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MULTIPLE SOLUTIONS FOR ELLIPTIC EQUATIONS WITH NONSTANDARD GROWTH CONDITIONS

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Abstract: In this paper, we investigate the existence and multiplicity of solutions for $p(x)$-Laplacian Dirichlet problem with nonstandard growth conditions by using the Ricceri variational principle. We verify the equation has at least three solution and obtain generalizes the corresponding result

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1. Introduction

The $p(x)$-laplacian Dirichlet problems are

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u = \lambda f(x,u) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where $\Omega \subset R^N (N \geq 3)$ is a bounded domain with smooth boundary, $\lambda > 0$ is a real number, $p(x)$ is a continuous function on $\overline{\Omega}$ with $\inf_{x\in\overline{\Omega}} p(x) > N$.

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The main interest in studying the $p(x)$-Laplace operator div $(|\nabla u|^{p(x)-2}\nabla u)$ of such problems arises from the studying of electrorheological fluids and elastic mechanics. The $p(x)$-Laplace operator is a generalization of the classical p-Laplace operator div $(|\nabla u|^{p-2}\nabla u)$ obtained in the case when p is a positive constant.

In the case, when $1 < p(x) < N$ for any $x \in \overline{\Omega}$, there were extensive studying in the last few decades with problems of type (P), but with different parameter λ and conditions which $f(x, u)$ satisfy. For example, when $\lambda = 1$ and $f(x, u) = f(x)$, Fan and Zhang in [8] established the equation:

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda f(x) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique weak solution.

In this paper, we assume $f(x, u)$ satisfies the following conditions:

(f_1) $f : \Omega \times R \to R$ is a Caratheodory function and satisfies

$$|f(x,t)| \leq c_1 + c_2|t|^{\alpha(x)-1}, \quad \forall (x,t) \in \Omega \times R\,,$$

where $\alpha \in C(\overline{\Omega})$, $\alpha(x) > 1$ and $1 < \alpha^+ = \max_{x\in\bar{\Omega}} \alpha(x) < p^- = \min_{x\in\bar{\Omega}} p(x)$.

(f_2) $f(x,t) < 0$ when $|t| \in (0,1)$;

$f(x,t) \geq M > 0$ when $|t| \in (t_0, \infty)$ $t_0 > 1$.

We will study equation (P) in the case when $p(x) > N$ for any $x \in \overline{\Omega}$, which exist at least three weak solutions in a variable exponent Sobolev space by using the main tool a three critical point theorem due to Ricceri (see Theorem 1 in [14]).

The paper is organized as follows. We will introduce some basic proposition and preliminary results in Section 2, including the variable exponent Lebesgue, Soboleve spaces and Ricceri's three-critical-points theorem. In Section 3, we will give the main result and its proof.

2. Preliminary Results

In order to discuss problem (P), we need some theories of Lebesgue-Sobolev space with variable exponent $W_0^{1,p(x)}(\Omega)$, which we call generalized Lebesgue Sobolev spaces. Firstly we state some basic properties of $W_0^{1,p(x)}(\Omega)$ which will be used later. For details, see the papers [2], [3], [4], [5], [7], [8], [10], [12], the book written by Musielak [13]. Let Ω be a bounded open subset of $\mathbf{R}^N$, denote $C_+(\overline{\Omega}) = \{h \mid h \in C(\overline{\Omega}),\ \ h(x) > 1 \text{ for all } x \in \overline{\Omega}\}$. For any $h \in C_+(\overline{\Omega})$, we define $h^+ = \max_{x\in\overline{\Omega}} h(x)$ and $h^- = \min_{x\in\overline{\Omega}} h(x)$. For any $p(x) \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space $L^{p(x)}(\Omega)$={ $u : u$ is a measurable real-valued function $\int_\Omega |u|^{p(x)}dx < \infty$} with the norm

$$\|u\|_{L^{p(x)}(\Omega)} = \|u\|_{p(x)} = \inf\left\{\mu > 0 : \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\right\}.$$

Let the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to R$ defined by $\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)}dx$. If $u \in L^{p(x)}(\Omega)$ then the following relations hold

$$|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^+}\ , \tag{2}$$

$$|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^-}\ . \tag{3}$$

Next, we define the variable exponent Sobolev space

$$W^{1,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega)\ : |\nabla u| \in L^{p(x)}(\Omega)\right\}$$

with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}\ .$$

Denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. Here, we always assume that $p^- > 1$, it has the following equivalent norm

$$\|u\| = \inf\{\mu > 0\ : \int_\Omega (\left|\frac{u(x)}{\mu}\right|^{p(x)} + \left|\frac{\nabla u(x)}{\mu}\right|^{p(x)})dx \le 1\}$$

on $W^{1,p(x)}(\Omega)$. Let

$$I(u) = \int_\Omega (|u|^{p(x)} + |\nabla u|^{p(x)})dx,$$

if $u \in W^{1,p(x)}(\Omega)$, then there are the following relations

$$\|u\| < 1(= 1, > 1) \Leftrightarrow I(u) < 1(= 1, > 1), \tag{4}$$

$$\|u\| > 1 \Rightarrow \|u\|^{p^-} \le I(u) \le \|u\|^{p^+}, \tag{5}$$

$$\|u\| < 1 \Rightarrow \|u\|^{p^+} \le I(u) \le \|u\|^{p^-}. \tag{6}$$

Theorem 2.1. *Let X be a separable and reflexive real Banach space; Φ: $X \to R$ a continuously G â teaux differentiable and sequentially weakly lower semicontinuous functional whose G â teaux derivative admits a continuous inverse on X^*; Ψ: $X \to R$ a continuously G â teaux differentiable functional whose G â teaux derivative is compact. Assume that:*

(i) $\lim_{\|u\|\to\infty}(\Phi(u) + \lambda\Psi(u)) = \infty$, *for all* $\lambda > 0$;

(ii) there are $r \in R$ *and* $u_0, u_1 \in X$ *such that* $\Phi(u_0) < 0 < \Phi(u_1)$;

(iii) $inf_{u\in\Phi^{-1}((-\infty,r])}\Psi(u) > \frac{(\Phi(u_1)-r)\Psi(u_0)+(r-\Phi(u_0))\Psi(u_1)}{\Phi(u_1)-\Phi(u_0)}$.

Then there exist an open interval $\Lambda \subset (0,\infty)$ *and a positive real number* q *such that each* $\lambda \in \Lambda$ *the equation* $\Phi'(u) + \lambda\Psi'(u) = 0$ *has at least three solutions in* X *whose norms are less than* q.

Proposition 2.2. (see [7], [10]) *The spaces* $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ *and* $W_0^{1,p(x)}(\Omega)$ *are separable and reflexive Banach spaces.*

Proposition 2.3. (see [7], [10]) see The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p^0(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p^0(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p^0(x)}(\Omega)$, $\int_\Omega |uv|\, dx \leq 2\, |u|_{p(x)}\, |v|_{p^0(x)}$.

Proposition 2.4. (see [6]) *If* $p_1, p_2 \in C_+(\overline{\Omega}), p_1(x) \leq p_2(x)$ *for any* $x \in \overline{\Omega}$, *then* $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$, *and the imbedding is continuous.*

Remark 1. If $N < p^- \leq p(x)$ for any $x \in \overline{\Omega}$, by Theorem 2.2 in [7] we deduce that $W_0^{1,p(x)}(\Omega)$ is continuously embedded in $W_0^{1,p^-}(\Omega)$. Since $N < p^-$ it follows that $W_0^{1,p^-}(\Omega)$ is compactly embedded in $C(\overline{\Omega})$. Thus, we deduce that $W_0^{1,p(x)}(\Omega)$ is compactly embedded in $C(\overline{\Omega})$. Defining $\|u\|_\infty = \sup_{x\in\overline{\Omega}} |u(x)|$, we find that there exists a positive constant $c > 0$ such that

$$\|u\|_\infty \leq c\|u\|, \qquad \forall u \in W_0^{1,p(x)}(\Omega).$$

3. Definition and the Main Result

In this part, we will prove that (P) exists three weak solutions for the general case.

Definition 3.1. We say $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of equation (P) if

$$\int_\Omega (|\nabla u|^{p(x)-2}\nabla u \nabla v + |u|^{p(x)-2}uv)dx - \lambda \int_\Omega f(x,u)vdx = 0$$

for any $v \in W_0^{1,p(x)}(\Omega)$.

Theorem 3.2. Assume that $\inf_{y\in\bar{\Omega}} p(y) > N$ for any $x \in \bar{\Omega}$ and $f(x,u)$ satisfies $(f_1), (f_2)$. Then there exist an open interval $\Lambda \subset (0,\infty)$ and a positive real number $\rho > 0$ such that each $\lambda \in \Lambda$ the equation (P) has at least three solutions whose norms are less than ρ.

Proof of Theorem 3.2. Let X denote the generalized Sobleve space $W_0^{1,p(x)}(\Omega)$.

Defining

$$F(x,t) = \int_0^t f(x,s)ds.$$

In order to use Ricceri's result (Theorem 2.1), we define the functions $\Phi, \Psi : X \to R$ by

$$\Phi(u) = \int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx, \quad \Psi(u) = -\int_\Omega F(x,u)dx\,.$$

Then according to the proof of Proposition 3.1 in [12], we obtain that $\Phi, \Psi \in C^1(X,R)$ with the derivatives given by

$$\langle \Phi^{'}(u), v\rangle = \int_\Omega (|\nabla u|^{p(x)-2}\nabla u \nabla v + |u|^{p(x)-2}uv)dx, \quad \langle \Psi^{'}(u), v\rangle = -\int_\Omega f(x,u)vdx$$

for all $u, v \in X$.

Thus, if there exists a $\lambda > 0$ such that u is a critical point of the operator $\Phi + \lambda\Psi$ that is to say $\Phi'(u) + \lambda\Psi'(u) = 0$, then we can deduce that $u \in X$ is a weak solution of equation (P) by Definition 3.1. Similar to the conclusion of Theorem 2.1, for proving our result, it is enough to verify Φ and Ψ satisfy the hypotheses of Theorem 2.1.

It is obvious that $(\Phi^{'})^{-1} : X^* \to X$ exists and continuous since $\Phi^{'} : X \to X^*$ is a homeomorphism by [8]. Moreover, $\Psi^{'} : X \to X^*$ is completely continuous because of assumption (f_1) from [8], which implies $\Psi^{'}$ is compact.

Next, we will verify that condition (i) in Theorem 2.1 is fulfilled. In fact, by relation (5), we have

$$\Phi(u) \geq \frac{1}{p^+}\int_\Omega (|\nabla u|^{p(x)} + |u(x)|^{p(x)})dx = \frac{1}{p^+}I(u) \geq \frac{1}{p^+}\|u\|^{p^-}, \ \forall\, u \in X, \ \|u\| > 1\,.$$

On the other hand, due to assumption (f_1), we have $\Psi(u) = -\int_\Omega F(x,u)dx = \int_\Omega -F(x,u)dx$ and $|F(x,t)| \leq c_1|t| + c_2\frac{1}{\alpha(x)}|t|^{\alpha(x)}$, therefore,

$$\begin{aligned}\Psi(u) &\geq -c_1\int_\Omega |u|dx - c_2\int_\Omega \frac{1}{\alpha(x)}|u|^{\alpha(x)} \\ &\geq -c_3\|u\| - c_2 \cdot \frac{1}{\alpha^+} \cdot (\int_\Omega (|u|^{\alpha^+} + |u|^{\alpha^-})dx = -c_3\|u\| - c_4(|u|_{\alpha^+}^{\alpha^+} + |u|_{\alpha^-}^{\alpha^-}).\end{aligned}$$

Using Remark 1, we know X is continuously embedded in $L^{\alpha^\pm}(\Omega)$. Furthermore, we can find two positive constants $d_1, d_2 > 0$ such that

$$|u|_{\alpha^+} \leq d_1\|u\|\,, \quad |u|_{\alpha^-} \leq d_2\|u\|, \quad \forall u \in X.$$

so $\Psi(u) \geq -c_3\|u\| - c_4 d_1 \|u\|^{\alpha^+} - c_4 d_2 \|u\|^{\alpha^-}$. It follows that

$$\Phi(u) + \lambda\Psi(u) \geq \frac{1}{p^+}\|u\|^{p^-} - \lambda c_3\|u\| - \lambda c_4(d_1\|u\|^{\alpha^+} + d_2\|u\|^{\alpha^-}), \quad \forall u \in X.$$

Since $1 < \alpha^+ < p^-$, it follows that $\lim_{\|u\|\to\infty}(\Phi(u) + \lambda\Psi(u)) = \infty$ and (i) is verified.

In the sequel, we verify that conditions (ii) and (iii) in Theorem 2.1 are satisfied.

By $F^{'}(x,t) = f(x,t)$ and assumption (f_2), $F(x,t)$ is increasing for $t \in (t_0, \infty)$, $t_0 > 1$ and decreasing for $t \in (0,1)$, uniformly with respect to x and $F(x,0) = 0$ is obvious. Because $F(x,t) \to \infty$ when $t \to \infty$ by (f_2). Then, there exists a real number $\delta > t_0$ such that

$$F(x,t) \geq 0 = F(x,0) \geq F(x,\tau), \qquad \forall \; x \in \Omega, \quad t > \delta, \quad \tau \in (0,1).$$

Because $\Omega \subset R^N$ is bounded, there exists a cube Q such that $\Omega \subset Q$. Suppose the length of Q is $L > 0$. Divide L into K equal parts, then Q is divided into K^N subcubes and $s = \frac{L}{K}$ is the length of small cubes. Note the number of small cubes which lie in Ω entirly is $S(K)$. It is obvious that $S(K) \to \infty$ when $K \to \infty$. We denote $Q(x_0, s)$ is a cube with center x_0 and its length is s, moreover, $Q(x_0, s) \subset \Omega$.

Let a, b be two real numbers such that $0 < a < \min\{1, c\}$ with c given in Remark 1 and $b > \delta$. When $t \in [0,a]$, we have $F(x,t) \leq F(x,0)$, it follows that

$$\int_\Omega \sup\nolimits_{0<t<a} F(x,t)dx \leq \int_\Omega F(x,0)dx = 0.$$

We define

$$v_i(x) = \begin{cases} b & \text{if} \quad x \in Q_i(x_i, \frac{s}{2}) \\ 0 & \text{if} \quad x \in Q_i(x_i, s) \setminus Q_i(x_i, \frac{s}{2} + \varepsilon) \end{cases}$$

in $Q_i = Q_i(x_i, s)$ $1 \leq i \leq S(K)$, $v_i(x) \in C_0^\infty(Q_i, [0,b])$ for $\varepsilon > 0$ and ε will be determined later.

Consider $u_0, u_1 \in X$, $u_0(x) = 0$ for any $x \in \Omega$ and

$$u_1(x) = \begin{cases} v_i & \text{if} \quad x \in Q_i \,, \\ 0 & \text{if} \quad x \in \Omega \setminus \overline{\bigcup_{i=1}^{S(K)} Q_i} \,. \end{cases}$$

We also define $r = \frac{1}{p^+}.(\frac{a}{c})^{p^+}$, Clearly, $r \in (0,1)$. A simple computation implies

$$\Phi(u_1) = \int_\Omega \frac{1}{p(x)}(|\nabla u_1(x)|^{p(x)} + |u_1(x)|^{p(x)})dx$$

$$\geq \frac{1}{p^+} \sum_{i=1}^{S(K)} \int_{Q_i} (|\nabla v_i(x)|^{p(x)} + |v_i(x)|^{p(x)}) dx$$

$$= \frac{1}{p^+} \sum_{i=1}^{S(K)} [\int_{Q_i(x_i,\frac{s}{2})} b^{p(x)} dx + \int_{Q_i(x_i,\frac{s}{2}+\varepsilon)\setminus Q_i(x_i,\frac{s}{2})} (|\nabla v_i(x)|^{p(x)} + |v_i(x)|^{p(x)})] dx$$

$$\geq \frac{1}{p^+} \sum_{i=1}^{S(K)} \int_{Q_i(x_i,\frac{s}{2})} b^{p(x)} dx \geq \frac{1}{p^+} \cdot \sum_{i=1}^{S(K)} (b^{p^-} \cdot (\frac{s}{2})^N) = \frac{1}{p^+} \cdot S(K) \cdot b^{p^-} \cdot (\frac{s}{2})^N$$

and especially

$$\Phi(u_0) = \Psi(u_0) = 0.$$

Next, we can choose K large enough such that $S(K) > \frac{r \cdot p^+}{b^{p^-} \cdot (\frac{s}{2})^N}$, so $\Phi(u_1) > r$. Thus, we obtain

$$\Phi(u_0) < r < \Phi(u_1)$$

and (ii) in Theorem 1 is verified.

On the other hand, we have

$$-\frac{(\Phi(u_1) - r)\Psi(u_0) + (r - \Phi(u_0))\Psi(u_1)}{\Phi(u_1) - \Psi(u_0)} = -r \cdot \frac{\Psi(u_{1)}}{\Phi(u_1)} = r \cdot \frac{-\Psi(u_1)}{\Phi(u_1)}\,.$$

Here

$$-\Psi(u_1) = \int_\Omega F(x, u_1(x)) dx = \int_{\bigcup_{i=1}^{S(K)} Q_i} F(x, u_1(x)) dx = \sum_{i=1}^{S(K)} \int_{Q_i} F(x, v_i(x)) dx$$

$$= \sum_{i=1}^{S(K)} \left(\int_{Q_i(x_i,\frac{s}{2})} F(x, b) dx + \int_{Q_i(x_i,\frac{s}{2}+\varepsilon)\setminus Q_i(x_i,\frac{s}{2})} F(x, v_i) dx \right).$$

Furthermore, $F(x, b) > 0$ and $b > \delta$. We let $\int_{Q_i(x_i,\frac{s}{2})} F(x, b) dx = b_i > 0$. There exists a constant $M > 0$ such that $|F(x, u_i)| \leq M$ in $Q_i(x_i, \frac{s}{2} + \varepsilon) \setminus Q_i(x_i, \frac{s}{2})$. Since $0 < u_i(x) < b$ in $Q_i(x_i, \frac{s}{2} + \varepsilon) \setminus Q_i(x_i, \frac{s}{2})$, it follows that

$$\int_{Q_i(x_i,\frac{s}{2}+\varepsilon)\setminus Q_i(x_i,\frac{s}{2})} F(x, u_i) dx \geq -M \cdot O(\varepsilon)\,.$$

Since

$$\left| Q_i(x_i, \frac{s}{2} + \varepsilon) \setminus Q_i(x_i, \frac{s}{2}) \right| = (\frac{s}{2} + \varepsilon)^N - (\frac{s}{2})^N = O(\varepsilon),$$

so

$$-\Psi(u) \geq \sum_{i=1}^{S(K)} (b_i - M \cdot O(\varepsilon)) = \sum_{i=1}^{S(K)} b_i - S(K) \cdot M \cdot O(\varepsilon)\,.$$

We can choose $\varepsilon > 0$ small enough such that $S(K) \cdot M \cdot O(\varepsilon) < \frac{1}{2}\sum_{i=1}^{S(K)} b_i$, it follows that

$$-\Psi(u) > \frac{1}{2}\sum_{i=1}^{S(K)} b_i > 0.$$

Then, we obtain

$$-\frac{(\Phi(u_1)-r)\Psi(u_0)+(r-\Phi(u_0))\Psi(u_1)}{\Phi(u_1)-\Psi(u_0)} = -r\cdot\frac{\Psi(u_1)}{\Phi(u_1)} = r\cdot\frac{-\Psi(u_1)}{\Phi(u_1)} > 0.$$

Next, we consider the case when $u \in X$ with $\Phi(u) \leq r < 1$. Since $\frac{1}{p^+}I(u) \leq \Phi(u) \leq r$, we obtain $I(u) \leq p^+ \cdot r = (\frac{a}{c})^{p^+} < 1$. It follows that $\|u\| < 1$ by (4). Furthermore, by (6), it is clear that

$$\frac{1}{p^+}\cdot\|u\|^{p^+} \leq \frac{1}{p^+}\cdot I(u) \leq \Phi(u) \leq r.$$

Thus, using Remark 1, for any $u \in X$ with $\Phi(u) \leq r$, we obtain

$$|u(x)| \leq c\cdot\|u\| \leq c\cdot(p^+\cdot r)^{\frac{1}{p^+}} = a, \qquad \forall \ x \in \Omega.$$

The above inequality shows

$$-\inf_{u\in\Phi^{-1}(-\infty,r]}\Psi(u) = \sup_{u\in\Phi^{-1}(-\infty,r]} - \Psi(u) \leq \int_\Omega \sup_{0\leq t\leq a}F(x,t)dx \leq 0.$$

It follows that

$$-\inf_{u\in\Phi^{-1}(-\infty,r]}\Psi(u) < r\cdot\frac{\int_\Omega F(x,b)dx}{\int_\Omega \frac{1}{p(x)}\cdot b^{p(x)}dx}.$$

That is

$$\inf_{u\in\Phi^{-1}(-\infty,r]}\Psi(u) > \frac{(\Phi(u_1)-r)\Psi(u_0)+(r-\Phi(u_0))\Psi(u_1)}{\Phi(u_1)-\Psi(u_0)},$$

which means condition (iii) in Theorem 2.1 is verified.

Since all the assumptions of Theorem 2.1 are satisfied. So the conclusion that there exists an open interval $\Lambda \subset (0,\infty)$ and a positive constant $\rho > 0$ such that for any $\lambda \in \Lambda$ the equation

$$\Phi'(u) + \lambda\Psi'(u) = 0$$

has at least three solutions in X whose norms are less than ρ. It follows that Theorem 3.2 holds. □

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