circles centers is no longer the $z-$ semi-axis but a curve Γ. The gradient of this pressure field differs from that corresponding to the rigid motion through a constant vector field which is parallel to the rotating plane and is arbitrarily fixed. As in Poiseuille flow between two fixed planes there is a pressure drop to allow a non-trivial flow (i.e. with non-zero velocity field), here we have the constant term of pressure gradient to determine a non-trivial flow (i.e. non-rigid rotation) and the deformation of the locus of the streamlines centers into the curve Γ.

Pao (see [6]) was one of the first authors who investigated the flow of an incompressible viscous fluid over a rotating disk when a circular magnetic field is imposed; in [3] the swirling flow is studied in the presence of a magnetic field normal to the disk. For motions confined between two parallel infinite plates rotating about two noncoincident axes perpendicular to them we refer to [4] where however the induced magnetic field is neglected entirely.

In the present paper we determine the exact solution for the steady flow of a homogeneous incompressible electrically conducting fluid in a halfspace bounded by a rigid infinite plane rotating with constant angular velocity $\mathbf{\Omega}$ about a fixed axis (z-axis) normal to it. An external magnetic field of constant strength H_0 is applied in the z-direction. The plane is supposed to be non-electrically conducting.

The exact asymmetric solution is found imposing on the rotating plane no-slip condition for the velocity field. As far as the magnetic field is concerned it is continuous across the boundary $z = 0$ because we suppose the halfspace $\mathcal{S}^- = \{(x, y, z) \in \mathbb{R}^3 : z < 0\}$ to be vacuum (free space) and the magnetic permeability of the fluid is taken equal to that of free space. Moreover we assume the fields bounded with respect to z.

We find that there is a boundary layer for the velocity (BLV) in which the flow is not a rigid rotation and there is a boundary layer (BLH) for the magnetic field $\mathbf{H}$ in which the angle $\varphi \in (0, \frac{\pi}{2})$ between $\mathbf{H}$ and the external magnetic field $\mathbf{H}_0$ depends on z. In other words $\varphi = \varphi(z)$ and it changes with z. As one can see by means of numerical simulations, (BLH) is much larger than (BLV) and if the angular velocity and the strength of H_0 increase, then the thickness of these boundary layers grows thinner. Moreover outside (BLV) the curve Γ tends, as $z \to +\infty$, to a straight line Γ_∞, parallel to the z-axis, the fluid tends to rotate as a rigid body about Γ_∞ while outside (BLH) the total magnetic field tends to the external one. It is interesting to remark that the constant pressure drop in the (x, y)-direction determines the translation of rotation axis from the z-axis, as $z \to +\infty$.

Finally this solution can be deduced as a limiting case, as $l \to +\infty$, of the solution to the problem relative to the strip $0 \leq z \leq l$. Therefore the solution relative to the halfspace, which has a much simpler form, can be seen as a good approximation to the solution relative to the strip when l is sufficiently large.

The paper is organized in the following manner:

In Section 2 we formulate the problem and prescribe the pressure field whose gradient differs from that of the rigid motion through a constant vector parallel to the plane $z = 0$.

In Section 3 we obtain the unique exact solution of the problem. It depends on the material constants, on H_0, Ω and on the constant part of the pressure gradient.

In Section 4 we illustrate some interesting consequences of the solution. In particular if $H_0 = 0$ we find the results of [7] again and we furnish some

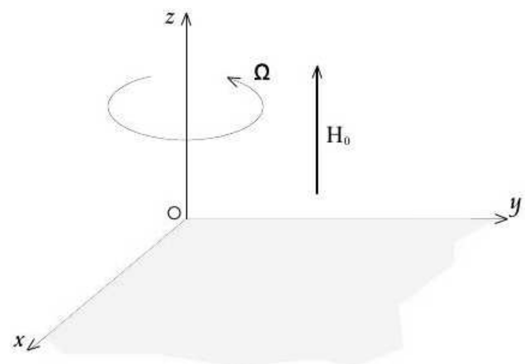

Figure 1: Flow description

numerical examples in the case of mercury at temperature $30°C$.

In Section 5 we obtain the solution when the fluid is confined between two parallel plates rotating about the same axis z normal to them. We show that, as $l \to +\infty$, this solution tends to the solution of the problem relative to the halfspace.

2. Statement of the Problem

Consider the stationary flow of a Newtonian fluid confined in the halfspace $\mathcal{S}$ bounded by a rigid infinite plane rotating with constant angular velocity $\mathbf{\Omega}$ about a fixed axis normal to it. A Cartesian rectangular coordinate system $Oxyz$, with the z-axis coincident with the axis of rotation, is introduced so that $\mathcal{S} = \{(x, y, z) \in \mathbb{R}^3 : z \geq 0\}$, $z = 0$ is the equation of the rigid wall and $\mathbf{\Omega} = \Omega(0, 0, 1)$, $\Omega > 0$, without loss of generality. We suppose the plane $z = 0$ to be electrically non-conducting.

The equations governing the steady flow of the electrically conducting homogeneous incompressible fluid upon which is impressed a uniform magnetic field $\mathbf{H}_0 = (0, 0, H_0)$, $H_0 > 0$, orthogonal to the plane (supposing the body forces to be conservative) are (see [2], [5])

$$\begin{aligned} &\nu \triangle \mathbf{v} - \mathbf{v} \cdot \nabla \mathbf{v} - \nabla p + \mu \mathbf{H} \cdot \nabla \mathbf{H} = \mathbf{0}, \\ &\eta \triangle \mathbf{H} + \nabla \times (\mathbf{v} \times \mathbf{H}) = \mathbf{0}, \qquad \text{on } \mathcal{S} \\ &\nabla \cdot \mathbf{v} = 0, \\ &\nabla \cdot \mathbf{H} = 0. \end{aligned} \tag{1}$$

In (1) $\mathbf{v}, \mathbf{H}$ are the velocity and magnetic fields respectively, $p = \frac{1}{\rho}\Big(p^* + \frac{\mu_e H^2}{2} + P\Big)$, p^* is the pressure field and $-\nabla P$ is the external body force (for sake of simplicity, in the sequel p will be called pressure field), ρ is the constant mass density; ν is the kinematic viscosity coefficient, $\mu = \frac{\mu_e}{\rho}$, μ_e magnetic permeability (it is not restrictive to take μ_e equal to the magnetic permeability of free space), $\eta = \frac{1}{\mu_e \sigma_e}$ is the magnetic resistivity and σ_e is the electrical conductivity.

To (1) we adjoin the boundary conditions

$$\mathbf{v} = \Omega(-y, x, 0), \quad \mathbf{H}_\tau = \mathbf{0}, \quad \text{at} \;\; z = 0. \tag{2}$$

Condition $(2)_1$ implies that the fluid adheres to the boundary $z = 0$ of $\mathcal{S}$ (no-slip condition); condition $(2)_2$ means that the tangential component $\mathbf{H}_\tau$ of the magnetic field is continuous across the boundary $z = 0$.

As it is easy to verify, under conditions (2), system (1) admits the simple solution (rigid body motion)

$$\mathbf{v}_R = \Omega(-y, x, 0), \quad \mathbf{H}_R = (0, 0, H_0), \quad p_R = \frac{1}{2}\Omega^2(x^2 + y^2) + p_0\,, \tag{3}$$

where p_0 is an arbitrary constant.

Moreover, as is well known, the streamlines in any $z =$ constant plane are concentric circles with center on the z-axis.

In our analysis we fix $f_0, g_0 \in \mathbb{R}$ arbitrarily and we assume that the fluid is subjected to a pressure field p given by

$$p = \frac{1}{2}\Omega^2[(x - f_0)^2 + (y - g_0)^2] + p_0\,, \tag{4}$$

where p_0 is an arbitrary inessential constant.

We notice that $\nabla p = \nabla p_R + \nabla p_\Gamma$ with $\nabla p_\Gamma = -\Omega^2(f_0, g_0, 0)$. This vector is parallel to the plane $z = 0$.

We shall search classical solutions $(\mathbf{v}, \mathbf{H})$ of (1) with p given by (4), such that:

i) the streamlines in any $z =$ constant plane are concentric circles;

ii) $(\mathbf{v}, \mathbf{H})$ satisfies the boundary conditions (2);

iii) $(\mathbf{v}, \mathbf{H}) \in \mathcal{M}$, $\mathcal{M}$ being the class of functions $(\mathbf{v}_\mathcal{M}, \mathbf{H}_\mathcal{M})$ which are bounded with respect to z, $z \in [0, +\infty)$, and $\mathbf{H}_\mathcal{M} \to (0, 0, H_0)$ as $z \to +\infty$, uniformly in x, y.

To this end, we shall seek sufficiently smooth solutions of (1), (2) belonging to $\mathcal{M}$ of the form

$$v_1 = -\Omega(y - g(z)), \quad v_2 = \Omega(x - f(z)), \quad v_3 = 0, \tag{5}$$

$$H_i = h_i(z), \quad i = 1, 2, \quad H_3 = h_3(z) + H_0 \quad \forall z \geq 0. \tag{6}$$

By virtue of assumptions (5), the space curve Γ, given by the locus of the points at which the velocity is zero in each of the plane parallel to the plane $z = 0$, has Cartesian equations

$$x = f(z), \quad y = g(z), \quad z \in [0, +\infty). \tag{7}$$

3. Exact Solution

First of all we note that assumptions (5) ensure that $\mathbf{v}$ is divergence free so that (1.3) is automatically verified. Assumptions (6) together with *iii*) and (1.4) imply

$$H_3 = H_0, \quad \forall z \geq 0\,, \tag{8}$$

so that the induced magnetic field reduces to $(h_1(z), h_2(z), 0)$.

Now substituting (5), (6) into (1.1), (1.2) and taking into account (4), (2), (8); we easily obtain

$$\begin{aligned} g'' + \frac{\mu H_0}{\nu \Omega}\, h_1' - \frac{\Omega}{\nu}\, f &= -\frac{\Omega}{\nu}\, f_0, \\ f'' - \frac{\mu H_0}{\nu \Omega}\, h_2' + \frac{\Omega}{\nu}\, g &= \frac{\Omega}{\nu}\, g_0, \\ h_1'' + \frac{\Omega H_0}{\eta}\, g' - \frac{\Omega}{\eta}\, h_2 &= 0, \qquad z \in (0, +\infty) \\ h_2'' - \frac{\Omega H_0}{\eta}\, f' + \frac{\Omega}{\eta}\, h_1 &= 0 \end{aligned} \tag{9}$$

together with

$$f(0) = g(0) = 0, \quad h_1(0) = h_2(0) = 0. \tag{10}$$

Putting

$$\mathcal{F} = f + i\, g, \quad \mathcal{H} = h_1 + i\, h_2, \quad \mathcal{F}_0 = f_0 + i\, g_0,$$

the equations (9) can be written as:

$$\begin{aligned} \mathcal{F}'' + i\, \frac{N}{H_0 k_2} \mathcal{H}' - i\, k_1 \mathcal{F} &= -i\, k_1 \mathcal{F}_0, \\ \mathcal{H}'' - i H_0 k_2 \mathcal{F}' + i\, k_2 \mathcal{H} &= 0, \end{aligned} \tag{11}$$

where $N = \frac{\mu H_0^2}{\nu \eta}, \quad k_1 = \frac{\Omega}{\nu}, \quad k_2 = \frac{\Omega}{\eta}$.

We notice that, from a physical point of view, $k_1 \gg k_2$ (for example, in the case of mercury at temperature $30°C$ one has $k_1 \simeq 10^6\, k_2$). Therefore, in the sequel, we shall suppose

$$k_1 - k_2 > 0.$$

After some calculations, from (11), we deduce that $\mathcal{F}$ satisfies the following 4-th order ODE:

$$\mathcal{F}^{IV} - [N + i(k_1 - k_2)]\mathcal{F}'' + k_1 k_2 \mathcal{F} = k_1 k_2 \mathcal{F}_0. \tag{12}$$

The roots of the associated characteristic equation are

$$m_1 = \alpha_1 + i\beta_1, \quad -m_1, \quad m_2 = \alpha_2 - i\beta_2, \quad -m_2\,,$$

with

$$\begin{aligned}
2\alpha_1 &= \Big\{\Big[(N+a)^2 + (k_1 - k_2 + b)^2\Big]^{1/2} + (N+a)\Big\}^{1/2},\\
2\beta_1 &= \Big\{\Big[(N+a)^2 + (k_1 - k_2 + b)^2\Big]^{1/2} - (N+a)\Big\}^{1/2},\\
2\alpha_2 &= \Big\{\Big[(N-a)^2 + (k_1 - k_2 - b)^2\Big]^{1/2} + (N-a)\Big\}^{1/2}\\
2\beta_2 &= \Big\{\Big[(N-a)^2 + (k_1 - k_2 - b)^2\Big]^{1/2} - (N-a)\Big\}^{1/2},\\
\sqrt{2}a &= \Big\{\Big[\Big(N^2 - (k_1 + k_2)^2\Big)^2 + 4N^2(k_1 - k_2)^2\Big]^{1/2} + \Big[N^2 - (k_1 + k_2)^2\Big]\Big\}^{1/2},\\
\sqrt{2}b &= \Big\{\Big[\Big(N^2 - (k_1 + k_2)^2\Big)^2 + 4N^2(k_1 - k_2)^2\Big]^{1/2} - \Big[N^2 - (k_1 + k_2)^2\Big]\Big\}^{1/2}.
\end{aligned} \tag{13}$$

Therefore the general solution of (12) is

$$\mathcal{F}(z) = C_1 e^{m_1 z} + C_2 e^{-m_1 z} + C_3 e^{m_2 z} + C_4 e^{-m_2 z} + \mathcal{F}_0, \qquad \forall z \geq 0,$$
$$C_i \in \mathbb{C} \quad (i = 1, 2, 3, 4).$$

In our case

$$\mathcal{F}(0) = 0, \qquad \mathcal{F} \text{ bounded as } \; z \to +\infty, \tag{14}$$

so that $\mathcal{F}(z)$ reduces to

$$\mathcal{F}(z) = C(e^{-m_1 z} - e^{-m_2 z}) + \mathcal{F}_0(1 - e^{-m_2 z}), \quad \forall z \geq 0\,, \tag{15}$$

where C is an arbitrary constant to be determined.

At this point we integrate (11.1) taking into account (15) and the hypothesis $\mathcal{H}(z) \to 0$ as $z \to +\infty$.

We get

$$\mathcal{H}(z) = \frac{H_0 k_2}{N}\Big[C\Big(-\frac{k_1 + im_1^2}{m_1}\, e^{-m_1 z} + \frac{k_1 + im_2^2}{m_2}\, e^{-m_2 z}\Big) + \mathcal{F}_0 \frac{k_1 + im_2^2}{m_2}\, e^{-m_2 z}\Big], \quad \forall z \geq 0. \tag{16}$$

The boundary condition $\mathcal{H}(0) = 0$ allows to determine the complex constant

C:

$$C = \frac{m_1(k_1 + im_2^2)}{(m_2 - m_1)(k_1 - im_1m_2)}\mathcal{F}_0. \tag{17}$$

Finally we can separate the real and imaginary parts of (15), (16) with C given by (17) to obtain

$$\begin{aligned}
f(z) &= f_0\{1 + e^{-\alpha_1 z}(u\cos\beta_1 z - v\sin\beta_1 z) - e^{-\alpha_2 z}[(1+u)\cos\beta_2 z + v\sin\beta_2 z]\}\\
&+g_0\{e^{-\alpha_1 z}(v\cos\beta_1 z + u\sin\beta_1 z) + e^{-\alpha_2 z}[(1+u)\sin\beta_2 z - v\cos\beta_2 z]\}\\
g(z) &= f_0\{-e^{-\alpha_1 z}(v\cos\beta_1 z + u\sin\beta_1 z) - e^{-\alpha_2 z}[(1+u)\sin\beta_2 z - v\cos\beta_2 z]\}\\
&+g_0\{1 + e^{-\alpha_1 z}(u\cos\beta_1 z - v\sin\beta_1 z) - e^{-\alpha_2 z}[(1+u)\cos\beta_2 z + v\sin\beta_2 z]\}
\end{aligned} \tag{18}$$

$$\begin{aligned}
h_1(z) &= \frac{H_0k_2}{N}f_0\Big[-e^{-\alpha_1 z}(c\cos\beta_1 z + d\sin\beta_1 z) + e^{-\alpha_2 z}(c\cos\beta_2 z - d\sin\beta_2 z)\Big]\\
&+\frac{H_0k_2}{N}g_0\Big[e^{-\alpha_1 z}(d\cos\beta_1 z - c\sin\beta_1 z) - e^{-\alpha_2 z}(d\cos\beta_2 z + c\sin\beta_2 z)\Big]\\
h_2(z) &= \frac{H_0k_2}{N}f_0\Big[e^{-\alpha_1 z}(-d\cos\beta_1 z + c\sin\beta_1 z) + e^{-\alpha_2 z}(d\cos\beta_2 z + c\sin\beta_2 z)\Big]\\
&+\frac{H_0k_2}{N}g_0\Big[-e^{-\alpha_1 z}(c\cos\beta_1 z + d\sin\beta_1 z) + e^{-\alpha_2 z}(c\cos\beta_2 z - d\sin\beta_2 z)\Big]
\end{aligned} \tag{19}$$

with

$$u = \frac{A_1(A_2 - A_1) + B_1(B_2 - B_1)}{(A_2 - A_1)^2 + (B_2 - B_1)^2}, \quad v = \frac{B_1A_2 - A_1B_2}{(A_2 - A_1)^2 + (B_2 - B_1)^2},$$

$$A_1 = \alpha_1(\alpha_2^2 - \beta_2^2) + \beta_1(2\alpha_2\beta_2 + k_1), \quad A_2 = \alpha_2(\alpha_1^2 - \beta_1^2) + \beta_2(2\alpha_1\beta_1 - k_1),$$

$$B_1 = -\beta_1(\alpha_2^2 - \beta_2^2) + \alpha_1(2\alpha_2\beta_2 + k_1), \quad B_2 = \beta_2(\alpha_1^2 - \beta_1^2) - \alpha_2(2\alpha_1\beta_1 - k_1),$$

$$c = uD_1 + vE_1 = (1+u)D_2 + vE_2, \quad d = uE_1 - vD_1 = (1+u)E_2 - vD_2\,,$$

$$D_1 = \frac{k_1\alpha_1 - \beta_1(\alpha_1^2 + \beta_1^2)}{\alpha_1^2 + \beta_1^2}, \qquad E_1 = \frac{-k_1\beta_1 + \alpha_1(\alpha_1^2 + \beta_1^2)}{\alpha_1^2 + \beta_1^2},$$

$$D_2 = \frac{k_1\alpha_2 + \beta_2(\alpha_2^2 + \beta_2^2)}{\alpha_2^2 + \beta_2^2}, \qquad E_2 = \frac{k_1\beta_2 + \alpha_2(\alpha_2^2 + \beta_2^2)}{\alpha_2^2 + \beta_2^2}. \tag{20}$$

We conclude this section summarizing the previous results in the following:

Theorem 1. *Let an electrically conducting homogeneous incompressible Newtonian fluid occupy the halfspace* $\mathcal{S} = \{(x, y, z) \in \mathbb{R}^3 : z \geq 0\}$ *bounded by the rigid plane* $z = 0$ *rotating about the fixed* z*-axis with constant angular velocity* $\boldsymbol{\Omega} = (0, 0, \Omega)$ $(\Omega > 0)$*. We suppose that a uniform magnetic field* $\mathbf{H}_0 = (0, 0, H_0)$ *orthogonal to the electrically non-conducting rotating plane is impressed upon the fluid and the body forces are conservative. Then, if the pressure field is given by (4), the velocity* $\mathbf{v}$ *and the total magnetic field* $\mathbf{H}$ *for* *the stationary flow of the fluid satisfying* i), ii), iii) *are given by* (5), (6), (8), (18), (19), (20).

We notice that $\forall(f_0, g_0)$ (i.e. for any choice of the pressure field of the kind (4)) there exists a unique magnetohydrodynamic flow satisfying the previous conditions.

4. Concluding Remarks

In this Section we discuss some properties of the solution and analyze some numerical examples.

1. First of all we observe that if $H_0 = 0$, we find the results concerning the Newtonian non-electrically conducting case (see [7]). Actually we have

$$\alpha_1 = \sqrt{\frac{\Omega}{2\nu}} = \beta_1 \equiv m, \qquad \alpha_2 = \sqrt{\frac{\Omega}{2\eta}} = \beta_2, \qquad u = -1, \qquad v = 0\,,$$

so that f, g of (18) reduce to f^*, g^* respectively given by

$$\begin{aligned} f^*(z) &= f_0(1 - e^{-mz}\cos mz) - g_0 e^{-mz}\sin mz\,, \\ g^*(z) &= f_0\, e^{-mz}\sin mz + g_0(1 - e^{-mz}\cos mz),\ \forall z \geq 0\,, \end{aligned} \tag{21}$$

which are the functions (7) of [7].

Further a simple calculation shows that $h_i \to 0, i = 1, 2$, as $H_0 \to 0$.

2. If we take the limit as $z \to +\infty$ in both members of (18), we get

$$\lim_{z\to+\infty} f(z) = f_0, \qquad \lim_{z\to+\infty} g(z) = g_0.$$

These results state that, as $z \to +\infty$, $\mathbf{v}$ differs from the rigid body velocity $\mathbf{v}_R$ through the constant vector $\mathbf{v}_0 = \Omega(g_0, -f_0, 0)$ orthogonal to the pressure drop ∇p_Γ and the curve Γ tends, as $z \to +\infty$, to the straight line Γ_∞, parallel to the z-axis, which includes the point of coordinates $(f_0, g_0, 0)$.

In other words we have that as $z \to +\infty$ the fluid rotates about the axis Γ_∞ and the induced magnetic field vanishes.

Hence the pressure drop ∇p_Γ in the (x, y)-direction determines the translation of the rotation axis from the z-axis as $z \to +\infty$.

Moreover we observe that the pressure field (4) does not influence the component of the magnetic field H_3; actually $H_3 = H_0$ as in the rigid body motion.

3. We notice that one can consider the problem in all space supposing the halfspace $\mathcal{S}^- = \{(x, y, z) \in \mathbb{R}^3 : z < 0\}$ to be vacuum (free space).

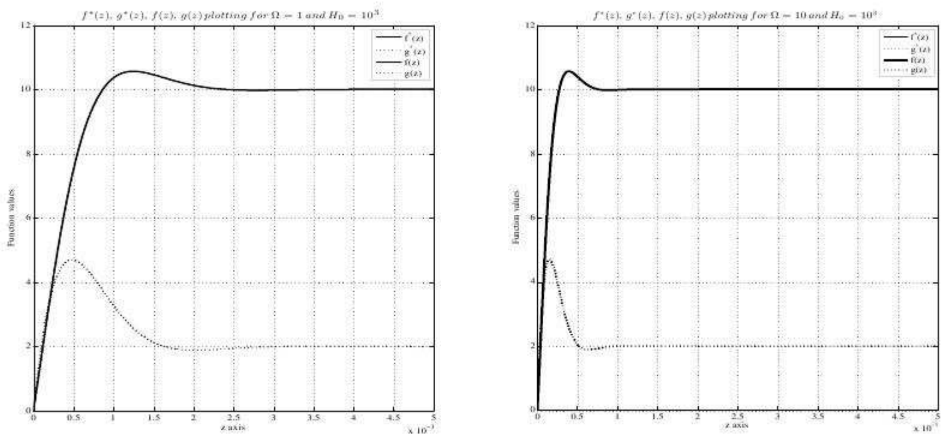

Figure 2: Plots showing the comparison between f, f^* and g, g^* when $H_0 = 10^3 A\,m^{-1}$

The solution of the problem $(\mathbf{v}, \mathbf{H})$ is such that

$$\mathbf{v} = (-\Omega(y - g(z)), \Omega(x - f(z)), 0), \quad \mathbf{H} = (h_1(z), h_2(z), H_0) \text{ in } \mathcal{S},$$

$$\mathbf{v} = \mathbf{0}, \quad \mathbf{H} = (0, 0, H_0) \text{ in } \mathcal{S}^- .$$

The usual conditions across the plane $z = 0$ for the magnetic field are satisfied.

4. Here we give some numerical examples in the case of mercury at temperature $30°C$.

The material constants have the values: $\mu_e = 4\pi \times 10^{-7}\ NA^{-2}$ (actually we can take μ_e equal to the magnetic permeability of free space), $\sigma_e = 10^6\ \Omega^{-1}m^{-1}$, $\nu = 1.2 \times 10^{-7}\ m^2 sec^{-1}$. These are typical values for many other liquid metals. Moreover, for mercury $\rho = 1.35 \times 10^4\ kg\,m^{-3}$.

The graphs are given for $f_0 = 10.0\,m$, $g_0 = 2.0\,m$, $H_0 = 10^3 A\,m^{-1}$ (induction magnetic field $\sim 10^{-2}$ Tesla) and $H_0 = 10^5 A\,m^{-1}$ (induction magnetic field ~ 1 Tesla) and supposing the angular velocity of the plane equal to $1\,rad\,sec^{-1}$ and to $10\,rad\,sec^{-1}$ (i.e. ~ 0.15, $\sim 1.5\,revolution/sec$ respectively).

We recall that the values given for f_0, g_0 are purely indicative because they are completely arbitrary.

First of all we notice that the thickness of the layer relative to $\mathbf{v}$ (BLV) in which the curve Γ is distorted is very thin (of the order of some millimetres).

Figure 2 shows that if $H_0 = 10^3\ A\,m^{-1}$ then the graphs of the functions f, f^*

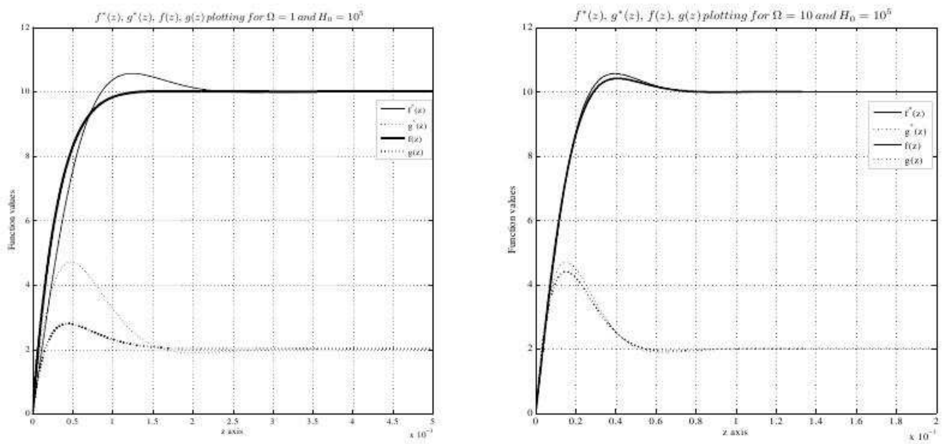

Figure 3: Plots showing the comparison between f, f^* and g, g^* when $H_0 = 10^5 A\, m^{-1}$

and g, g^* are practically coincident even if the angular velocity Ω increases.

Figure 3 shows that the graphs of the functions f, f^* and g, g^* are no more coincident when H_0 increases; so, if H_0 increases then the influence of the magnetic field is more significant.

Moreover we can see that if the angular velocity increases (Figure 3.b) then the difference between the graphs, even for intense magnetic fields, is less manifest.

Finally, from these figures, we conclude that velocity field presents a boundary layer (BLV) whose thickness depends on H_0 and Ω. More precisely when $\Omega = 1$ it grows thinner when H_0 increases; if $\Omega = 10$ (i.e. the angular velocity grows) then the thickness of the layer does not change in significant manner even if H_0 increases.

Figures 4, 5, 6, 7 show the behavior of h_1, h_2 near zero (Figures a) and for z sufficiently large (Figures b).

We can see that the strength of the induced magnetic field is much smaller than H_0; the angle $\varphi \in (0, \frac{\pi}{2})$ between the total magnetic field $\mathbf{H}$ and the external magnetic field $\mathbf{H}_0$ changes with z in a boundary layer (BLH) whose thickness depends on H_0 and Ω. This thickness decreases when H_0, Ω increase as for the boundary layer relative to the velocity field. The width of (BLH) is much larger than the width of (BLV).

We can note that φ changes fast near the boundary while outside the bound-

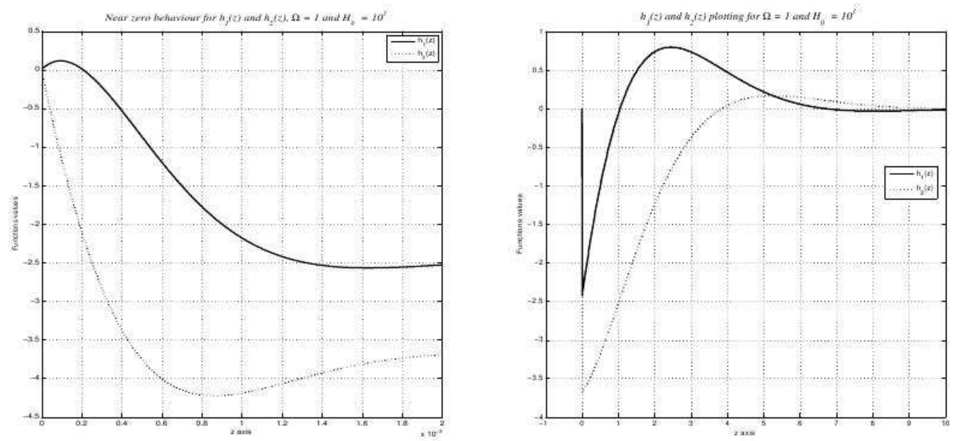

Figure 4: Plots showing the behavior, near zero (Figure 4.a) and for z sufficiently large (Figure 4.b), of $h_1,\ h_2$ when $H_0 = 10^3 A\,m^{-1}$ and $\Omega = 1\ rad/sec$

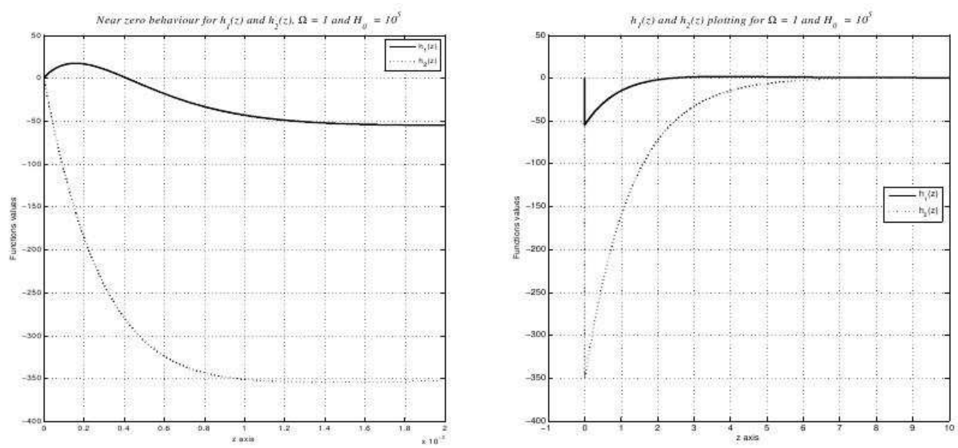

Figure 5: Plots showing the behavior, near zero (Figure 5.a) and for z sufficiently large (Figure 5.b), of $h_1,\ h_2$ when $H_0 = 10^5 A\,m^{-1}$ and $\Omega = 1\ rad/sec$

ary layer, the total magnetic field reduces to $\mathbf{H}_0$ which is parallel to the z-axis.

Figures 8-9 show the curve Γ (and its projections) for the values of H_0 and Ω above considered.

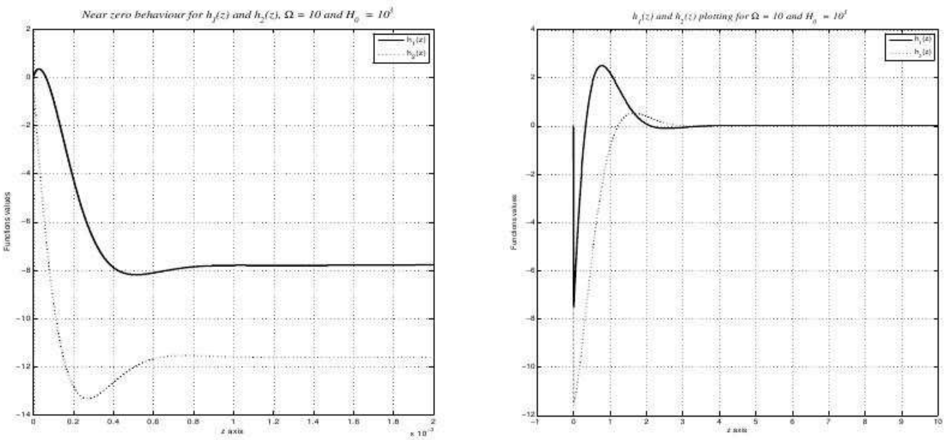

Figure 6: Plots showing the behavior, near zero (Figure 6.a) and for z sufficiently large (Figure 6.b), of $h_1,\ h_2$ when $H_0 = 10^3 A\,m^{-1}$ and $\Omega = 10\ rad/sec$

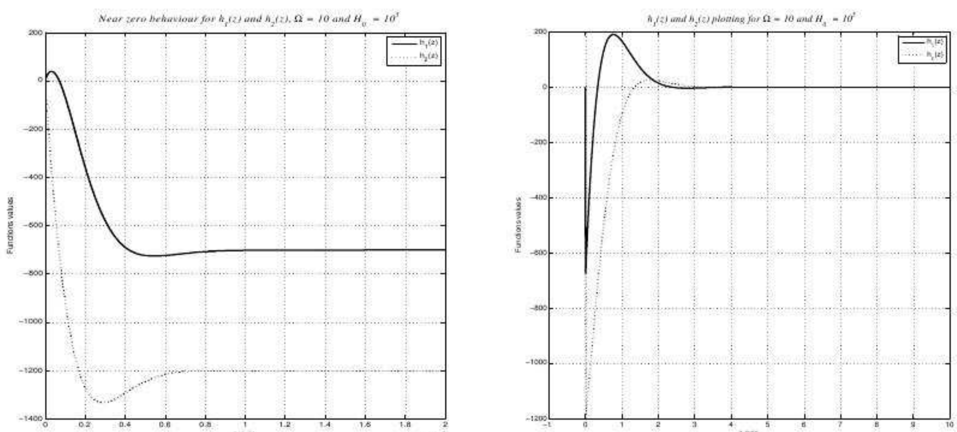

Figure 7: Plots showing the behavior, near zero (Figure 7.a) and for z sufficiently large (Figure 7.b), of $h_1,\ h_2$ when $H_0 = 10^5 A\,m^{-1}$ and $\Omega = 10\ rad/sec$

These figures are similar to those of the Newtonian case ($N = 0$), see [7], apart from the thickness of the boundary layer which depends on the presence of the external magnetic field.

We observe that the fluid tends to move as a rigid body outside of a layer,

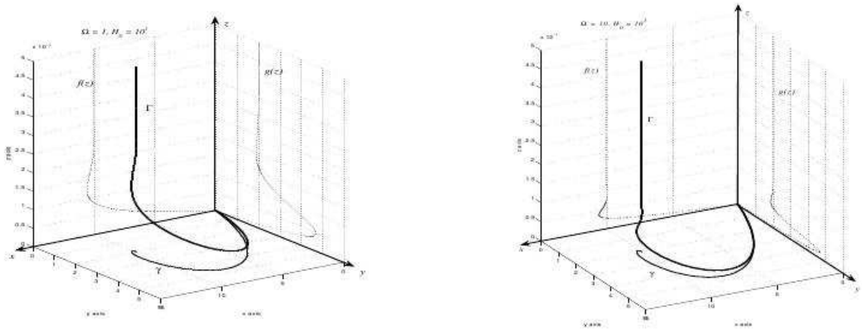

Figure 8: Curve Γ and its projections plots with $H_0 = 10^3 A\, m^{-1}$

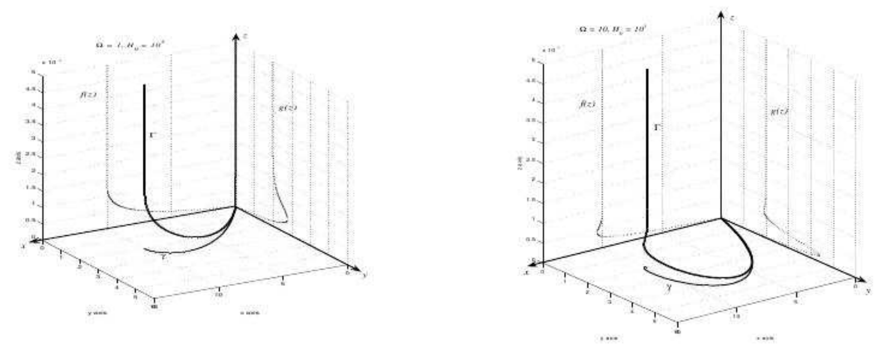

Figure 9: Curve Γ and its projections plots with $H_0 = 10^5 A\, m^{-1}$

whose thickness depends on H_0 and on Ω. As one can see, if the angular velocity and the strength of H_0 increase then the thickness of the flow boundary layer grows thinner.

5. Exact Solutions between Two Rigid Rotating Planess

In this section we assume that the fluid is confined between two rigid planes $z = 0$, and $z = l$, $l > 0$, rotating with the same angular velocity $\mathbf{\Omega}$ about the z-axis and that it is submitted to the pressure field p given by (4).

These flow problems has relevance to the determination of the material moduli in viscometric experiments (rheometers).

In order to determine $(\mathbf{v}, \mathbf{H})$ of the form (5), (6) we have to integrate system

(9) in $(0, l)$.

We impose at $z = 0$ and $z = l$ the boundary conditions (2), i.e.

$$\begin{aligned} f(0) = g(0) = 0, \quad & h_1(0) = h_2(0) = 0, \\ f(l) = g(l) = 0, \quad & h_1(l) = h_2(l) = 0. \end{aligned} \tag{22}$$

As in Section 3, we obtain $H_3(z) = H_0, \quad \forall z \in [0, l]$.

The general solution $\mathcal{F} = f + ig, \quad \mathcal{H} = h_1 + ih_2$ of (9) is given by:

$$\begin{aligned} \mathcal{F}(z) &= C_1 e^{m_1 z} + C_2 e^{-m_1 z} + C_3 e^{m_2 z} + C_4 e^{-m_2 z} + \mathcal{F}_0, \\ \mathcal{H}(z) &= \frac{H_0 k_2}{N} \Big[\frac{k_1 + i\, m_1^2}{m_1} \Big(C_1 e^{m_1 z} - C_2 e^{-m_1 z} \Big) \\ &\qquad + \frac{k_1 + i\, m_2^2}{m_2} \Big(C_3 e^{m_2 z} - C_4 e^{-m_2 z} \Big) \Big], \end{aligned} \tag{23}$$

where $C_i \in \mathbb{C}$ are arbitrary constants and $z \in [0, l]$.

The boundary conditions (22) furnish:

$$C_1 = \frac{a_2 \mathcal{F}_0 \{ a_1 [\cosh(m_2 l) - 1](1 + e^{-m_1 l}) + a_2 \sinh(m_2 l)(e^{-m_1 l} - 1) \}}{(a_1 - a_2)^2 \cosh(m_1 + m_2) l - (a_1 + a_2)^2 \cosh(m_1 - m_2) l + 4 a_1 a_2},$$

$$C_2 = \frac{a_2 \mathcal{F}_0 \{ a_1 [\cosh(m_2 l) - 1](1 + e^{m_1 l}) + a_2 \sinh(m_2 l)(1 - e^{m_1 l}) \}}{(a_1 - a_2)^2 \cosh(m_1 + m_2) l - (a_1 + a_2)^2 \cosh(m_1 - m_2) l + 4 a_1 a_2},$$

$$C_3 = \frac{a_1 \mathcal{F}_0 \{ a_1 \sinh(m_1 l)(e^{-m_2 l} - 1) + a_2 [\cosh(m_1 l) - 1](1 + e^{-m_2 l}) \}}{(a_1 - a_2)^2 \cosh(m_1 + m_2) l - (a_1 + a_2)^2 \cosh(m_1 - m_2) l + 4 a_1 a_2},$$

$$C_4 = \frac{a_1 \mathcal{F}_0 \{ a_1 \sinh(m_1 l)[1 - e^{m_2 h}] + a_2 [\cosh(m_1 l) - 1](1 + e^{m_2 l}) \}}{(a_1 - a_2)^2 \cosh(m_1 + m_2) l - (a_1 + a_2)^2 \cosh(m_1 - m_2) l + 4 a_1 a_2},$$

where a_1 e a_2 are

$$a_1 = \frac{k_1 + i\, m_1^2}{m_1}, \qquad a_2 = \frac{k_1 + i\, m_2^2}{m_2}.$$

As it can be verified, as $l \to +\infty$ we have that

$$C_1,\ C_3 \to\ 0, \quad C_2 \to\ C, \quad C_4 \to\ -(C + \mathcal{F}_0).$$

Therefore the solution of problem (9), (10) can be deduced as a limiting case, as $l \to +\infty$, of the solution to the problem relative to the strip $0 \le z \le l$. So the solution relative to the halfspace (which is much simpler to be determined) can be seen as a good approximation to the solution relative to the strip when l is sufficiently large.

As one can see, cumbersome and long calculations show that our results are in accordance with those obtained in [7] in the sense that if $H_0 = 0$ we find again the results concerning the Newtonian non-electrically conducting case.

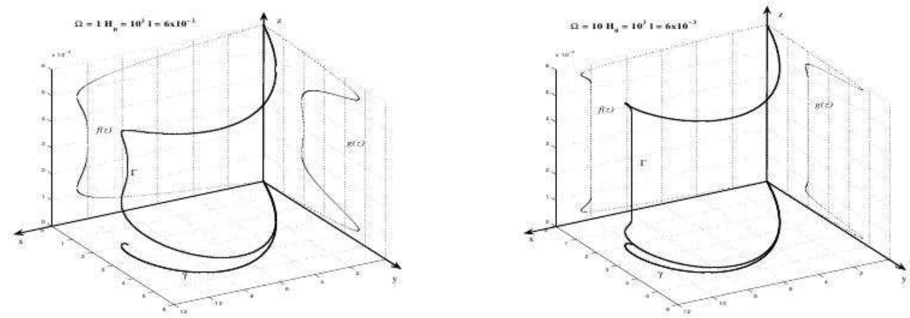

Figure 10: Curve Γ and its projections plots with $H_0 = 10^3 A\, m^{-1}$

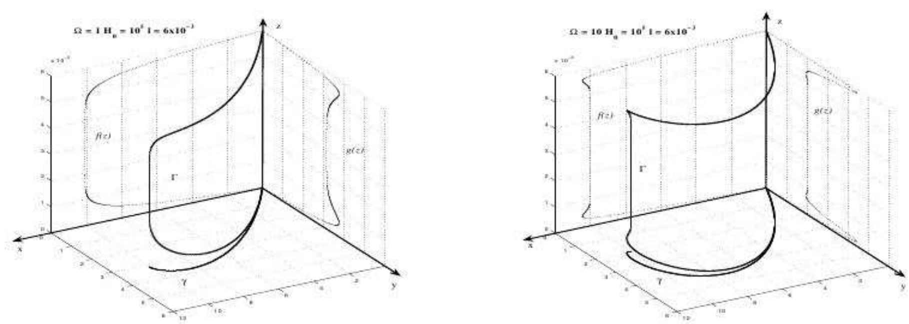

Figure 11: Curve Γ and its projections plots with $H_0 = 10^5 A\, m^{-1}$

Figures 10 and 11 show Γ and its projections in the case of the strip of width $l = 6 \times 10^{-3} m$ for the mercury considered in Remark 4. Figures 12 and 13 give the behavior of the induced magnetic field. In order to underline the influence of the magnetic field H_0, we have considered a strip with a very small width.

If we compare Figures 10, 11 with Figures 8, 9 we can see that near the plane $z = 0$ the development of Γ is the same. As in the case of the halfspace, the shape of Γ can be controlled by changing the values of H_0, Ω; moreover, in this case also the width of the strip influences the shape of Γ. Also in the case of the strip, the influence of the magnetic field becomes less manifest when the angular velocity increases.

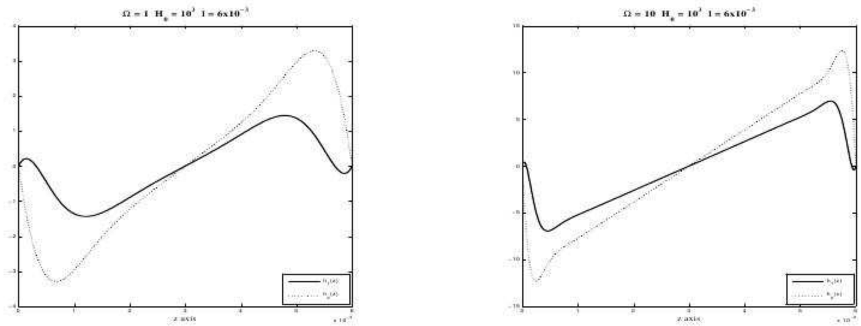

Figure 12: Plots showing the behavior of $h_1,\ h_2$ when $H_0 = 10^3 A\, m^{-1}$

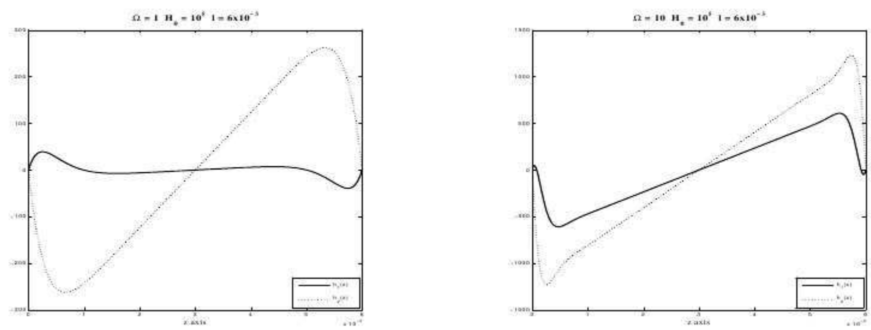

Figure 13: Plots showing the behavior of $h_1,\ h_2$ when $H_0 = 10^5 A\, m^{-1}$

Acknowledgements

This work was performed under the auspices of G.N.F.M. of INdAM and supported by Italian MIUR.

References

[1] R. Berker, A new solution of the Navier-Stokes equation for the motion of a fluid contained between two parallel plates rotating about the same axis, *Arch. Mech. Stos.*, **31** (1980), 265-280.

[2] A.C. Eringen, G.A. Maugin, *Electrodynamics of Continua II - Fluids and Complex Media*, Springer-Verlag, Berlin (1990).

[3] H.A. Jasmine, J.S. Gajjar, Convective and absolute instability in the incompressible boundary layer on a rotating disk in the presence of a uniform magnetic field, *J. of Eng. Math.*, **52** (2005), 337-353.

[4] H.K. Mohanty, Hydromagnetic flow between two rotating disks with noncoincident parallel axes of rotation, *The Phys. of Fluids*, **15** (1972), 1456-1458.

[5] S.I. Pai, *Modern Fluidmechanics*, Van Nostrand (1981).

[6] H.J. Pao, Magnetohydrodynamic flows over a rotating disk, *AIAA J.*, **6** (1968), 1285-1291.

[7] M.C. Patria, On flow and stability of a Newtonian fluid past a rotating plane, *Meccanica*, **26** (1991), 135-141.

[8] T. von Karman, Über laminare und turbulente Reibung, *Z.A.M.M.*, **1** (1921), 233-252.