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METRIC REGULARITY AND SUBDIFFERENTIAL OF A FUNCTION AT A POINT FOR A CLASS OF FUNCTIONS

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Abstract: The property of metric regularity of a multifunction can be emphatized in several ways. The Frechet differential at a point, the strict differential at a point or the strict slope at a point of a function can be directly linked to metric regularity. The condition which involve the metric regularity are mentioned in [9], [5], [8] and [2]. This article emphasizes the connection between the subdifferential of a function at a point and the metric regularity for new multifunctions.

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1. Preliminaries

There are several equivalent definitions of metric regularity on Banach spaces. The concept has been presented in many basic papers and articles. The theory of metric regularity is an extension of two classical results: the Lyusternik Theorem and the Graves surjection functions Theorem. Developments in nonsmooth analysis in the 1980s and 1990s paved the way for a number of far-reaching extensions of these results. The well-known Lyusternik-Graves Theo-

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rem assures the property of metric regularity for differentiable functions at each point from the graph, proving through the equality $\operatorname{reg} F(\bar{x}|\bar{y}) = \operatorname{reg} DF(\bar{x})$ exactly the modulus of metric regularity [2]. Additionally, the Robinson-Ursescu Theorem [2] assures the metric regularity at the graph points for the multifunctions which have a closed and conex graph. The following two equivalent formulations of the Banach-Schauder Theorem are esential for the understanding of what follows. Let X and Y be a Banach spaces and let $A : X \to Y$ be a bounded linear operator mapping X onto Y. Then:

(a) there is a $N > 0$ such that $\mathbb{B}_Y \subset A(N\mathbb{B}_X)$ which means that the unit ball in Y is covered by the image of the ball of radius N in X, and

(b) there is a $K > 0$ such that for any $(x, y) \in X \times Y$ we have $d(x, A^{-1}(y)) \leq K\|y - Ax\|$, where $A^{-1}(y)$ is the inverse image of y under A and d is the metric.

It is proven that the higher bound of N and K is the same, given through $\|(A^*)^{-1}\|$; that is the norm of the inverse of the adjoint operator of A.

2. Backgroud in Metric Regularity

2.1. Terminology, Notations, Definitions. Equivalence Definitions

Unless otherwise stated, we always assume set-valued maps to be closed-valued, that is, we assume that all sets $F(x)$ are closed, where $F : X \rightrightarrows Y$ is a set-valued function and X, Y are Banach spaces. The sets

$$\operatorname{dom} F = \{x|F(x) \neq \Phi\} \text{ and } \operatorname{Im}, F = \cup\{F(x) : x \in X\} \tag{2.1}$$

are respectively called the domain and the image of F. It will be presumed that F is a self application, meaning that the domain of F is nonempty and $F(X) \subset Y$ with strict inclusion. A set-valued map is often identified by its graph

$$Gr\ f = \{(x, y) \in X \times Y|y \in F(x)\}. \tag{2.2}$$

The inverse of F is defined by

$$F^{-1}(y) = \{x \in X|y \in F(x)\}. \tag{2.3}$$

From their definition, $\operatorname{dom} F^{-1} = \operatorname{Im} f$. Moreover, if $f : X \to \bar{\mathbb{R}}$ and the multifunction

$$x \mapsto \{\alpha \in \mathbb{R}|\alpha \geq f(x)\} \tag{2.4}$$

is defined, its graph is called epigraph of f and it is defined by

$$\operatorname{epi} f = \{(x, \alpha) \in X \times \mathbb{R}|\alpha \geq f(x)\}. \tag{2.5}$$

The symbol $d(\cdot,\cdot)$ is used for distance. Occasionally, we shall consider the parametric distance $d_\alpha((x,y),(u,v)) = d(x,u) + \alpha d(y,v)$ with $\alpha > 0$, on the product space $X \times Y$. Let us also introduce the closed ball of center x and radius r as

$$B(x,r) = \{y | d(x,y) \le r\}. \tag{2.6}$$

Definition 2.1. (see [9]) Let V be a subset of the set $X \times Y$. We say that F is metrically regular on V if then there exists $K > 0$ such that

$$(x,y) \in V \Rightarrow d(x, F^{-1}(y)) \le K d(y, f(x)). \tag{2.7}$$

The smallest K for which (2.7) holds will be called the norm of metric regularity on F and written $\mathrm{Reg}_\vee F$.

Definition 2.2. (see [9]) F is metrically regular near $(\bar{x},\bar{y}) \in Gr\ F$ if for some $\epsilon > 0$ it is metrically regular on the set $V = B(\bar{x},\epsilon) \times B(\bar{y},\epsilon)$. The lower bound of such K in this case will be called the norm of metric regularity of F near $(\bar{x},\bar{y})$ and written Reg $F(\bar{x},\bar{y})$.

The following proposition assured the equivalence of Definitions 2.1 and 2.2.

Proposition 2.1. (see [9]) *A set-valued map is metrically regular near $(\bar{x},\bar{y}) \in Gr\ F$ if and only if is metrically regular on the set*

$$V = \{(x,y) \in B(\bar{x},\epsilon) \times B(\bar{y},\epsilon) : d(y, F(x)) \le \epsilon\}$$

for some $\epsilon > 0$.

Definition 2.3. (see [9]) We say that F covers on V at a linear rate if there is a $K > 0$ such that

$$(x,y) \in V,\ v \in F(x) \text{ and } d(v,y) < \epsilon \Rightarrow (\exists) u : d(u,x) \le Kt \text{ and } y \in F(u). \tag{2.8}$$

The lower bound of those K for which (2.8) holds will called the norm of covering of F on V, and its inverse, the constant of covering, written $Sur_\vee F$.

Definition 2.4. (see [9]) Let $W \subset X \times Y$. We say that F is pseudo-Lipschitz on W if there is a $K > 0$ such that

$$(x,y) \in W \text{ and } y \in F(u) \Rightarrow d(y, F(x)) \le K d(x,u). \tag{2.9}$$

The smallest K for (2.9) holds is called the pseudo-Lipschitz norm of F on W.

The following proposition assured the equivalence of Definitions 2.3 and 2.4.

Proposition 2.2. (see [9]) *The following statements are equivalent:*

(a) F is regular on V;

(b) F covers on V at a linear rate;

(c) F^{-1} *is pseudo-Lipschitz on* $W = \{(x,y)|(x,y) \in V\}$.

Moreover, the norms of regularity and covering of F *on* V *and the pseudo-Lipschitz norm of* F^{-1} *on* W *are equal. In particular,* $\operatorname{Reg}_{\vee} F \cdot Sur_{\vee} F = 1$.

The graph of a function allows a characterization of its metric regularity. Only the definition will be mentioned here.

Definition 2.5. (see [9]) We say that F is graph-regular on V with norm not greather then K if

$$d(x, F^{-1}(y)) \leq d_k((x,y), GrF) \tag{2.10}$$

for all $(x,y) \in V$.

Another characterization of the metrical regularity is obtained with the help of the strict slope of a function at a point (Theorem 2 from [9] and Theorem 3 from [9], p. 516 to be seen). In the following subsection, we shall provide a characterization of the subdifferential of a function at a point. With the help of these, we will be able to give in Section 3 a metrical regularity result at a maximum point.

2.2. Subgradient. Subdifferential. Normals

Definition 2.6. (see [4]) Let Ω be a nonempty subset of X.

(i) Given $x \in \Omega$ and $\epsilon \geq 0$ define the set of ϵ-normals to Ω at x by

$$\hat{N}_\epsilon(x;\Omega) = \left\{ x^* \in X^* : \limsup_{u \xrightarrow{\Omega} x} \frac{< x^*, u - x >}{\|u - x\|} \leq \epsilon \right\}. \tag{2.11}$$

If $x \notin \Omega$, we put $\hat{N}_\epsilon(x;\Omega) = \Phi$. When $\epsilon = 0$ elements of (2.11) are called Frechet normals and their collection, denoted by $\hat{N}(x;\Omega)$, is the prenormal cone to Ω at x;

(ii) Let $\bar{x} \in \Omega$. Then $x^* \in X^*$ is a basing limiting normal to Ω at $\hat{x}$ if there are sequences $\epsilon_k \to 0$, $\epsilon_k > 0$, $x_k \xrightarrow{\Omega} \bar{x}$ and $x_k^* \xrightarrow{\omega^*} x^*$ such that $x_k^* \in \hat{N}_{\epsilon_k}(x_k;\Omega)$ for all $k \in \mathbb{N}$. The collection of such normals

$$N(\bar{x};\Omega) = \limsup_{\substack{x \to x^* \\ \epsilon \searrow 0}} \hat{N}_\epsilon(x;\Omega) \tag{2.12}$$

is the normal cone to Ω at $\bar{x}$. Put $N(\bar{x};\Omega) = \Phi$, for $\bar{x} \notin \Omega$.

Definition 2.7. (see [4]) Consider a function $\varphi : X \to \bar{\mathbb{R}}$ and a point

$\bar{x} \in X$ with $|\varphi(\bar{x})| < \infty$. The set

$$\partial\varphi(\bar{x}) = \{x^* \in X^* | (x^*, -1) \in N((\bar{x}, \varphi(\bar{x}); epi\, \varphi))\} \tag{2.13}$$

is the subdifferential of φ at x and its elements are basic subgradient of φ at this point.

Remark 2.1. If $\varphi = d_\Omega : X \to \mathbb{R}$, $d_\Omega(x) = \inf_{u \in \Omega} \|x - u\|$ and $\Omega \subset X$, the expression of the subdifferential at $\bar{x} \in \Omega$ is given by:

Proposition 2.3. (see [4]) *We have*

$$\{x^* | < x^*, x - \bar{x} > \leq \varphi(x) - \varphi(\bar{x})\} = \partial\, \varphi(\bar{x}) = \hat{N}(\bar{x}, \Omega) \cap \mathbb{B}^*, \tag{2.14}$$

where $\mathbb{B}^*$ *is closed unit ball of the dual space.*

Note that, in this case, the relation that defines the subgradient extends the one that characterizes the Gateaux derivative at $\bar{x}$ of a convex and Gateaux differentiable function $f : X \to \mathbb{R}$

$$< f'(\bar{x}), x - \bar{x} > \leq f(x) - f(\bar{x}),\ \forall x \in X. \tag{2.15}$$

In particular, if $\Omega = \{0\}$, the associated function $\varphi = \|\cdot\|$ (i.e. $\varphi(x) = \|x\|$) is convex and continuous. So, according to a well known result (see, for instance, [9, p. 104]) φ has a subdifferential (in the sense of convex analysis), expressed by the first term of (2.14). It may be also characterized as follows.

Proposition 2.4. *Assume that* $(X, \|\cdot\|)$ *is a Banach space. Then*

$$\partial\, \varphi(x) = \{x^* \in X^* : \|x^* \leq 1\|\}, if\ x = 0,$$

$$\partial\, \varphi(x) = \{x^* \in X^* : \|x^*\| = 1, < x^*, x > = \|x\|\}, if\ x \neq 0\,.$$

Proof. i) If $x = 0$, $x^* \in \partial\, f(0)$ if and only if $\|y\| \geq < x^*, y >$, $(\forall) y \in X$; that is equivalent to

$$\|x^*\|_{X^*} = \sup_{y \neq 0} \frac{< x^*, y >}{\|y\|} \leq 1\,.$$

ii) If $x \neq 0$ and $x^* \in X^*$ such that $< x^*, x > = \|x\|_X$ and $1 = \|x^*\|_{X^*} = \sup\limits_{y \neq 0} \frac{<x^*,y>}{\|y\|}$ so that $\frac{<x^*,y>}{\|y\|} \leq 1$, $(\forall) y \in X$, $y \neq 0$. Consequently:

$$\|y\| - \|x\| \geq < x^*, y - x >,\ (\forall) y \in X \text{ and } x^* \in \partial\, f(x)\,.$$

Conversely, for $x^* \in \partial\, f(x)$, we have $-\|x\| = \|0\| - \|x\| \geq - < x^*, y >$, so $< x^*, x > \geq \|x\|$. We also have the inequality

$$\|x\| = \|2x\| - \|x\| \geq < x^*, 2x - x > = < x^*, x >,$$

followed by $< x^*, x > = \|x\|$. For some $y \in X$ and $\lambda > 0$ it holds $\|y + x/\lambda\| - 1/\lambda\|x\| \geq < x^*, y >$. Setting λ to tend towards $+\infty$ we get for all $y \in X$ that

$\|y\| \geq < x^*, y >$, so $\|x^*\|_{X^*} = 1$. □

3. Main Results

In the following, X, Y are normed spaces.

Definition 3.1. (see [4]) Let $f : X \to \bar{\mathbb{R}}$ be a function and let $\bar{x} \in dom\, f$. The quantity

$$|\nabla f|(\bar{x}) = \limsup_{\substack{u \to \bar{x} \\ u \neq \bar{x}}} \frac{|f(\bar{x}) - f(u)|^+}{d(\bar{x}, u)} \tag{3.1}$$

is called the strong slope of f at $\bar{x}$.

For example, for a Frechet differentiable function on a normed linear space, $|\nabla f|(\bar{x})$ is equal to the norm of $\nabla f(\bar{x})$.

Definition 3.2. (see [4]) Let $F : X \rightrightarrows Y$ set-valued map and $(x, y) \in GrF$. The coderivative of F at (x, y) is the set-valued map $D^*F(x, y) : Y^* \to X^*$ defined by

$$D^*F(x, y)(y^*) = \{x^* \in X^* | (x^*, -y^*) \in N(GrF, (x, y))\}$$

We have the following result:

Theorem 3.1. (see [9]) *If ∂ is the subdifferential of f, then for any f and any x we have*

$$|\nabla f|(x) \leq \inf\{\|x^*\| : x^* \in \partial f(x)\}. \tag{3.2}$$

Remark 3.1. If X is a Asplund space and f is lower semicontinuous and open $U \subset X$, then we have equality in (3.2) for all $x \in U$.

Theorem 3.2. (see [9]) *If X and Y are finite dimensional spaces, then F is regular near $(\bar{x}, \bar{y}) \in GrF$ if and only if*

$$Ker\; D^*F(\bar{x}, \bar{y}) = \{0\}, \tag{3.3}$$

where $Ker\; D^*F(\bar{x}, \bar{y}) = \{y^* \in Y^* : 0 \in D^*F(\bar{x}, \bar{y})(y^*)\}$ and D^* is the coderivative associated with limiting subdifferential.

Proposition 3.1. *If $f : X \to \bar{\mathbb{R}}$, $\bar{x} \in X$ so that:*

(i) f convex, proper, lower semicontinuous,

(ii) $Gr\ f$ closed,

(iii) $(\exists) U \subset X$ for which $f(u) \leq f(\bar{x})$, $(\forall) u \in U$, U neighborhood of $\bar{x}$,

then f is pseudo-Lipschitzian on $V = \{(\bar{x}, \bar{y}) | y \in f(u), y \neq f(\bar{x})\}$ or f^{-1} has *the property of metric regularity on* $W = \{(y, \bar{x}) | (\bar{x}, y) \in V\}$ *and* $\text{Reg}_W f^{-1} = inf\{\|x^*\| : x^* \in \partial f(\bar{x})\}$.

Proof. From f convex, proper, lower semicontinuous, we have that $int[dom(f)] \subseteq dom(\partial f)$ (see [9, p. 105]), so $\partial f(\bar{x})$ is nonempty and because $\bar{x}$ is point of local maximum, we have $f(u) - f(\bar{x}) \geq x^*(x - \bar{x})$. It implies that $f(u) - f(\bar{x}) \geq -x^*(\bar{x} - u)$, so $f(\bar{x}) - f(u) \leq x^*(\bar{x} - u)$ and $f(\bar{x}) - f(u) \geq 0$. We have $f(\bar{x}) - f(u) \leq |x^*(\bar{x} - u)| \leq \|x^*\| \|\bar{x} - u\|$, meanig $d(y, f(\bar{x})) \leq \|x^*\| d(\bar{x}, f^{-1}(y))$, $(\forall) y \in f(U)$ and by the Definition 2.1 we have the conclusion ☐

Remark 3.2. If $\bar{x}$ is a point of global minimum, from Theorem 1.6 from [4] f is not metrically regular. An example is given by $f : X \to [0, \infty)$, $f(x) = |x|$ and $\bar{x} = \bar{y} = 0$, for which $\partial f(0)$ is nonempty, even convex and $\bar{y} = 0 \notin$ *int dom* f^{-1}. Another example could be $f : \mathbb{R} \to [0, \infty)$, $f(x) = x^2$ and $\bar{x} = \bar{y} = 0$; $\partial f(0)$ is non-empty, the function f being convex. The functionals $x^*(t) = 0$ and $x^*(t) = 2t - 1$ are in $\partial f(0)$. The characterization of regularity is given by Theorem 1.6 from [4]. The same situation takes place if $\bar{x}$ is global maximum point.

Next we will characterize the behavior of new multifunctions classes towards the metric regularity. We need:

Definition 3.3. (see [7]) The result of the composition of $G : Y \rightrightarrows Z$ and $F : X \rightrightarrows Y$ is given by the multifunction $G \circ F : X \rightrightarrows Z$ characterized by

$$(G \circ F)(x) = G(F(x)) = \cup\{G(y) : y \in F(x)\}. \tag{3.4}$$

Definition 3.4. (see [7]) The inverse of $G \circ F$ is given by $(G \circ F)^{-1} : Z \rightrightarrows X$ characterized by

$$(G \circ F)^{-1}(z) = F^{-1}(G(z)) = \cup\{F^{-1}(u) : u \in G^{-1}(z)\}. \tag{3.5}$$

Definition 3.5. (see [7]) For X_1 and X_2 two sets we define the projections on X_1 and X_2 through

$$\Pi_{X_i} : X_1 \times X_2 \to X_i,\ \Pi_{X_i}((x_1, x_2)) = x_i,\ i \in \{1, 2\}. \tag{3.6}$$

Proposition 3.2. *If* $F : X \rightrightarrows Y$, $G : Y \rightrightarrows Z$, $G \circ F : X \rightrightarrows Z$ *are multifunctions so that:*

i) F *is metrically regular on* $V \subset X \times Y$, *of norm* k,

ii) G *is metrically regular on* $W \subset Y \subset Z$, *of norm* r,

iii) $F(\Pi_X V) \subset \Pi_Y W$, $G^{-1}(\Pi_Z W) \subset \Pi_Y V$

then $G \circ F$ *is metrically regular on* $U = (\Pi_X V) \times (\Pi_Z W)$ *of norm* $k \cdot r$.

Proof. Let $(x, z) \in U$. From Definition 2.1 for the metric regularity of F, we have the following:

$$d(x,(G\circ F)^{-1}(z)) = d(x,F^{-1}(G^{-1}(z))) \le k\cdot d(F(x),G^{-1}(z))$$

and from the metric regularity of G on W we have

$$k\cdot d(F(x),G^{-1}(z)) \le k\cdot r\cdot d(z,(G\circ F)(x))\,.$$

The iii) condition assures the correctness of the use of Definition 2.1 for F and G multifunctions on V and W sets. □

Definition 3.6. (see [7]) If $F_1, F_2 : X \rightrightarrows Y$ are multifunctions, we have their reunion, the multifunction $F_1 \cup F_2 : X \rightrightarrows Y$ given by

$$(F_1 \cup F_2)(x) = F_1(x) \cup F_2(x). \tag{3.7}$$

From Proposition 4.5 from [7], for the multifunction defined by (3.7), we have

$$(F_1 \cup F_2)^{-1}(B) = F_1^{-1}(B) \cup F_2^{-1}(B), (\forall)B \in \mathcal{P}(Y). \tag{3.8}$$

The property of metric regularity is preserved for the (3.7) multifunction and we have

Proposition 3.3. *Let $V \subset X \times Y$ on which F_1, F_2 are metrically regular of norms k_1 and respectively k_2. Then $F_1 \cup F_2$ is metrically regular on V and the norm is given by*

$$Reg_V F_1 \cup F_2 = \begin{cases} 0, & \text{if } [(F_1\cup F_2)(\Pi_X V)] \cap [\Pi_Y V] \text{ is nonempty}\,, \\ k, & \text{otherwise}\,, \end{cases} \tag{3.9}$$

where $k = inf\{\rho : \rho \in A\}$ and

$$A = \{\rho : \rho d(z,(F_1\cup F_2)(z)) \ge \max\{k_1 d(z,F_1(x)), k_2 d(z,F_2(x))\}\}.$$

Proof. For $z \in [(F_1 \cup F_2)(\Pi_X V)] \cap [\Pi_Y V]$, we have $(x,z) \in V$, $z \in (F_1 \cup F_2)(x)$ and $d(x,(F_1\cup F_2)^{-1}(z)) = d(x,(F_1^{-1}(z)\cup F_2^{-1}(z))) = 0$ since $z \in (F_1 \cup F_2)(x)$. On the other hand, $d(z,(F_1\cup F_2)(x)) = 0$. Remarkably, in this case, we can also have $F_1(x)\cap F_2(x)$ empty set and the expression of the regularity norm cannot be made with the help of A because A is empty. If $[(F_1 \cup F_2)(\Pi_X V)] \cap [\Pi_Y V]$ is empty, $d(z,(F_1 \cup F_2)(x)) > 0$ and because $(F_1 \cup F_2)(x) \supset F_1(x)$ we have $d(z,(F_1\cup F_2)(x)) \le d(z,F_1(x))$ for $(x,z) \in V$. In this situation we have $d(x,(F_1\cup F_2)^{-1}(z)) = d(x,(F_1^{-1}(z)\cup F_2^{-1}(z))) = \min\{d(x,F_1^{-1}(z)),$ $d(x,F_2^{-1}(z))\} \le \frac{1}{2}(k_1 d(z,F_1(x)) + k_2 d(z,F_2(x))) \le \rho d(z,(F_1\cup F_2)(x))$. Finally, A has a positive infimum because it is a subset of $\mathbb{R}_+$. □

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